Homework Set 2 – Solutions

DUE: Thursday May 5

1. One version of the "tired light" hypothesis states that the universe is not expanding, but that photons lose energy per unit distance

$$\frac{dE}{dr} = -KE\,.$$

Show that this hypothesis gives a distance-redshift relation that is linear in the limit $z \ll 1$. What value of K gives a Hubble constant $h = 0.7$? Give some arguments against the "tired light" hypothesis?

1. Tired Light

The redshift of light emitted with wavelength λ_{em} and observed to have wavelength λ_{obs} is defined via

(1) $$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{\lambda_{obs}}{\lambda_{em}} - 1.$$

Meanwhile, the energy of a photon is of course

(2) $$E_\gamma = h\nu = \frac{hc}{\lambda},$$

and the solution to the given / assumed differential equation is just

(3) $$E(r) = E_0 e^{-Kr} = \left(\frac{hc}{\lambda_{em}}\right) e^{-Kr}.$$

When this photon has travelled a distance D and is observed, this becomes

(4) $$\frac{hc}{\lambda_{obs}} = \left(\frac{hc}{\lambda_{em}}\right) e^{-KD}$$

Rearranging and substituting the expression for z from equation 1, we get

(5) $$e^{KD} = \frac{\lambda_{obs}}{\lambda_{em}} = 1 + z \quad \Rightarrow \quad KD = \ln(1+z)$$

For $z \ll 1$, the right hand side may be approximated by the lowest-order term in the Taylor series expansion, namely $\ln(1+z) \approx z$, and so in this limit

(6) $$z \approx KD$$

Comparing this to Hubble's Law, we see that $K = H_0/c = h/c$ 100 km/s/Mpc, and so for $h = 0.7$, we obtain $K = 0.23$/Gpc. That is, a photon would lose $(1/e)$ of its energy roughly every 4 Gpc. – John Forbes

Besides the lack of energy conservation and physical motivation, the "tired light" relation between redshift and distance is completely different from the Hubble law except at small redshifts – but Type 1a supernovae and other measurements of distance and redshift separately from nearby back to the Cosmic Background Radiation are in good agreement with the predictions of the standard ΛCDM cosmology with current cosmological parameters, and therefore incompatible with "tired light." – added by Joel

2. Consider a "power-law" cosmology, where the scale factor $a(t) = (t/t_0)^n$ where n is some number. For example, for the Einstein-de Sitter cosmology, $n = 2/3$.

(a) Verify that the Hubble radius is $d_H \equiv c/H_0 = ct_0/n$. Consider a galaxy that radiated at emission redshift z_e, corresponding to scale factor $a_e = 1/(1+z_e)$, light that we see today. Using

$$d_p(t_0) = r_e = \int_{t_e}^{t_0} \frac{dt}{a(t)} \quad ,$$

show that the proper distance today to this galaxy is

$$d_p(t_0) = \frac{n}{1-n} d_H (1 - {a_e}^{(1-n)/n})$$

and that the proper distance at the time of emission was

$$d_p(t_e) = a_e r_e \quad .$$

Note that the emission distance $d_p(t_e)$ vanishes for $t_e = 0$ (the Big Bang) and for $t_e = t_0$. Check that these formulas agree with the usual results for the E-dS case.

2.1. **Distances.** Given this scale factor-time relation, we can directly compute the derivative of the scale factor,

(7)
$$\dot{a}(t) = nt_0^{-1}(t/t_0)^{n-1} = na(t)/t = n\frac{a(t)}{t_0 a(t)^{1/n}} = nt_0^{-1}a(t)^{(n-1)/n}$$

and so with the definition of the Hubble parameter, we have

(8)
$$d_H = c/H_0 = c/H(t_0) = c\left(\frac{a(t_0)}{\dot{a}(t_0)}\right) = c\frac{1}{nt_0^{-1}} = ct_0/n$$

The scale factor-time relation also makes the comoving distance easy to calculate

(9)
$$d_p(t_0) = r_e = \int_{t_e}^{t_0} \frac{cdt}{a(t)} = \int_{t_e}^{t_0} c\left(\frac{t_0}{t}\right)^n dt = \frac{ct_0}{-n+1}\left(\frac{t}{t_0}\right)^{-n+1} |_{t_e}^{t_0} = \frac{nd_H}{1-n}\left(1 - ((t_e/t_0)^n)^{(1-n)/n}\right),$$

which, when compared with the assumed form of a(t) is just

(10)
$$d_p(t_0) = r_e = d_H \frac{n}{1-n}\left(1 - a_e^{(1-n)/n}\right)$$

The proper distance at a particular time is in general given by

(11)
$$d_p(t) = \int_{em}^{obs} ds = \int_0^{r_e} a(t)dr_e = a(t)\int_{t_{em}}^{t_{obs}} cdt/a(t) = a(t)r_e$$

and so at the time of emission, the proper distance is simply

(12)
$$d_p(t_e) = a(t_e)r_e = a_e r_e$$

For an Einstein-deSitter Universe, $n = 2/3$, which gives $d_p(t_0) = 2d_H(1-a_e^{1/2})$ and $d_p(t_e) = 2d_H a_e(1-a_e^{1/2})$. These results agree with e.g. Ryden's equations 5.60 and 5.61 so long as one replaces $a_e = 1/(1+z)$ and $d_H = c/H_0$.

(b) Show using Hubble's law that the velocity of this galaxy away from us today (i.e., at $t = t_0$) is

$$v(t_0) = \frac{nc}{1-n}(1 - {a_e}^{(1-n)/n})$$

and that the galaxy's velocity at the time of emission t_e was

$$v(t_e) = \frac{nc}{1-n}({a_e}^{(n-1)/n} - 1) \quad .$$

Note that when z_e is small (i.e., a_e is near 1), both recession velocities reduce to $v \approx cz_e$, as expected. (A graph of these velocities vs. redshift for the E-dS case was shown in class.)

2.2. **Velocities.** Hubble's Law is the statement that proper distance is proportional to recessional velocity, which must be true in an isotropically expanding universe where velocities are dominated by the Hubble flow

(13) $$v(t) = H(t)d_p(t) = \frac{\dot{a}(t)}{a(t)}a(t)r_e = nt_0^{-1}a(t)^{(n-1)/n}r_e = (c/d_H)a(t)^{(n-1)/n}\ r_e$$

Evaluating this expression at the present day,

(14) $$v(t_0) = (c/d_H)r_e = \frac{cn}{1-n}\left(1 - a_e^{(1-n)/n}\right)$$

and at the time of emission,

(15)

$$v(t_e) = (c/d_H)a(t_e)^{(n-1)/n}\ r_e = \frac{cn}{1-n}a_e^{(n-1)/n}\left(1 - a_e^{(1-n)/n}\right) = \frac{cn}{1-n}\left(a_e^{(n-1)/n} - 1\right).$$

(c) Show that the distance to the particle horizon is $d_H n/(1-n)$. Show that when $n > 1/2$ the radius of the observable universe is larger than the Hubble radius, and that in the limit $n \to 1$ there is no particle horizon. (The case $n = 1$ corresponds to an empty universe, also called the Milne cosmology.)

2.3. **Particle Horizon Distance.** The particle horizon is like an object from which light was emitted at $a_e, t_e \to 0$ and is observed today. The proper distance to such an object is

(16) $$d_p(t_0) = d_H\frac{n}{1-n}\left(1 - 0^{(1-n)/n}\right) = d_H\frac{n}{1-n}$$

If we consider the regime where $0 \leq n < 1$, then for increasing n, the numerator monotonically increases and the denominator monotonically decreases, hence the overall expression must monotonically increase. For $n = 1/2$, we have $d_p(t_0) = d_H$, so clearly for $n > 1/2$, it follows that $d_p(t_0) > d_H$. As n approaches 1, $d_p(t_0)$ approaches infinity, which physically means that an object which emitted light at the big bang whose light is just now reaching us must be infinitely far away, in other words light has had time to reach us from all closer objects, and so if every object in the universe is a closer object, there is no particle horizon, i.e. no distance beyond which we cannot see.

(d) Show that Hubble's law implies that the velocity at the particle horizon is $cn/(1-n)$. Show that the velocity of the particle horizon itself is $c/(1-n)$, and that this means that the particle horizon sweeps past the galaxies at the particle horizon at the speed of light.

2.4. **Particle Horizon Velocity.** Using the formula from part b for $v(t_0)$ with $a_e = 0$ gives

$$v_{recessional}(t_0) = \frac{cn}{1-n}\left(1 - 0^{(1-n)/n}\right) = \frac{cn}{1-n} \tag{17}$$

The velocity of the particle horizon itself may be found by taking the time derivative of the proper distance

$$v = \frac{d}{dt}d_p(t) = \frac{d}{dt}\left(a(t)\int_0^t \frac{cdt'}{a(t')}\right) = \frac{d}{dt}\left(a(t)\frac{nd_H}{1-n}(t/t_0)^{1-n}\right) = \frac{d}{dt}\left(\frac{nd_H}{1-n}(t/t_0)\right) \tag{18}$$

Evaluating the derivative is now trivial, and using the value of the Hubble radius confirmed in part a, we have

$$v = \frac{n}{1-n}\frac{d_H}{t_0} = \frac{n}{1-n}\frac{c}{n} = \frac{c}{1-n} \tag{19}$$

Now if we take the difference between the recessional velocity, which is the velocity relative to us of galaxies currently at the particle horizon, and the velocity of the horizon itself, we will get the velocity of the particle horizon relative to those galaxies

$$v_{rel} = v - v_{recessional} = \frac{c}{1-n} - \frac{cn}{1-n} = \frac{c}{1-n}(1-n) = c. \tag{20}$$

– John Forbes

3. Short calculations:

(a) If a neutrino has mass m_ν and decouples at $T_{\nu d} \sim 1$ MeV, show that the contribution of this neutrino and its antiparticle to the cosmic density today is (Dodelson Eq. 2.80)

$$\Omega_\nu = \frac{m_\nu}{94h^2 \text{eV}} \quad .$$

3). The neutrino number density can be found from the photon number density as:

$$n_\nu = \frac{3}{11} n_\gamma$$

Given the current temperature of the universe, this works out to ≈ 112 cm^{-3}. At late times the energy density of massy neutrinos ρ_ν reduces to $m_\nu n_\nu$, giving the following equation for neutrino density:

$$\begin{aligned}
\Omega_\nu = \frac{\rho_\nu}{\rho_{cr}} &= m_\nu \frac{8\pi G n_\nu}{3H_0^2} \\
&= m_\nu \frac{8\pi G n_\nu}{3} \left(\frac{h_P/2\pi}{2.133 \times 10^{-33} h \text{ eV}} \right)^2 \\
&= m_\nu \frac{2G n_\nu h_P^2}{3\pi} \frac{1}{(2.133 \times 10^{-33} h \text{ eV})^2} \\
&\approx \frac{m_\nu}{94h^2 \text{ eV}}
\end{aligned}$$

– Elizabeth Lovegrove

(b) Verify that $\eta_b \equiv n_b/n_\gamma$ is given by (Dodelson Eq. 3.11; Weinberg, *Cosmology*, pp. 168-169)

$$\eta_b = 5.5 \times 10^{-10} \left(\frac{\Omega_b h^2}{0.020} \right) \quad .$$

3.2. **Baryon Fraction.** As in the previous problem, we recall that $n_\gamma(T) = 2\zeta(3)\pi^{-2}T^3$. Since the total number of baryons and photons is more or less conserved, we can evaluate this for the present day temperature of the CMB and compare to the baryon fraction. The baryon number density must be related to the cosmic baryon fraction via

$$\Omega_b = \frac{\rho_b}{\rho_{cr}} = \frac{m_b n_b}{8.098 \times 10^{-11} h^2 eV^4} \tag{27}$$

Adopting the proton mass as the typical mass of a baryon, we obtain

(28)

$$\eta_b = \frac{n_b}{n_\gamma} = \frac{8.098 \times 10^{-11} h^2 eV^4 \Omega_b / m_p}{2\zeta(3)\pi^{-2}(2.725K)^3} \left(\frac{11605K}{1eV} \right)^3 \left(\frac{m_p}{938 \times 10^6 eV} \right) = 2.74 \times 10^{-8} \Omega_b h^2$$

so factoring out two percent from the baryon energy density,

$$\eta_b = 5.48 \times 10^{-10} \left(\frac{\Omega_b h^2}{.02} \right) \tag{29}$$

– John Forbes

(c) Verify the time-temperature relation (Dodelson Eq. 3.30)

$$t = 132\,\mathrm{sec}\,(0.1\mathrm{MeV}/T)^2 \quad .$$

The time-temperature relation is found from:

$$\frac{1}{T}\frac{dT}{dt} = -\sqrt{\frac{8\pi G\rho}{3}}$$

When decays become important and e^+e^- annihilation is complete, the energy density becomes:

$$\rho = 3.36\frac{\pi^2}{30}T^4$$

Combining these two and integrating yields:

$$\begin{aligned}\frac{1}{T}\frac{dT}{dt} &= -T^2\sqrt{\frac{8\pi G(3.36\pi^2/30)}{3}}\\ t &= \frac{1}{2\pi T^2}\sqrt{\frac{90}{26.88\pi G}}\\ &\approx 132\ \mathrm{s}\left(\frac{0.1\ \mathrm{MeV}}{T}\right)^2\end{aligned}$$

– Elizabeth Lovegrove

(d) Calculate the redshift of matter-radiation equality z_{eq} in terms of Ω_m and h. Assume that the photon temperature today is $T_\gamma = 2.73\mathrm{K}$ and use the fact, derived in class (and Weinberg, *Cosmology*, Eq. 3.1.21), that the total energy density in radiation (i.e., photons and three species of neutrinos of negligible mass) after e^+e^- annihilation is $\rho_r = 1.68\rho_\gamma$, where ρ_γ is the photon energy density.

To find the redshift of matter-radiaton equality, we set the densities of matter and radiation equal:

$$\begin{aligned}\frac{\Omega_r}{a^4} &= \frac{\Omega_m}{a^3}\\ a &= \frac{\Omega_r}{\Omega_m}\\ &= \frac{1.68}{\Omega_m}\frac{\rho_\gamma}{\rho_{cr}}\\ &= \frac{1.68}{\Omega_m}\frac{\pi^2}{15}\frac{T^4}{\rho_{cr}}\\ &= \frac{13.44\pi^3 G}{45k_h^2}\frac{T^4}{\Omega_m h^2}, H = k_h h\\ &= \frac{4.15\times10^{-5}}{\Omega_m h^2} \quad = (1+z_{eq})^{-1}\end{aligned}$$

Taking $\Omega_m = 0.27$ and $h = 0.7$, $z_{eq} \approx 3200$. – added by Joel

4. It is possible that the universe contains a quantum field called "quintessence" which in the simplest version has an equation of state parameter $w_Q = p_Q/\rho_Q$ with energy density ρ_Q positive (of course) but pressure p_Q negative. Suppose that the universe contains nothing but pressureless matter, i.e. with $w_m = 0$, and quintessence, with $w_Q = -3/4$. The current density parameter of matter is $\Omega_m \approx 0.3$ and that of quintessence is $\Omega_Q = 1 - \Omega_m$. At what scale factor a_{mQ} will the energy density of quintessence and matter be equal? Solve the Friedmann equation to find $a(t)$ for this universe. What is $a(t)$ in the limit $a \gg a_{mQ}$? What is the current age of the universe, expressed in terms of H_0 and $\Omega_{m,0}$?

4. Quintessence

In general, the fluid energy equation implies that for a material of EoS parameter w, the energy density evolves as

(39) $$\rho = \rho_0 a^{-3(1+w)}$$

where of course we're familiar with the usual $w = 1/3$, $w = 0$, and $w = -1$ cases corresponding to photons, cold matter, and a cosmological constant. This means that for this quintessence-matter universe, the Friedmann equation is

(40) $$H^2 = H_0^2 \left(\Omega_{Q,0}/a^{3/4} + \Omega_{m,0}/a^3\right)$$

Setting the two energy densities equal and solving for a, we get

(41) $$a_{mQ} = (\Omega_{m,0}/\Omega_{Q,0})^{4/9} \approx 0.69$$

for a flat universe with $\Omega_{m,0} = 0.3$. From the Friedmann equation, we have

(42) $$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_Q a^{5/4} + \Omega_m/a}}$$

With the aid of Mathematica and the definition $x = a/a_{mQ}$, this evaluates to

(43) $$H_0 t = \frac{4}{9}\frac{a^{3/2} B_{-x^{9/4}}(2/3, 1/2)}{(-1)^{2/3} x^{3/2}\sqrt{\Omega_m}} = \frac{4}{9}\frac{a_{mQ}^{3/2}}{\sqrt{\Omega_m}(-1)^{2/3}} B_{-x^{9/4}}(2/3, 1/2)$$

where the incomplete beta function is defined by

(44) $$B_z(a,b) = \int_0^z t^{a-1}(1-t)^{b-1} dt$$

Again with Mathematica's help, I find that the leading order term in an expansion about infinity is

(45) $$\frac{B_{x^{9/4}}(2/3, 1/2)}{(-1)^{2/3}} \approx 6x^{3/8}$$

so for $a \gg a_{mQ}$,

(46) $$H_0 t \approx \frac{8}{3}\frac{a_{mQ}^{3/2}}{\sqrt{\Omega_m}}(a/a_{mQ})^{3/8}$$

or simplifying a bit,

(47) $$a \approx \left(\frac{3}{8} H_0 t \sqrt{\Omega_m}\right)^{8/3} a_{mQ}^{-3}$$

Meanwhile, evaluating equation 43 numerically for $x = 1/0.69$, the age of the universe today is

(48) $$t_0 = \frac{0.51}{H_0\sqrt{\Omega_m}}$$

– John Forbes

Here's an alternative solution by Elizabeth Lovegrove

4). The fluid equation states that:

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0$$

must hold for each component of the universe, where ϵ is energy density. Integrating this equation gives the evolution of energy density as:

$$\epsilon = \epsilon_0 a^{-3(1+w)}$$

where w is the state parameter. Assuming a universe with a density of pressureless matter $\Omega_m = 0.3, w_m = 0$ and quintessence $\Omega_q = 1 - \Omega_m, w_q = -3/4$, these densities therefore decline as $\Omega_m a^{-3}, \Omega_q a^{-3/4}$. They are equal when:

$$\Omega_m a^{-3} = (1 - \Omega_m)a^{-3/4}$$
$$a^{9/4} = \frac{\Omega_m}{1 - \Omega_m}$$
$$a_{mq} = \left(\frac{\Omega_m}{1 - \Omega_m}\right)^{4/9} \Rightarrow z = 0.457$$

Solving the Friedmann equation gives:

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2(\Omega_m a^{-3} + \Omega_q a^{-3/4})$$
$$\int \frac{\mathrm{d}a}{a\sqrt{\Omega_m a^{-3} + \Omega_q a^{-3/4}}} = H_0 t$$
$$H_0 t = \frac{2a\sqrt{\frac{\Omega_q}{\Omega_m}a^{9/4} + 1}\,{}_2F_1\left(\frac{1}{2}, \frac{2}{3}, \frac{5}{3}, -\frac{\Omega_q}{\Omega_m}a^{9/4}\right)}{3\sqrt{\frac{\Omega_m + \Omega_q a^{9/4}}{a}}}$$

where ${}_2F_1()$ is a hypergeometric function. When $a \gg a_{mq}$, quintessence dominates, and the Friedmann equation reduces to:

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2(\Omega_q a^{-3/4})$$
$$H_0 t = \frac{8}{3\Omega_q}a^{3/8}$$

The current age of the universe can be found by setting $a = 1$ and solving for t using the full solution to the Friedmann equation; this cosmology gives an age of $t = 0.920923/H_0 =$ 12.86 billion years.

5. Suppose that the neutron decay time were $\tau_n = 1890$ s instead of $\tau_n = 890$ s, with all other physical parameters unchanged. Estimate Y_p, the primordial mass fraction of nucleons in ^{4}He, assuming for simplicity that all available neutrons are incorporated into ^{4}He.

5. Half Hour Neutrons

The first place the neutron lifetime enters into the calculation is via the rate for neutron to proton conversion.

$$\lambda_{np} = \frac{255}{\tau_n x^5}(12 + 6x + x^2) \tag{49}$$

where x is the inverse of the temperature in units of the difference in mass between the neutron and proton, about 1 MeV. Assuming this mass difference is not affected by the change in neutron lifetime, the relevant value of x where conversions become inefficient is still about 1. This conversion rate is relevant through the evolution equation for the neutron abundance,

$$\frac{dX_n}{dx} = \frac{x\lambda_{np}}{H(x=1)}\left(e^{-x} - X_n(1 + e^{-x})\right) \tag{50}$$

Doubling the neutron's lifetime approximately halves the RHS of this equation, but the value which sets the helium abundance is this value at large values of x, for which the size of λ_{np} serves only to increase or (as in this case) decrease the speed with which the solution approaches its asymptotic value. Thus I conclude that the primary effect of altering the neutron's lifetime will simply be from the alteration of the decay rate, while the asymptotic neutron abundance neglecting decays will be similar to its previous value.

The time-temperature relation is also relatively unaffected, so the fraction of neutrons which survive during a period of time where decays are the primary mechanism affecting neutron abundance is

$$\exp\left[-\Delta t/\tau_n\right] \approx \exp\left[-\left(\frac{132}{1890}\right)\left(\frac{0.1}{0.07}\right)^2\right] = 0.867 \tag{51}$$

The left-hand side simply reflects the definition of neutron decay time if the primary process is decay with some small probability of decay per neutron per unit time. The 132 is the scaling from the time-temperature relation, as is the fact that the temperature ratio, 0.1/0.07 is squared. A tenth of an MeV is simply the scaling of the time-temperature relation, while 0.07 MeV is the temperature at which nucleosynthesis begins. Thus

$$Y_p \approx 2X_n(T_{nuc}) \approx 2 \times 0.15 \times 0.867 = 0.26 \tag{52}$$

where 0.15 is the same asymptotic value of X_n neglecting decays as is used in Dodelson for the reasons mentioned above. Thus I estimate that a doubling of the neutron's lifetime results in an increase in the helium mass fraction from about 23% to 26%.

– John Forbes