Homework Set 3
DUE: Monday February 25

1. Fluctuation Power Spectrum

a. Explain physically why $P_k \propto k$ is the scale-invariant power spectrum expected for the initial density fluctuations.

b. Explain why the CDM power spectrum $P(k)$ is maximum near a specific wave number $k_{\rm max}$ and why it asymptotically approaches $P_k \propto k^{-3}$ at $k \gg k_{\rm max}$. What is the physical origin of the scale corresponding to $k_{\rm max}$?

c. Express the CDM power spectrum, qualitatively via a schematic plot, in terms of the rms density fluctuation $\delta_{\rm rms}$ on a mass scale M. How do we learn from this function about the hierarchical sequence of structure formation?

Top-Hat Model for Galaxy formation

Assume that a proto-galaxy is a sphere of uniform density $\rho_p(t)$, whose time evolution can be described by a bound-closed Friedmann model (i.e. a “mini-universe” with $k = 1$ and $\Lambda = 0$). Assume that this sphere is embedded in a background universe which is Einstein-deSitter (i.e. $k = 0$, $\Lambda = 0$) of mean density $\rho(t)$. (At early times, the E-dS model is always a good approximation, since the dark energy contribution only became important at fairly low redshift.) We wish to determine the way the density contrast ρ_p/ρ evolves in time. Following is a guide, step by step.

2. From a small density perturbation till maximum expansion

a) The Friedmann equation of an Einstein-deSitter model in the matter era is

$$\dot{a}^2 = \frac{2a^*}{a} - k, \quad a^* \equiv \frac{4\pi}{3} G\rho_0 a_0^3 \,,$$

where ρ_0 and a_0 are the values of the universal density and expansion factor today. Write the implicit solution for the universal expansion factor $a(t)$ in terms of the mass constant a^* and the conformal time η [defined by $d\eta \equiv dt/a(t)$], namely write the expressions for $a(\eta)$ and $t(\eta)$. Do the same for the perturbation, where you denote the corresponding quantities as a_p, a_p^*, η_p, etc.

b) Relate the solutions inside the perturbation and in the background by demanding that the physical time t is the same in both. Use this to relate η to η_p, and then to express a in terms of η_p (rather than η). Recall that we defined $a^* \propto \rho_0 a_0^3 = \rho a^3$ (and a_p^* in analogy), and show that

$$\frac{\rho_p}{\rho} = \frac{9(\eta_p - \sin\eta_p)^2}{2(1 - \cos\eta_p)^3}.$$

c) Define the density perturbation by

$$\frac{\delta\rho}{\rho} \equiv \frac{\rho_p - \rho}{\rho},$$

and use Taylor expansions to show that in the linear regime, when the perturbation is small, $\delta\rho/\rho \ll 1$, namely at early times, $\eta_p \ll 1$, the perturbation growth rate is

$$\frac{\delta\rho}{\rho} \propto t^{2/3}.$$

Compare to what we obtained using linear perturbation analysis.

d) Show that at maximum expansion, when the perturbation turns around, the density contrast is

$$\frac{\rho_p}{\rho} = \frac{9\pi^2}{16} \simeq 5.5$$

Note that this is true for any spherical perturbation, no matter when it reaches its maximum expansion.

3. Dark-matter collapse

a) Let the mass inside the perturbation be M, and its radius at maximum expansion be R_{max}. Assume that the kinetic energy at maximum expansion is zero (namely no non-radial motions). Assume that the collapse ends in virial equilibrium, where the kinetic energy equals half the potential energy (absolute value):

$$V^2 = \frac{GM}{R_{vir}}.$$

Use energy conversation during the collapse of dark matter to show that

$$R_{vir} = \frac{1}{2} R_{max}.$$

What is the corresponding growth of density inside the halo between maximum expansion and virialization?

b) What is the density contrast in the virialized halo relative to the background cosmological density at the time of virialization? In addition to the two factors already computed above, we have to include the decrease of the cosmological density between the time of maximum expansion (t_{max}) and the time of virialization (t_{vir}). Take this time to be roughly the time of collapse of the closed "mini-universe", namely when $\eta_p = 2\pi$. Show that the density contrast at virialization is

$$\frac{\rho_p}{\rho} \simeq 176.$$

4. The epoch of galaxy formation

a) Let the observed mean density in a galactic halo be ρ_{vir}, when the cosmological density today is ρ_0. Based on the above computation, what is the epoch of formation (namely virialization) of this halo? Express it in terms of redshift z_{vir} (recall $1+z = a_0/a$), and alternatively in terms of time t_{vir}/t_0.

b) Express ρ_0 in terms of Ω_m and the Hubble constant h (where $H_0 \equiv 100h\,\mathrm{km\,s^{-1}Mpc^{-1}}$). Show that

$$(1+z)_{vir} \simeq 6 \left(\frac{\rho_{vir}}{10^{-24}\, g\, cm^{-3}} \right)^{1/3} (\Omega_m h^2)^{-1/3}.$$

c) A halo is observed to have a flat rotation curve with velocity $V = 220\,\mathrm{km\,s^{-1}}$ and a virial radius of $R = 100h^{-1}kpc$. What can we say about its formation epoch?

d) The gas loses energy by radiation and by dissipation during the collapse. By observing the density of the gas (and stellar) component today, what can we say about the epoch of galaxy formation?

5. Please write the current title and brief description of your term project. List the main references that you are studying for it.