Homework Set 1 - Solutions

DUE: Thursday April 21

1. Cosmological constant. (a) Calculate the value of the cosmological constant Λ, and also calculate $\Omega_\Lambda = \Lambda/3H_0^2$, the corresponding vacuum energy density in units of critical density ρ_c. Assume that $\Omega_m = 0.3$ and that the curvature vanishes ($k = 0$). Express Λ and $\rho_\Lambda \equiv \Omega_\Lambda \rho_c$ in several different units, including natural units ($\hbar = c = 1$; take $\rho_\Lambda \equiv \Omega_\Lambda \rho_c$ to have units of eV^4).

(b) The acceleration due to the cosmological constant equals $\Lambda r/3$. Numerically compare that to the sun's gravitational acceleration $GM_\odot/r^2$ in order to find the distance r at which they are equal.

1(a) $k=0 \Rightarrow \Omega_\Lambda = 1 - \Omega_m = 0.7$, so $\rho_\Lambda = \Omega_\Lambda \rho_c$

$\rho_c = 3H_0^2/8\pi G = 1.879 \times 10^{-29}\,g\,cm^{-3}\,h^2 = 1.054 \times 10^4\,eV\,cm^{-3}\,h^2$

$= 8.099 \times 10^{-47}\,GeV^4\,h^2$ (see Kolb & Turner Appendix A), $h = 0.7$

$\Rightarrow \rho_\Lambda = \underline{6.4 \times 10^{-30}\,g\,cm^{-3}} = \underline{2.8 \times 10^{-11}\,eV^4} = (2.3 \times 10^{-3}\,eV)^4$

In the Friedmann-Lemaître equation $\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}$, the LHS has dimensions $(\text{time})^{-2}$, so $\frac{\Lambda}{3}$ must also.

Since $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$, $\Lambda = 3H_0^2\Omega_\Lambda = 3(0.7)(3.0856 \times 10^{17}\,s)^{-2}\,h^2$

$\Lambda = \underline{1.1 \times 10^{-35}\,s^{-2}} = 4.7 \times 10^{-66}\,eV^2 = 1.2 \times 10^{-56}\,cm^{-2} = 1.2 \times 10^{-52}\,m^{-2}$

$= 1.1 \times 10^{-20}\,lyr^{-2}$

(b) Subtracting $E(ii) - G(00) \Rightarrow \frac{2\ddot a}{a} = -\frac{8\pi}{3}G\rho - 8\pi G p + \frac{2}{3}\Lambda$. Thus the acceleration of the scale factor due to Λ is $\ddot a = \frac{\Lambda}{3}a$.

If we define distance $r = au$, where u = comoving coordinate, this $\Rightarrow \ddot r = \frac{\Lambda}{3}r$ (the factor c^2 is included when we don't take $c=1$).

(In General Relativity texts, Einstein's equations are shown to imply a generalized Poisson equation $\nabla^2\phi = 4\pi G\rho - \Lambda c^2$, corresponding to a force law $F = -\frac{GM}{r^2} + \frac{\Lambda c^2}{3}r$.)

$$\frac{GM_\odot}{r^2} = \frac{\Lambda c^2}{3}r \Rightarrow r^3 = \frac{3GM_\odot}{\Lambda c^2} = \frac{3(6.67 \times 10^{-8}\,cm^3\,\cancel{g^{-1}}\,\cancel{s^{-2}})(1.989 \times 10^{33}\,\cancel{g})}{1.1 \times 10^{-35}\,\cancel{s^{-2}}}$$

$$r^3 = 3.62 \times 10^{61}\,cm^3,$$

$$r = \underline{3.3 \times 10^{20}\,cm} = 107\,pc = 349\,lyr$$

2. For a flat universe with $\Omega_{m,0} < 1$ and positive cosmological constant $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$, the density contributions of the matter and cosmological constant are equal when the scale factor has the value $a_{m\Lambda} = (\Omega_{m,0}/\Omega_{\Lambda,0})^{1/3}$. This equals 0.75 for the Benchmark Model: $\Omega_{m,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$. Show that for this case the Friedmann equation can be integrated to give the expression

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln[y^{3/2} + \sqrt{1+y^3}]\,,$$

where $y \equiv a/a_{m\Lambda}$. Show that for $a \ll a_{m\Lambda}$, this reduces to

$$a(t) \approx (\frac{3}{2}\sqrt{\Omega_{m,0}} H_0 t)^{2/3}\,,$$

and for $a \gg a_{m\Lambda}$, it reduces to

$$a(t) \approx a_{m\Lambda} \exp(\sqrt{1-\Omega_{m,o}} H_0 t)\,.$$

Show finally that the age of the universe today in this case is

$$t_0 = \frac{2}{3H_0\sqrt{1-\Omega_{m,o}}} \ln\left[\frac{\sqrt{1-\Omega_{m,o}}+1}{\sqrt{\Omega_{m,0}}}\right],$$

and that for the Benchmark Model this is $t_0 = 0.964 H_0^{-1}$.

2). Assuming $\Omega_m < 1, \Omega_\Lambda = 1 - \Omega_m, a_{m\Lambda} = (\Omega_m/\Omega_\Lambda)^{1/3}$:

$$\begin{aligned}
\frac{H^2}{H_0^2} &= \Omega_m a^{-3} + \Omega_\Lambda \\
\frac{da}{dt}\frac{1}{a} &= H_0\sqrt{\Omega_m a^{-3} + \Omega_\Lambda} \\
\int \frac{1}{a}\frac{da}{\sqrt{\Omega_m a^{-3} + \Omega_\Lambda}} &= \int H_0 \, dt \\
H_0 t &= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(2(\sqrt{\Omega_\Lambda\Omega_m + \Omega_\Lambda^2 a^3} + \Omega_\Lambda a^{3/2})\right) + C \\
&= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(2\sqrt{\Omega_\Lambda\Omega_m}(\sqrt{1+y^3} + y^{3/2})\right) + C \\
&= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(\sqrt{1+y^3} + y^{3/2}\right) + \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(2\sqrt{\Omega_\Lambda\Omega_m}\right) + C \\
t = 0 \implies a = y = 0 \implies C &= -\frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(2\sqrt{\Omega_\Lambda\Omega_m}\right) \\
H_0 t &= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(\sqrt{1+y^3} + y^{3/2}\right)
\end{aligned}$$

For $a \ll a_{m\Lambda}, y \to 0$:

$$\begin{aligned}
H_0 t &= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(\sqrt{1+y^3} + y^{3/2}\right) \\
&= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(1 + y^{3/2}\right) \\
&\approx \frac{2}{3\sqrt{\Omega_\Lambda}} (y^{3/2}) \\
a^{3/2} &= a_{m\Lambda}^{3/2} \frac{3\sqrt{\Omega_\Lambda} H_0 t}{2} \\
a &= \left(\frac{3}{2}\sqrt{\Omega_m} H_0 t\right)^{2/3}
\end{aligned}$$

For $a \gg a_{m\Lambda}, y \to \infty$:

$$\begin{aligned}
H_0 t &= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(\sqrt{1+y^3} + y^{3/2}\right) \\
&\approx \frac{2}{3\sqrt{\Omega_\Lambda}} \ln y^{3/2} + y^{3/2} \\
2y^{3/2} &= \exp\left(\frac{3H_0 t\sqrt{\Omega_\Lambda}}{2}\right) \\
a &\approx a_{m\Lambda} \exp(\sqrt{\Omega_\Lambda} H_0 t)
\end{aligned}$$

For the present time, $a = 1, y = \frac{1}{a_{m\Lambda}}$:

$$\begin{aligned}
H_0 t &= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(\sqrt{1 + \left(\frac{1}{a_{m\Lambda}}\right)^3} + \left(\frac{1}{a_{m\Lambda}}\right)^{3/2}\right) \\
&= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(\sqrt{\frac{\Omega_\Lambda + \Omega_m}{\Omega_m}} + \sqrt{\frac{\Omega_\Lambda}{\Omega_m}}\right) \\
&= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(\frac{\sqrt{\Omega_\Lambda} + 1}{\sqrt{\Omega_m}}\right)
\end{aligned}$$

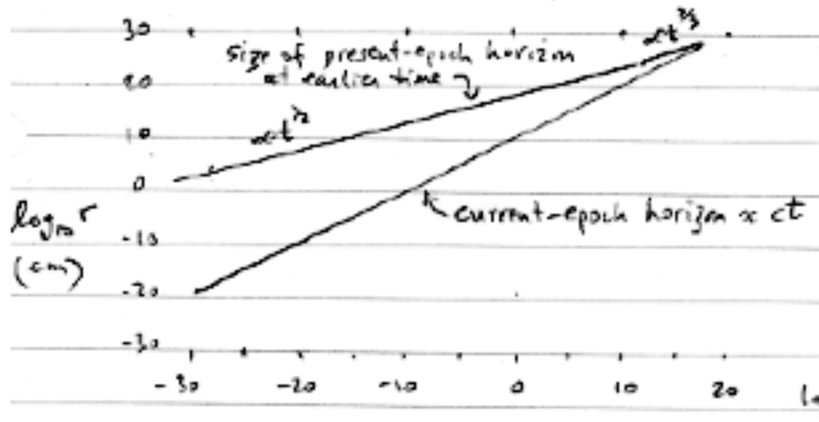
For the Benchmark model, $\Omega_m = 0.3, \Omega_\Lambda = 0.7 \implies t_0 = 0.964099.$

3. Popularizations of cosmology often talk about the "size of the universe" at some earlier time, but they usually mean by this the size that the present-epoch horizon was at that time. (For example, in the *Cosmic Voyage* IMAX film, Rocky Kolb says that everything that we can presently see was once only as big as a marble that he holds in his hand.) To clarify this, make a log-log plot in which the horizontal axis is time since the Big Bang, from 10^{-30} s to 10^{20} s, and the vertical axis is length, and plot curves representing both (a) the size in the past of the present-epoch horizon, and (b) the size of the current horizon (i.e., the distance that light has travelled since the Big Bang), as a function of the time since the Big Bang. (These curves cross at t_0.) Use any cosmology that you like, for example Einstein-de Sitter, but specify which one you are using. Briefly discuss some implications of your plot.

3. The present-epoch horizon in the EdS model is $2cH_0^{-1} = 3ct_0$. Its size at any earlier time was $2cH_0^{-1}a(t) = 3ct_0 a(t)$, where during the matter-dominated era $a(t) = (t/t_0)^{2/3}$ and during the earlier radiation-dominated era $a \sim t^{1/2}$.

The current-epoch horizon in the EdS model is $2cH^{-1} = 3ct$ during the matter-dominated era. As the problem said, one can just regard this as approximately ct.

The matter-radiation equality is at $z_{eq} \approx 3500$ for the Benchmark Model. For EdS with $\Omega_{m,0} = 1$, $t_{eq} = 3.4 \times 10^{10} (\Omega_0 h^2)^{-2}$ s, and for $h = 0.5$ this is $t_{eq} = 5.4 \times 10^{11}$ s; $a_{eq} = 1.66 \times 10^{-4}$ for $\Omega_0 = 1$, $h = 0.5$.



Take $t_0 = \frac{2}{3} H_0^{-1} = \frac{2}{3} 9.8 h^{-1}$ Gyr

$= 13$ Gyr for $h = \frac{1}{2}$

$= 4.1 \times 10^{17}$ s.

$ct_0 = 1.2 \times 10^{28}$ cm

$$\left(\frac{t_{eq}}{t_0}\right)^{2/3} = \left(\frac{5.4 \times 10^{11}}{4.1 \times 10^{17}}\right)^{2/3} = 1.2 \times 10^{-4}$$

The most important implication is that the present horizon was composed of many causally disconnected patches at earlier times.

4. Geometry. (a) Show that if $k = 0$ and the scale factor a grows as $t^{2/3}$, the apparent angular sizes of distant objects of the same linear size have a minimum at $z = 1.25$. (b) Under the same assumptions, suppose that a galaxy is observed at $z = 1.25$. For what fraction of the Hubble time has its light been travelling toward us? (*Be careful to answer what this question asks; don't just calculate something similar.*)

4. In this Einstein-de Sitter case, the proper distance to an object at scale factor a_e is $r_e = \int_0^{r_e} dr = \int_{t_e}^{t_0} \frac{dt}{a} = \int_{a_e}^{1} \frac{da}{a^2 H}$

Here $H^2 = \frac{H_0^2}{a^3}$ or $H = H_0 a^{-3/2}$ so $r_e = \int_{a_e}^{1} \frac{da}{H_0 a^{1/2}} = \frac{2}{H_0}(1 - a_e^{1/2})$

(as derived in class). The angle θ subtended by length L at r_e is

$\theta = \frac{L}{a_e r_e} = \frac{L H_0}{2} \frac{1}{a_e - a_e^{3/2}}$. $\frac{d\theta}{da} = 0 \Rightarrow 0 = 1 - \frac{3}{2} a^{1/2}$, $a^{1/2} = \frac{2}{3}$, $a = \frac{4}{9}$

$a = \frac{4}{9} = \frac{1}{1+z} \Rightarrow 1+z = \frac{9}{4}$, $z = \frac{5}{4} = \underline{1.25}$ as claimed.

Light emitted at $z_e = 1.25$ has been travelling toward us a time

$t_0 - t_e = t_0(1 - a_e^{3/2}) = t_0\left(1 - \frac{8}{27}\right) = \frac{19}{27} t_0 = \left(\frac{19}{27}\right)\left(\frac{2}{3} t_H\right) = \frac{38}{81} t_H$

$= \underline{0.469\, t_H}$ (Note that $t_0 = \frac{2}{3} H_0^{-1}$ for this case.)

5. Consider a galaxy of physical (visible) size 5 kpc. What angle would this galaxy subtend if situated at redshift 0.1? 1? 5? Do the calculation in a flat universe, first with zero cosmological constant, and then in the Benchmark Model with $\Omega_{m,0} = 0.3$. (*You are welcome to use Ned Wright's Javascript cosmology calculator – see below.*)

5. Use Ned Wright's calculator http://www.astro.ucla.edu/~wright/CosmoCalc.html

redshift	EdS	5 kpc =	Benchmark	5 kpc =
0.1	1.757 kpc/"	2.85"	1.844 kpc/"	2.71"
1	6.081	0.82"	8.008 kpc/"	0.62"
5	4.095	1.22"	6.281 kpc/"	0.80"

Answers for EdS ↗ for Benchmark ↗

Basic pattern: as z increases, the angular size first diminishes, and then grows larger. These numbers were calculated for h = 0.7 ($h_{70} = 1$). For any other value of the Hubble parameter, multiply by $h_{70} = h/0.7$

6. The astronomical convention is that the relationship between apparent magnitude m and absolute magnitude M is

$$m - M = 5\log(\frac{d_L}{10pc}) + K$$

where d_L is the luminosity distance and K is a correction for the redshifting of the spectrum of the source (Dodelson eq. 2.81). Plot $m - M$ as a function of redshift for a flat matter-dominated universe (this can be done analytically) and for the Benchmark Model (you need to evaluate numerically a 1D integral). Neglect the K correction. Compare with a plot showing high-redshift supernova data, for example Dodelson's Fig. 1.7.

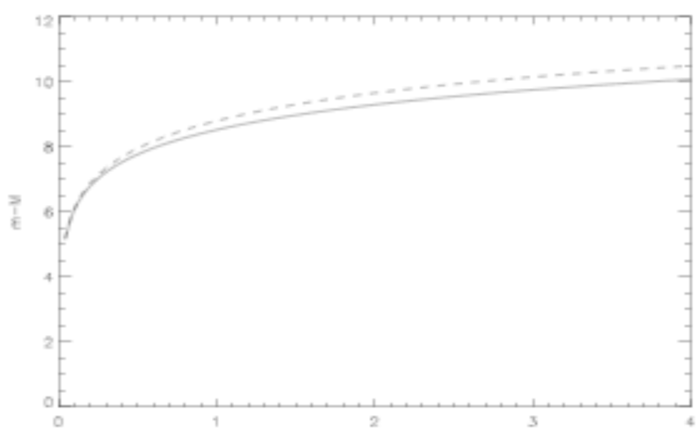
6). Starting with the equation for χ derived in problem 4 for a flat, matter-dominated universe, we can get d_L for this cosmology:

$$\begin{aligned} \chi &= 3ct_0\left(1 - \frac{1}{\sqrt{1+z}}\right) \\ \implies d_L = \frac{\chi}{a} &= \frac{2c}{H_0 a}\left(1 - \sqrt{a}\right) \\ &= \frac{2c(1+z)}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right) \end{aligned}$$

For the Benchmark Model ($\Omega_m = 0.3, \Omega_\Lambda = 0.7$), the equation is more complicated:

$$\begin{aligned} \chi &= c\int_{t_e}^{t_0}\frac{dt}{a} \\ \frac{da}{dt}\frac{1}{a} &= H_0\sqrt{\frac{\Omega_m}{a^3} + \Omega_\Lambda} \\ dt &= \frac{da}{aH_0}\frac{1}{\sqrt{\frac{\Omega_m}{a^3} + \Omega_\Lambda}} \\ \implies \chi &= \frac{c}{H_0}\int_a^1 \frac{da}{a^2}\frac{1}{\sqrt{\frac{\Omega_m}{a^3} + \Omega_\Lambda}} \end{aligned}$$

which must be integrated numerically. The results are shown below in Figure 2 for $\Omega_m = 1$ (solid) and Benchmark Model (dashed) over the range $z = [0, 4]$.



7. Suppose astronomers measure the age of a galaxy at redshift $z = 2.5$. How old would this galaxy have to be (at the time the light from it was emitted) in order to rule out the hypothesis that $\Omega_M = 1$ with negligible vacuum and radiation energy density. Use $H_0 = 70$ km/sec/Mpc. (This is Weinberg, *Cosmology*, page 569, problem 4.)

7). Under the assumption of a flat, matter-dominated universe, the scale function goes as:

$$\frac{da}{dt}\frac{1}{a} = H_0(\frac{\Omega_M}{a^3}) = \frac{H_0}{\sqrt{a^3}}$$

$$\int a^{1/2}\,da = \int H_0\,dt$$

$$\frac{2a^{3/2}}{3H_0} = t$$

$$\frac{2}{3H_0(1+z)^{3/2}} = t$$

Using $H_0 = 70$ km/s/Mpc, a galaxy at $z = 2.5$ can be at most $1.422 * 10^9$ yrs old.