1. Cosmological constant. (a) Calculate the value of the cosmological constant $\Lambda$, and also calculate $\Omega_\Lambda = \Lambda / 3H_0^2$, the corresponding vacuum energy density in units of critical density $\rho_c$. Assume that $\Omega_m = 0.3$ and that the curvature vanishes ($k = 0$). Express $\Lambda$ and $\rho_\Lambda \equiv \Omega_\Lambda \rho_c$ in several different units, including natural units ($\hbar = c = 1$; take $\rho_\Lambda \equiv \Omega_\Lambda \rho_c$ to have units of $eV^4$).

(b) The acceleration due to the cosmological constant equals $\Lambda r c^2 / 3$. Numerically compare that to the sun’s gravitational acceleration $GM_\odot / r^2$ in order to find the distance $r$ at which they are equal.

2. For a flat universe with $\Omega_{m,0} < 1$ and positive cosmological constant $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$, the density contributions of the matter and cosmological constant are equal when the scale factor has the value $a_{m\Lambda} = (\Omega_{m,0}/\Omega_{\Lambda,0})^{1/3}$. This equals 0.75 for the Benchmark Model: $\Omega_{m,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$. Show that for this case the Friedmann equation can be integrated to give the expression

$$H_0 t = \frac{2}{3\sqrt{1 - \Omega_{m,0}}} \ln \left[ y^{3/2} + \sqrt{1 + y^3} \right],$$

where $y \equiv a/a_{m\Lambda}$. Show that for $a \ll a_{m\Lambda}$, this reduces to

$$a(t) \approx \left( \frac{3}{2} \sqrt{\Omega_{m,0}} H_0 t \right)^{2/3},$$

and for $a \gg a_{m\Lambda}$, it reduces to

$$a(t) \approx a_{m\Lambda} \exp(\sqrt{1 - \Omega_{m,0}} H_0 t).$$

Show finally that the age of the universe today in this case is

$$t_0 = \frac{2}{3H_0 \sqrt{1 - \Omega_{m,0}}} \ln \left[ \sqrt{1 - \Omega_{m,0}} + 1 \right],$$

and that for the Benchmark Model this is $t_0 = 0.964 H_0^{-1}$.

3. Popularizations of cosmology often talk about the “size of the universe,” but they usually mean by this the size that the present-epoch horizon was at some earlier time. (For example, in the Cosmic Voyage IMAX film, Rocky Kolb says that everything that we can presently see was once only as big as a marble that he holds in his hand.) To clarify
this, make a log-log plot in which the horizontal axis is time since the Big Bang, from $10^{-30}$ s to $10^{20}$ s, and the vertical axis is length, and plot curves representing both (a) the size in the past of the present-epoch horizon, and (b) the size of the current horizon (i.e., the distance that light has travelled since the Big Bang), as a function of the time since the Big Bang. (These curves cross at $t_0$.) Use any cosmology that you like, for example Einstein-de Sitter, but specify which one you are using. Briefly discuss some implications of your plot.

4. Geometry. (a) Show that if $k = 0$ and the scale factor $a$ grows as $t^{2/3}$, the apparent angular sizes of distant objects of the same linear size have a minimum at $z = 1.25$. (b) Under the same assumptions, suppose that a galaxy is observed at $z = 1.25$. For what fraction of the Hubble time has its light been travelling toward us?

5. Consider a galaxy of physical (visible) size 5 kpc. What angle would this galaxy subtend if situated at redshift 0.1? 1? 5? Do the calculation in a flat universe, first with zero cosmological constant, and then in the Benchmark Model with $\Omega_{m,0} = 0.3$.

6. The astronomical convention is that the relationship between apparent magnitude $m$ and absolute magnitude $M$ is

$$m - M = 5 \log\left(\frac{d_L}{10\text{pc}}\right) + K$$

where $d_L$ is the luminosity distance and $K$ is a correction for the redshifting of the spectrum of the source (Dodelson eq. 2.81). Plot $m - M$ as a function of redshift for a flat matter-dominated universe (this can be done analytically) and for the Benchmark Model (you need to evaluate numerically a 1D integral). Neglect the K correction. Compare with a plot showing high-redshift supernova data, for example Dodelson’s Fig. 1.7.

Note: Ned Wright’s Cosmology web page, with many useful links, is

http://www.astro.ucla.edu/~wright/cosmolog.htm ;

his Javascript distance calculator is at