Physics 129

Nuclear and Particle Astrophysics
Winter 2014

Lecture 2 - January 8
Cross Sections and Decay Rates

Weekly Discussion Thursdays
6-7:45 pm in ISB 231-235
(except when the SPS meets)

Perkins (p. 8) says "Isospin symmetry in nuclear physics results from the near coincidence in the light quark masses."

Table 1.4 Constituent quark masses

Flavour	Quantum number	Approximate rest-mass, GeV/c^2
up or down	—	0.31
strange	$S=-1$	0.50
charm	$C=+1$	1.6
bottom	$B=-1$	4.6
top	$T=+1$	175

In high energy collisions, quarks can be temporarily separated from their gluon retinues, and the "current" quark masses that then apply are only a few MeV for the u and d quarks.

Isospin is a flavor symmetry with the same SU(2) group structure as spin. Members of the same isospin multiplets have similar masses. Isospin multiplets can be

	singlets (I=0),	doublets (I=1/2),	triplets (I=1), or	quartets (I=3/2).
Examples:	Λ, Ω^-	$(p, n), (K^+, K^0)$	(π^-, π^0, π^+) $(\Sigma^-, \Sigma^0, \Sigma^+)$	$(\Delta^-, \Delta^0, \Delta^+, \Delta^{++})$
		(uud, udd) $(u\bar{s}, d\bar{s})$	$(d\bar{u}, u\bar{u}+d\bar{d}, u\bar{d})$ (dds, uds, uus)	(uuu, uud, udd, ddd)

Perkins (p. 8) also says

In the strong interactions between the quarks, the flavour quantum number is conserved, and is denoted by the quark symbol in capitals. For example, a strange s quark has a strangeness quantum number $S = -1$, while a strange antiquark $\bar{s}$ has $S = +1$. Thus, in a collision between hadrons containing u and d quarks only, heavier quarks can be produced, but only as quark–antiquark pairs, so that the net flavour is conserved. In weak interactions, on the contrary, the quark flavour may change, for example, one can have transitions of the form $\Delta S = \pm 1$, $\Delta C = \pm 1$, etc. As an example, a baryon called the lambda hyperon of $S = -1$ decays to a proton and a pion, $\Lambda \to \mathrm{p} + \pi^-$, with a mean lifetime of 2.6×10^{-10} s, typical of a weak interaction of $\Delta S = +1$. This decay would be expressed as $sud \to uud + d\bar{u}$ in quark nomenclature.

Physical Quantity	**Interaction** Strong	Electromagnetic	Weak
momentum	+	+	+
energy (incl. mass)	+	+	+
ang. momentum	+	+	+
electric charge	+	+	+
quark flavour	+	+	–
lepton number*	./.	+	+
parity	+	+	–
charge conjugation	+	+	–
strangeness	+	+	–
isospin	+	–	–
baryon number	+	+	+

*the lepton number is not relevant for strong interactions

Table from Grupen, *Astroparticle Physics*

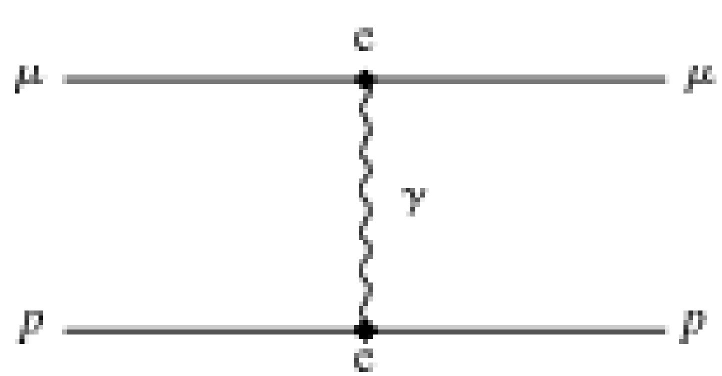
Feynman diagrams correspond to rules for calculating quantum mechanical amplitudes T_{if}. Such diagrams essentially consist of **external legs** (particles entering and leaving the interaction), **internal lines** (virtual particles not on the mass shell -- i.e., not satisfying $(p^\mu)^2 \equiv E^2 - \mathbf{p}^2 = m^2$), **vertices** (joining external or internal lines), and **loops** of virtual particles.

Vertices correspond to **coupling constants**, such as **e** for electromagnetic couplings of charged particles to photons. Note that $\alpha \equiv e^2/(4\pi\hbar) \approx 1/137$.

Internal lines correspond to propagators, of the form $1/(q^2 - m^2)$, where $q^\mu = p_2{}^\mu - p_1{}^\mu$ is the 4-momentum flowing through the internal line, and $q^2 = (q^0)^2 - \mathbf{q}^2$.

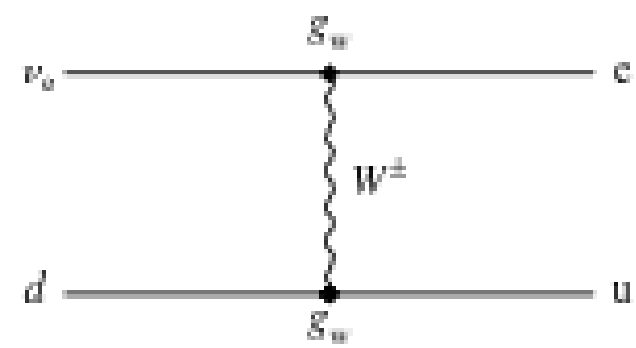
Thus, for example, the

electromagnetic interaction below



corresponds to amplitude $-\,e^2/q^2$
($m^2 = 0$ since the photon is massless)

The weak interaction below



corresponds to amplitude

$e^2/(M_W{}^2 - q^2)$

The Coulomb potential corresponding to this is determined by Fourier transformation from momentum to position space; it's the familiar

$$V_{\mathrm{Coulomb}}(r) = -\alpha/r$$

If instead the particle exchanged were a scalar boson of mass m_π, the potential would instead be

$$V(r) \propto (1/r)\exp(-r/r_0)$$

where $r_0 = \hbar/m_\pi c$, the Compton wavelength of the pion. (See Perkins eq. 1.6a and Appendix B.) This argument was given by Yukawa in 1935, and from the ~1 fm range of the strong interaction he deduced that $m_\pi \sim 100$ MeV. When a particle of this mass was discovered in secondary cosmic rays in 1937, it was thought to be the pion. Only later was it realized that the cosmic ray particle was not strongly interacting; it was actually the muon. The pion was not discovered until 1947.

If the exchanged particle were the $W^\pm$ or Z^0 bosons, of mass 80 or 91 GeV, the range r_0 would be much shorter, about 0.0025 fm. The amplitude is of order $e^2/(M_W^2 - q^2)$, and for processes like beta decay ($n \longrightarrow p + e^- + \bar{\nu}_e$) in which $|q^2| << M_W^2$, this is $e^2/M_W^2 \approx G_F \approx 10^{-5}\ \mathrm{GeV}^{-2}$, the weak interaction strength introduced by Enrico Fermi (1934).

The strong force between quarks due to gluons is more complicated, since massless gluons interact strongly with themselves. The corresponding potential can be approximated as a sum of two terms

$$V_{color}(r) = -(4/3)(\alpha_s/r) + k\,r$$

where for the strong interaction $\alpha_s \sim 1$, much bigger than $\alpha \approx 1/137$ for electromagentism. For large quark separation r, the $k\,r$ term is dominant, where k has the enormous value $k \approx 0.85$ GeV/fm = 136,000 J/m. This is responsible for "quark confinement" -- we never see quarks separated by as much as a fm. But in the high-energy early universe, theory indicates that there was a phase transition to a "quark-gluon plasma" state in which quarks and gluons were not confined. There is some experimental evidence for this in high-energy nuclear collisions.

Gravity, with $V_{Grav}(r) = -Gm_1m_2 / r$, is a long-range force like electromagnetism, since the graviton, a spin-2 boson, is massless. Gravity, electromagnetism, and the strong interaction can lead to bound states of massive, charged, and strongly interacting particles. But because of the very rapid decrease in the weak force with distance, the weak force does not lead to bound states. Instead, the charged weak force, due to $W^{\pm}$ exchange, leads to particle transformations such as $e^- \longrightarrow \nu_e$ or $u \longrightarrow d$. The neutral weak force, due to Z^0 exchange, leads to parity violation (discussed in Perkins Chapter 3).

The rate W at which a quantum mechanical process initial $\longrightarrow$ final happens is given in terms of the amplitude T_{if} for this process by "Fermi's Golden Rule"

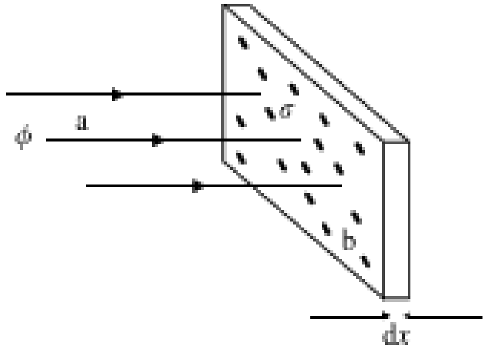
$$W = \left(\frac{2\pi}{\hbar}\right) |T_{\mathrm{if}}|^2 \rho_{\mathrm{f}}$$

where $\rho_f = dN/dE_f$ is the energy density of final states, as discussed in Perkins p. 23. The resulting expression for the differential cross section for a process a + b $\longrightarrow$ c + d in the center-of-momentum (CM) reference frame (where $\boldsymbol{p}_a + \boldsymbol{p}_b = \boldsymbol{p}_c + \boldsymbol{p}_d = 0$) is

$$\frac{d\sigma}{d\Omega}(a + b \rightarrow c + d) = \frac{1}{4\pi^2\hbar^4} |T_{\mathrm{if}}|^2 \frac{p_f^2}{v_{\mathrm{i}}\, v_{\mathrm{f}}}$$

The cross section is the reaction rate per target particle per unit incident flux.

Here $\boldsymbol{p}_f = \boldsymbol{p}_c$ and $E_f = E_a + E_b = E_c + E_d$ is the total energy in the CM frame, and v_i and v_f are the relative velocities of the initial and final particles.



The reduction in the number of incident particles $n = n_a$ after passing through an absorber of thickness dx is

$$dn = -\, n\, \sigma\, n_b\, dx$$

This integrates to $n(x) = n(0)\exp(-\sigma n_b x)$, which means that the number decreases by a factor $1/e$ in a distance = "mean free path" = $\lambda = 1/(\sigma n_b)$, a familiar expression from statistical mechanics.

The differential cross section

$$\frac{d\sigma}{d\Omega}(a + b \rightarrow c + d) = \frac{1}{4\pi^2\hbar^4} |T_{if}|^2 \frac{p_f^2}{v_i v_f}$$

simplifies in the "extreme relativistic" case when the particle velocities are nearly equal to the speed of light *c* so their momenta are much larger than their masses. Then $E_f = 2p_f$ and setting $\hbar = 1$ and $s = E_f^2$ the above expression becomes

$$\frac{d\sigma}{d\Omega} = |T_{if}|^2 \frac{s}{64\pi^2}$$

In this case, q^2, the square of the 4-momentum transfer $q^\mu = (p^c - p^a)^\mu$, is given by $|q^2| = 2p_f^2(1 - \cos\theta)$, where θ is the angle between the incoming direction $\boldsymbol{p}_a$ and the outgoing direction $\boldsymbol{p}_c$. Then $dq^2 = p_f^2\, d\Omega/\pi$ and the expression above becomes

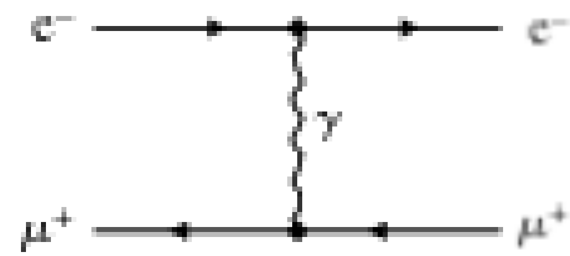
$$\frac{d\sigma}{dq^2} = \frac{|T_{if}|^2}{16\pi}$$

(We will discuss 4-vectors and relativistic calculations further in the next lecture, following Perkins Chapter 2.)

Following Perkins, I will next give approximate expressions for the cross sections for various examples of a + b ⟶ c + d elementary particle processes. Recall the basic formula in the extreme relativistic limit:

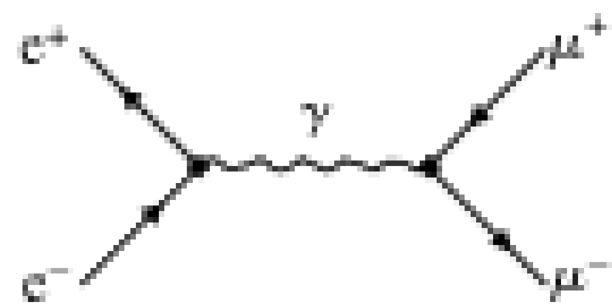
$$\frac{d\sigma}{d\Omega} = |T_{if}|^2 \frac{s}{64\pi^2}$$

For $e^-\mu^+ \longrightarrow e^-\mu^+$ the Feynman diagram is



The amplitude $T_{if} = e^2 / |q^2| = 4\pi\alpha / |q^2|$ gives $$\frac{d\sigma}{d\Omega} \sim \frac{\alpha^2 s}{q^4}$$ *Rutherford cross section*

For $e^+e^- \longrightarrow \mu^+\mu^-$ the Feynman diagram is a rotated version of the one above, with outgoing $\mathbf{e^-}$ becoming incoming $\mathbf{e^+}$ and incoming $\boldsymbol{\mu^+}$ becoming outgoing $\boldsymbol{\mu^-}$. Here $|q^2| = s$, so $T_{if} = e^2/s$ and the differential cross section is



crossed diagram

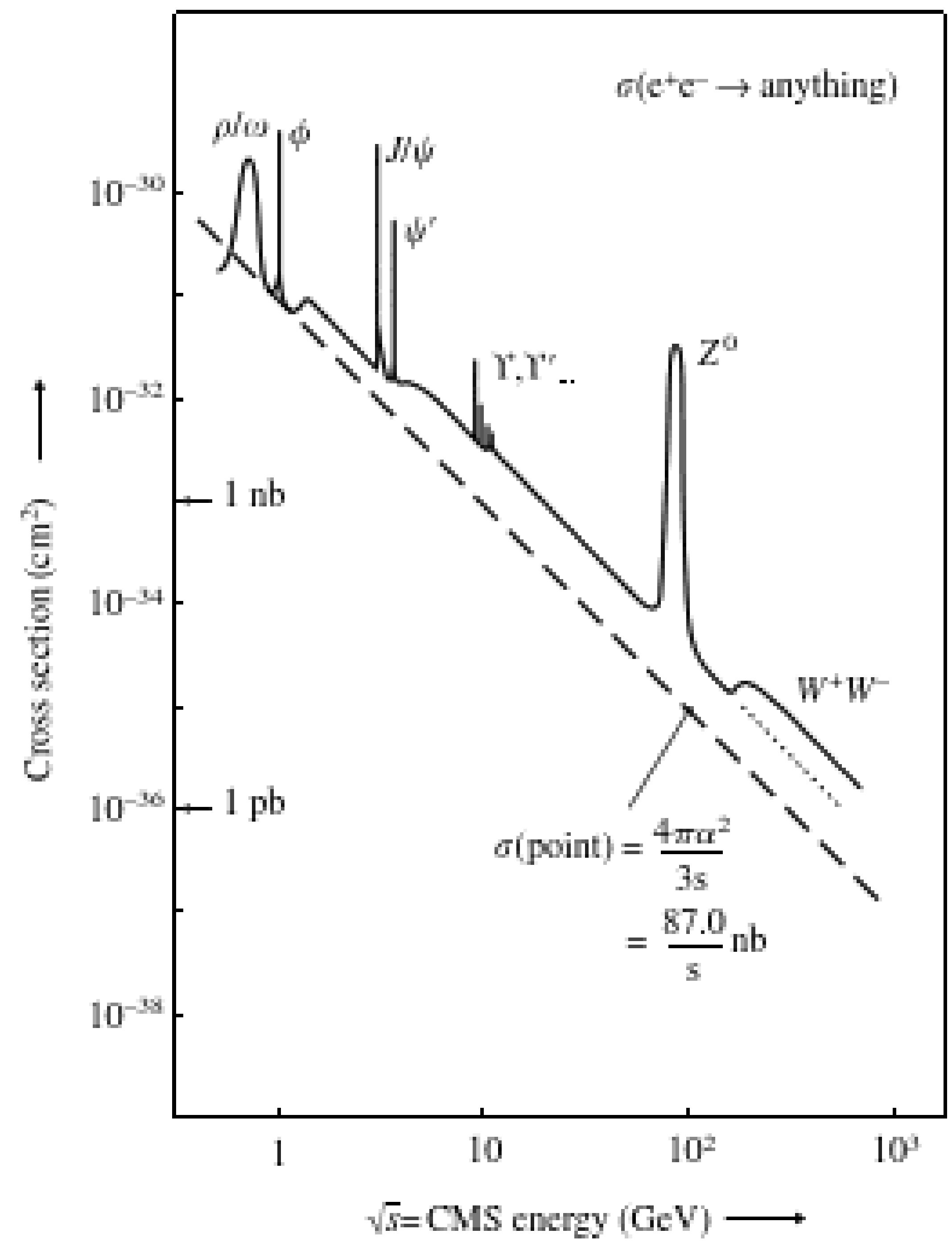
$$\frac{d\sigma}{d\Omega} \sim \frac{\alpha^2}{s}$$

The corresponding total cross section is $$\sigma = \frac{4\pi\alpha^2}{3s}$$

For **$e^+e^- \longrightarrow Q^+Q^- \longrightarrow$ hadrons** the cross section is just the same as for $e^+e^- \longrightarrow \mu^+\mu^-$ except for an additional factor proportional to the sums of the squares of the charges Q of the quarks with masses low enough to be produced with the CM energy available, i.e. with $E \geq 2m_Q$. We also have to take into account that all 3 colors of quarks can be produced. The relevant calculation is given in Perkins's Example 1.3 (and a homework problem is to correct this).

As expected from this argument, the experimental measurements of the cross section for $e^+e^- \longrightarrow$ hadrons , shown at right, are represented by long straight lines parallel to the 1/s dependance of $e^+e^- \longrightarrow \mu^+\mu^-$, labeled σ(point) in the diagram.

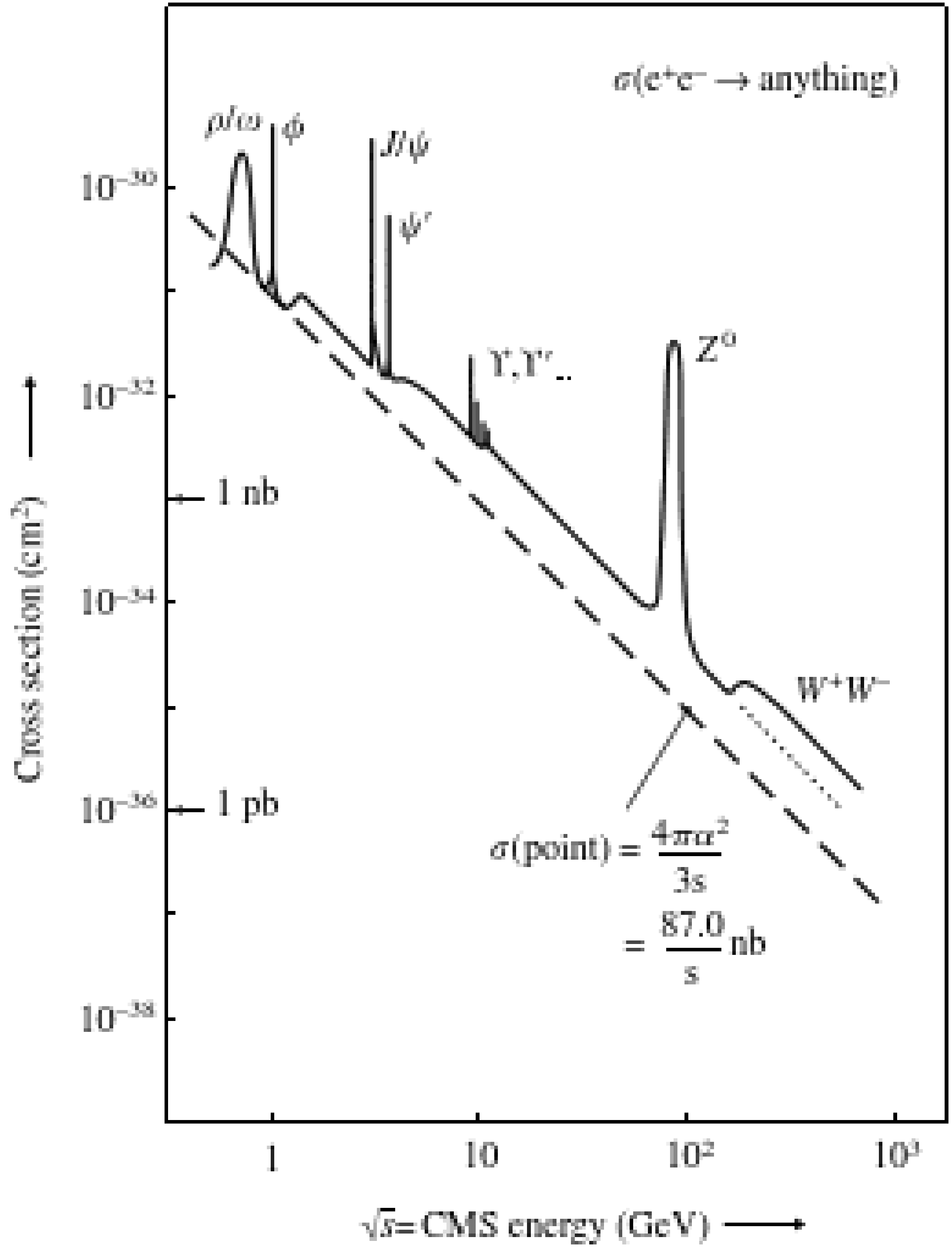
But note that these straight lines are interuppted by superimposed peaks, corresponding to the initial e^+e^- having just enough energy to make unstable particles P, i.e. $e^+e^- \longrightarrow P \longrightarrow$ hadrons.



The superimposed peaks, correspond to the initial e^+e^- having just enough energy to make unstable particles *P*, i.e. $e^+e^- \longrightarrow P \longrightarrow$ hadrons.

The first wide peak corresponds to $e^+e^- \longrightarrow \rho/\omega$, i.e. a combination of two spin-1 mesons ρ and ω, both made out of ordinary u and d quarks, and having similar masses and short lifetimes.

The $\varphi = s\bar{s}$ is made out of strange quarks, with a constituent mass about 0.5 GeV, so the φ is produced when the CM energy is about 1 GeV. At higher energies s-quark pairs are produced readily, leading to the first jump in the cross section.



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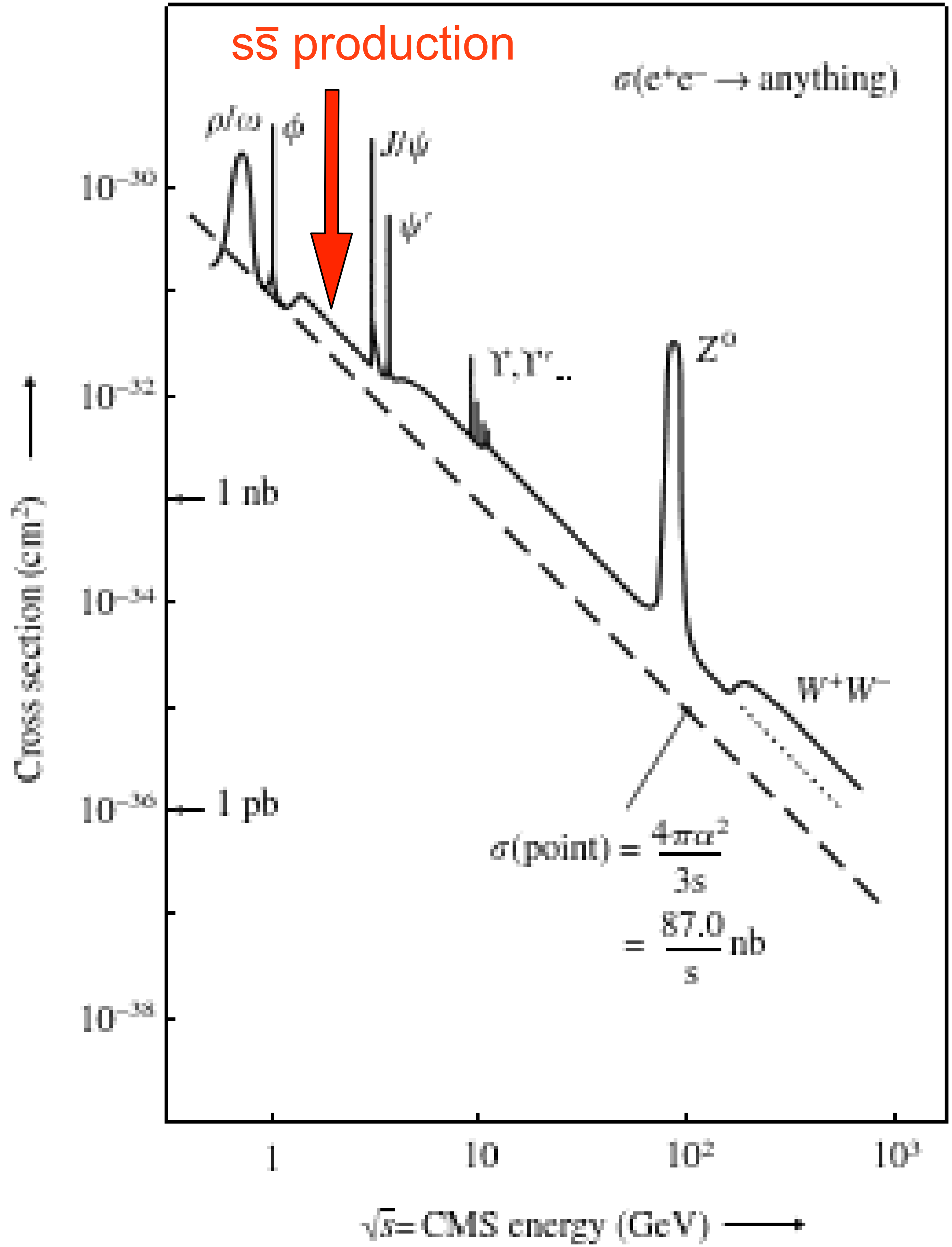
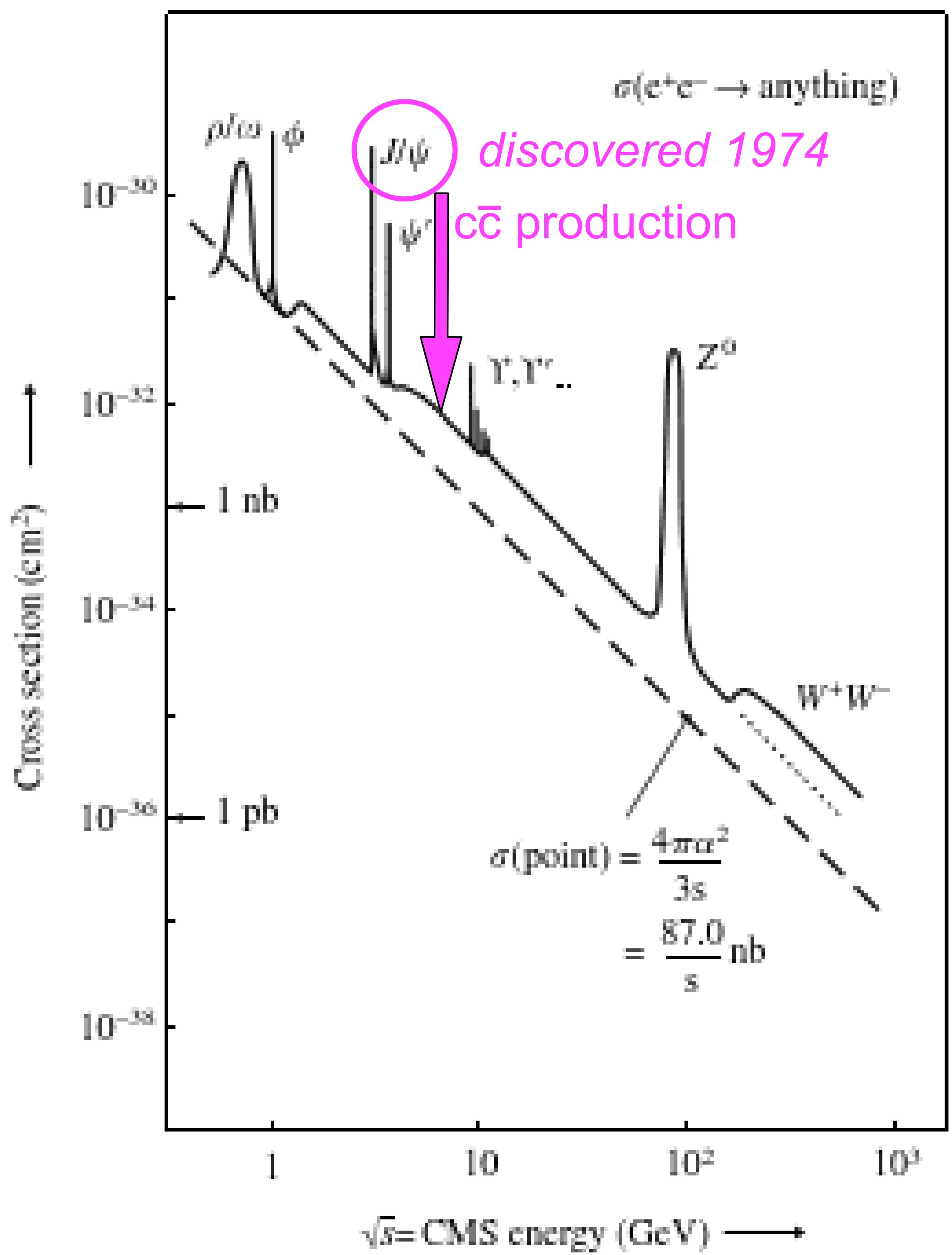


The next peak is J/Ψ = $c\bar{c}$ at a CM energy of about 3.1 GeV, indicating that the mass of the charm quark c is about 1.6 GeV. In 1972 I co-authored with Ben Lee and Sam Trieman the first paper that calculated the charm quark mass using the then brand new electroweak gauge theory, and we found that it had to be about 1 to 2 GeV in order to agree with a weak interaction process involving K mesons, so it was great when the J/Ψ mass turned out to agree with our predictions. We had used a SO(3) rather than the now-standard SU(2)xU(1) variant of electroweak theory, but it turned out that SU(2)xU(1) gave the same prediction.



The next peak is upsilon $\Upsilon = b\bar{b}$ when the bottom quark makes an entrance.

The Z^0 bump appears when the initial e^+e^- collision has CM energy equal to the 91 GeV mass of the Z^0 boson.

When the e^+ and e^- each have energy greater than the mass of the $W^\pm$, the total cross section jumps again.

The quarks and $W^\pm$ appear to be point-like particles!

Table 1.4 Constituent quark masses

Flavour	Quantum number	Approximate rest-mass, GeV/c^2
up or down	—	0.31
strange	$S = -1$	0.50
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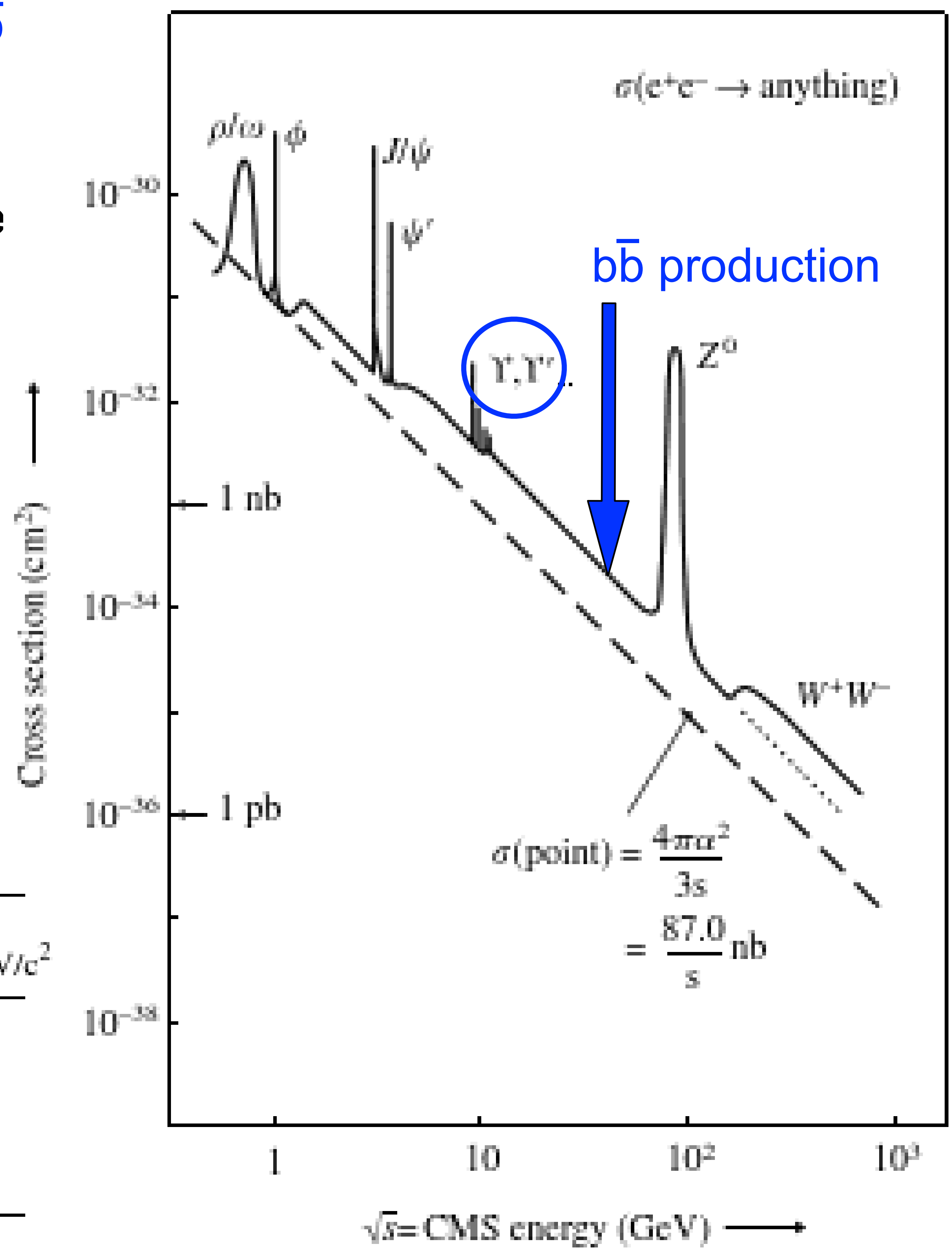
top	$T = +1$	175



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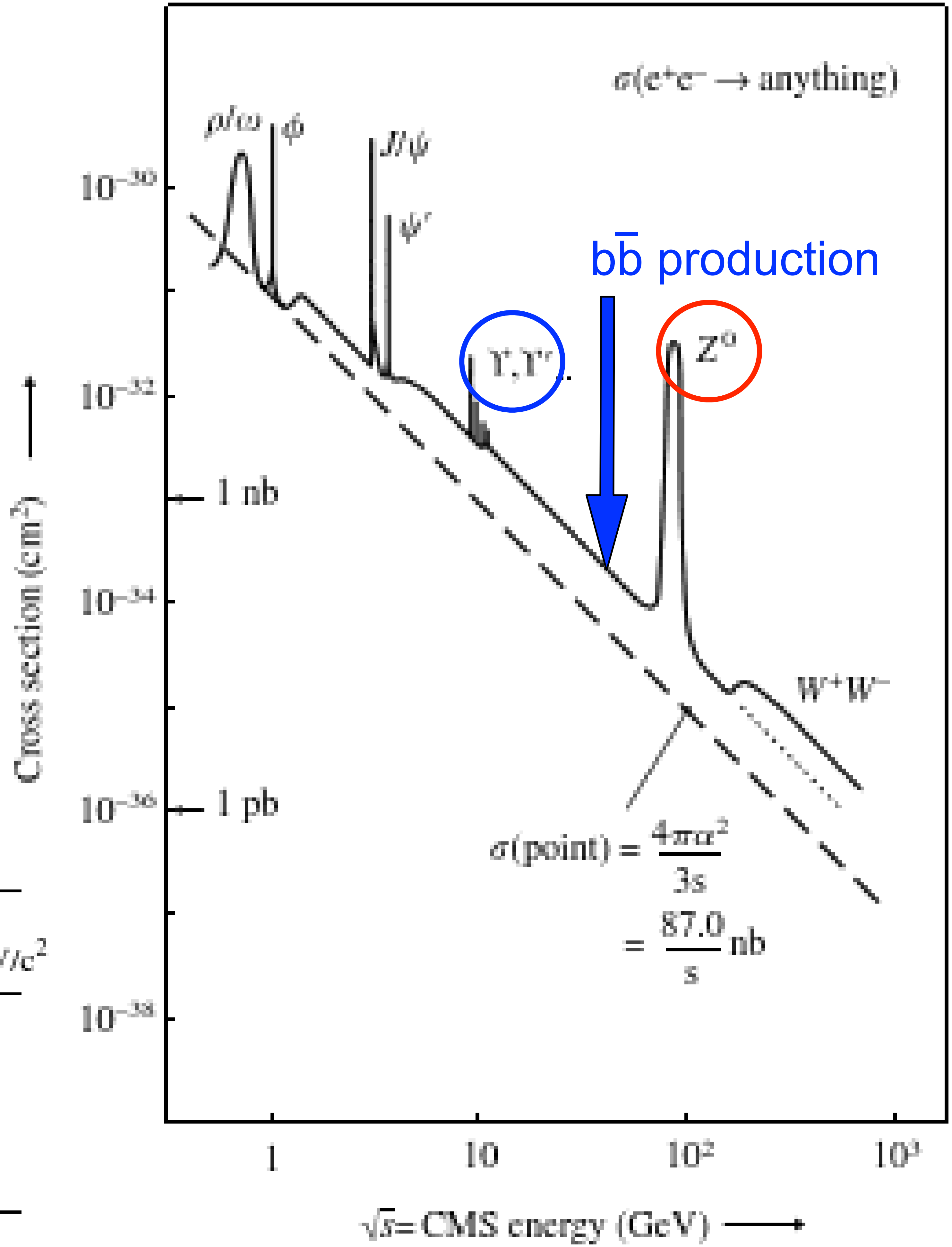
The quarks and $W^\pm$ appear to be point-like particles like e^+e^- !



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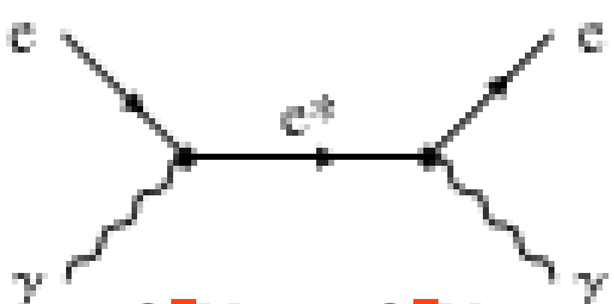
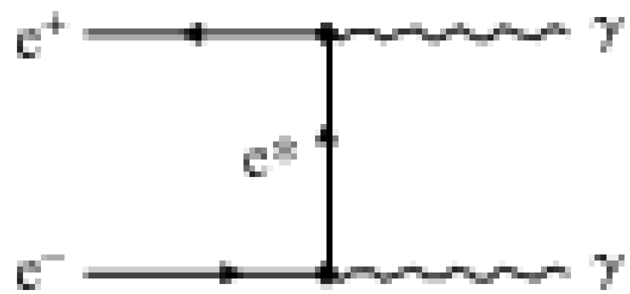
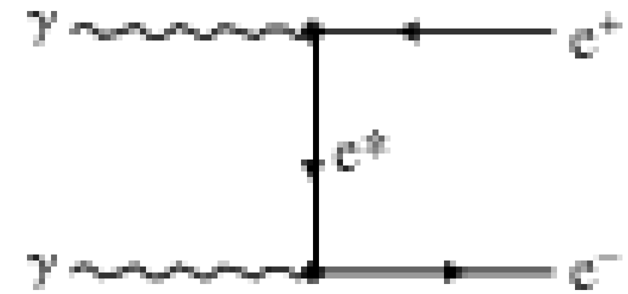


$e^-\gamma \longrightarrow e^-\gamma$

Compton scattering



$e^+e^- \longrightarrow \gamma\gamma$



$\gamma\gamma \longrightarrow e^+e^-$

The above Feynman diagrams are all related to each other by crossing. Because in this case a virtual electron (represented by e*) rather than a photon is the intermediate state, there is an extra logarithm of energy in the cross section compared with the previous cases. The cross sections in the limit of high energy s >> m^2 are given in Perkins eq. (1.26):

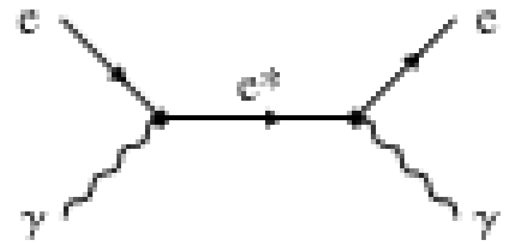
$$\sigma\left(e^+e^- \to \gamma\gamma\right) = \left(\frac{2\pi\alpha^2}{s}\right)\left[\ln\left(\frac{s}{m^2}\right) - 1\right]$$

$$\sigma\left(\gamma\gamma \to e^+e^-\right) = \left(\frac{4\pi\alpha^2}{s}\right)\left[\ln\left(\frac{s}{m^2}\right) - 1\right]$$

$$\sigma\left(\gamma e \to \gamma e\right) = \left(\frac{2\pi\alpha^2}{s}\right)\left[\ln\left(\frac{s}{m^2}\right) + \frac{1}{2}\right]$$

In $e^+e^- \longrightarrow \gamma\gamma$ there are two indistinguishable particles in the final state compared with $\gamma\gamma \longrightarrow e^+e^-$, so the phase-space volume is halved.

$e^- \gamma \longrightarrow e^- \gamma$



Compton scattering

The same high-energy formula $$\sigma(\gamma e \to \gamma e) = \left(\frac{2\pi\alpha^2}{s}\right)\left[\ln\left(\frac{s}{m^2}\right) + \frac{1}{2}\right]$$

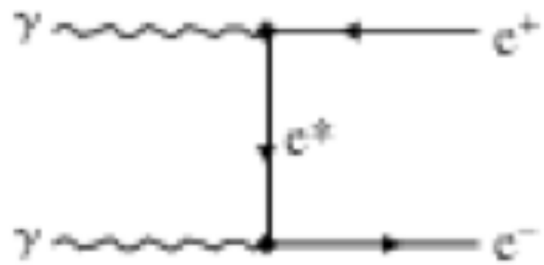
applies to "inverse" Compton scattering, when it's the initial electron rather than the initial photon that has the high energy. This process is the main way that high energy gamma rays are produced in distant galaxies known as "blazars" in which a jet of radiation happens to be pointed in our direction from a central supermassive black hole.

In the low-energy limit of Compton scattering, the electron mass replaces s, so $\sigma \sim \alpha^2/m_e^2$. The exact formula for this "Thomson cross section" is

$$\sigma(\gamma e \to \gamma e)_{\text{Thomson}} = \frac{8\pi\alpha^2}{3m_e^2} = 0.666 \text{ barns}$$

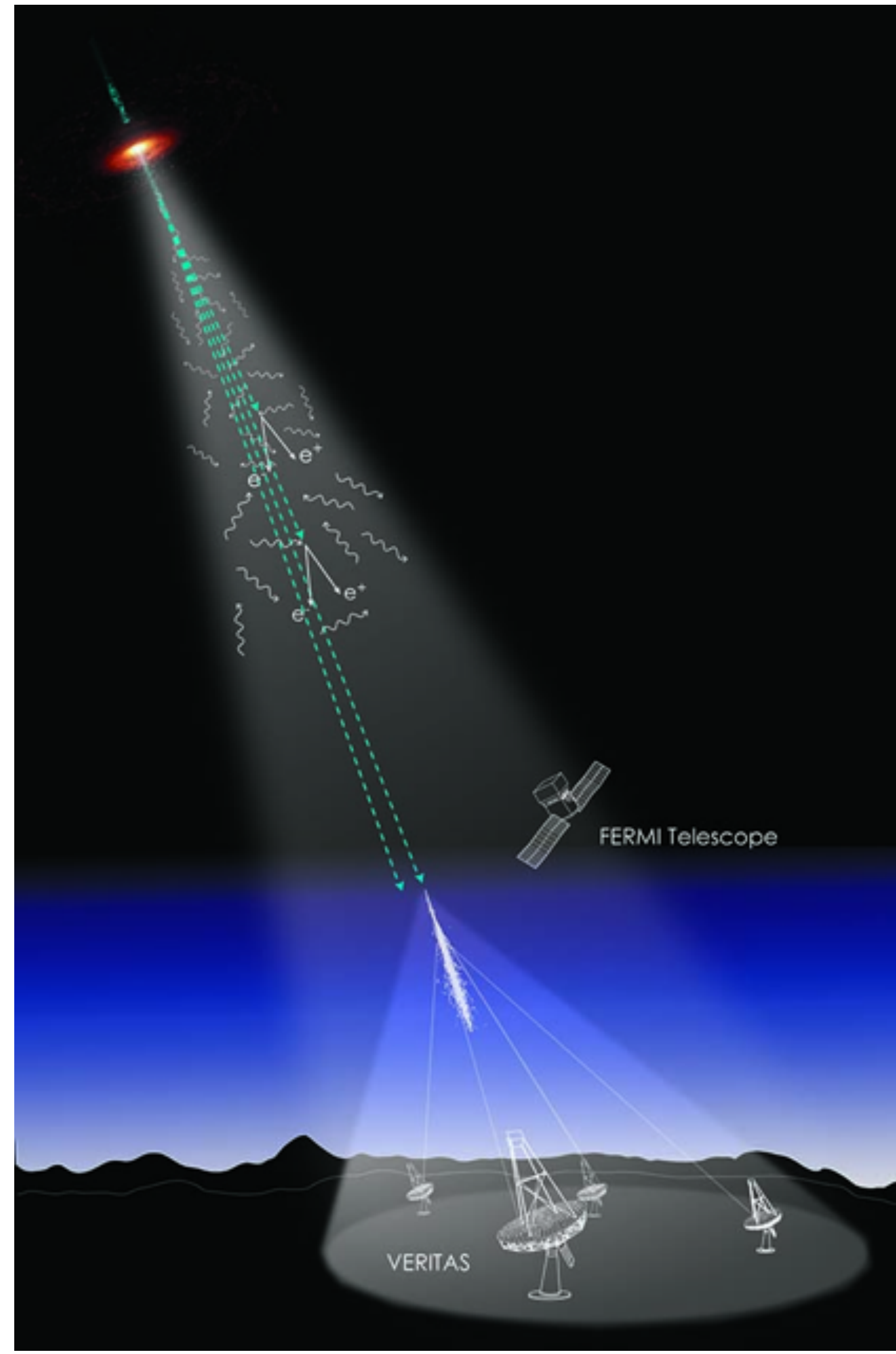
Recall that a barn is defined as 1 barn ≡ $(10 \text{ fm})^2 = 10^{-28} \text{ m}^2 = 10^{-24} \text{ cm}^2$.

$\gamma\gamma \longrightarrow e^+e^-$

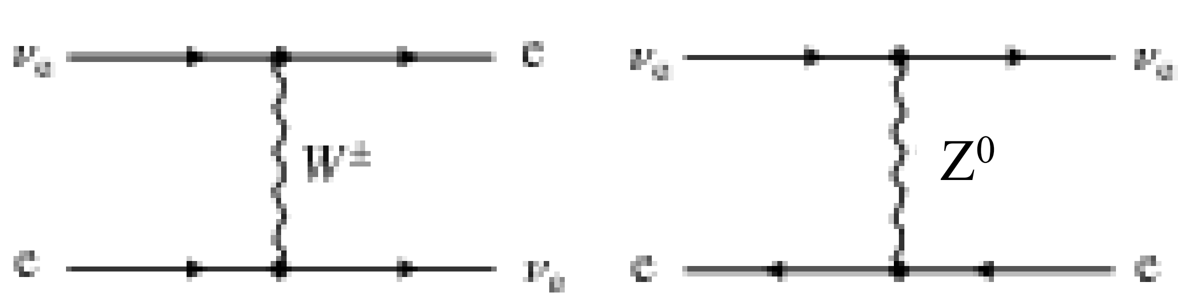


$$\sigma\left(\gamma\gamma \rightarrow e^+e^-\right) = \left(\frac{4\pi\alpha^2}{s}\right)\left[\ln\left(\frac{s}{m^2}\right) - 1\right]$$

The energy threshold for this process is $s_{th} = 4m_e^2$, and $\sigma \sim \beta\alpha^2/s$, where $\beta = (1 - 4m_e^2/s)^{1/2}$ is the CM velocity of the produced electron and positron. This process is the main way that high energy gamma rays from blazars are removed on their way to us, by interacting with low-energy photons (“extragalactic background light,” EBL) radiated as starlight or as radiation from cool dust, and producing e^+e^- pairs.



Energetic gamma rays (dashed lines) from a distant blazar strike photons of extragalactic background light (wavy lines) in intergalactic space, annihilating both gamma ray and photon. Different energies of EBL photons waylay different energies of gamma rays, so comparing the attenuation of gamma rays at different energies from different spacecraft and ground-based instruments indirectly measures the spectrum of EBL photons.



$e^- \nu_e \longrightarrow e^- \nu_e$

This process can happen via either W exchange for e-type neutrinos or Z exchange for all three types of neutrinos. The former diagram leads to the cross section

$$\sigma \sim \frac{g_w^4 s}{M_W^4} \sim \mathrm{G}_\mathrm{F}^2 s$$

where $g_W \sim e$ and $g_W^2/M_W^2 = \mathrm{G_F}$. An exact calculation gives

$$\sigma\left(\nu_e e \rightarrow \nu_e e\right) = \frac{\mathrm{G}_\mathrm{F}^2 s}{\pi}$$

$e^+ e^- \longrightarrow \nu\nu$

For the crossed reaction $e^+e^- \longrightarrow$ neutrino + anti-neutrino, W exchange can make electron-type neutrino and anti-neutrino, but Z exchange can produce all three types of neutrinos. The cross section in the high energy limit is

$$\sigma\left(e^+ e^- \rightarrow \nu_e \bar{\nu}_e\right) = \frac{\mathrm{G}_\mathrm{F}^2 s}{6\pi}$$

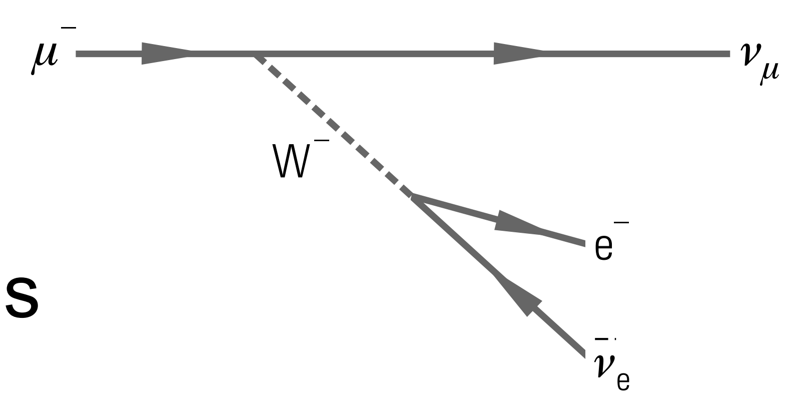
These processes are important both at accelerators, in the early universe, and in supernovae.

Decays and Resonances

In addition to the a + b ⟶ c + d processes we've been discussing, decays a ⟶ b + c or a ⟶ b + c + d can also occur. The rate W_a for such processes is related to the lifetime τ_a of the particle by $W_a = 1/\tau_a$.

In cases like $e^+e^- \longrightarrow P \longrightarrow$ hadrons, where P is an unstable particle, or "resonance," the energy uncertainty or "width" in energy of the resonance Γ is related to W by $\Gamma = \hbar W$, so in natural units the width Γ equals the rate W.

An example is muon decay, $\mu^- \longrightarrow \nu_\mu + e^- + \bar{\nu}_e$



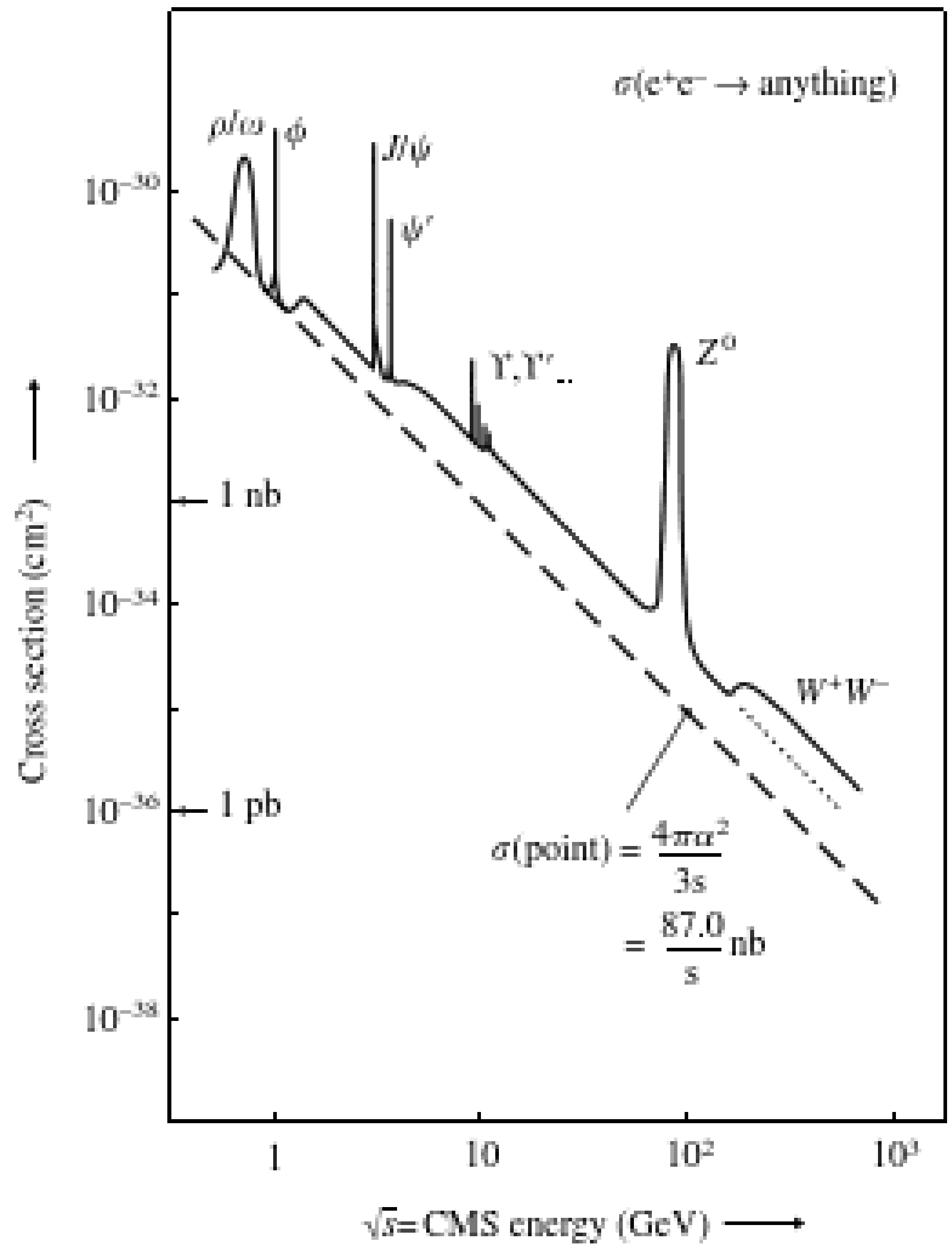
The amplitude T is proportional to $e^2/M_W^2 = G_F$, so the rate W must be proportional to G_F^2, which has dimensions of Mass^{-4}. Since Γ has dimensions of energy or mass, $\Gamma \propto G_F^2\, Q^5$, where Q is the largest energy involved. In the case of muon decay, $Q \sim m_\mu$, and a detailed calculation gives

$$\Gamma\left(\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu\right) = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

Resonances

We discussed cases like $e^+e^- \longrightarrow P \longrightarrow$ hadrons, where P is an unstable particle, or "resonance," represented by the peaks in the figure. The wave function for the decaying particle P is given by

$$\psi(t) = \psi(0)\ \exp(-i\omega_R t)\ \exp\left(-\frac{t}{2\tau}\right)$$
$$= \psi(0)\ \exp\left\{\frac{-t\,(iE_R + (\Gamma/2))}{\hbar}\right\}$$



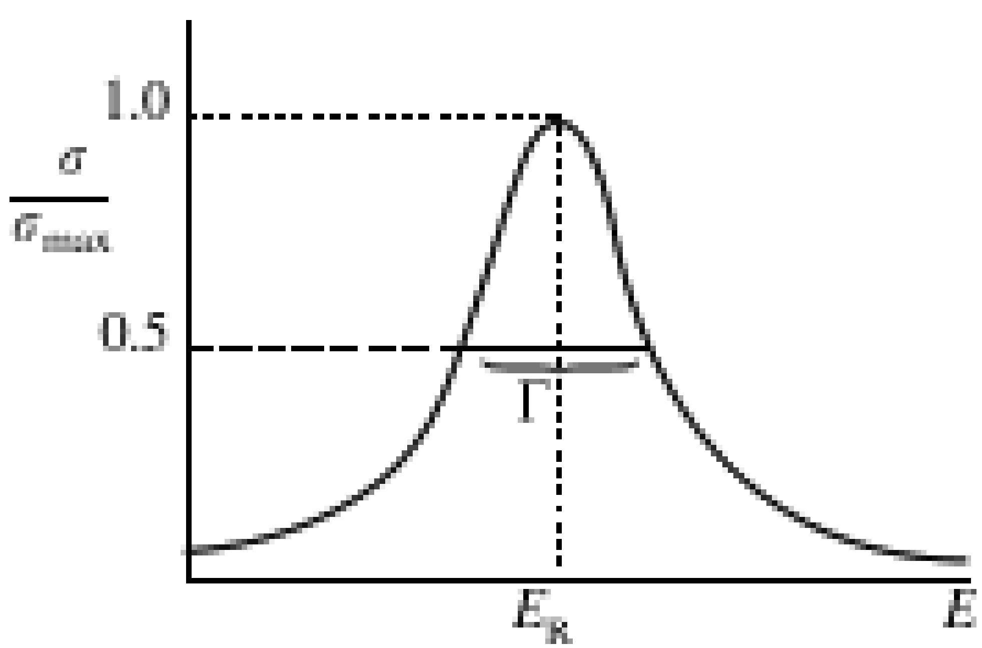
Here the central energy $E_R = \hbar\omega_R$, and the width $\Gamma = \hbar/\tau$. The intensity $I(t) = \Psi^*\,\Psi = I(0)\exp(-t/\tau)$, so the average lifetime is τ. This can also be written as $I(t) = I(0)\,2^{-t/t_{1/2}}$, which implies that the "half-life" $t_{1/2} = \tau \ln 2 = 0.693\,\tau$.

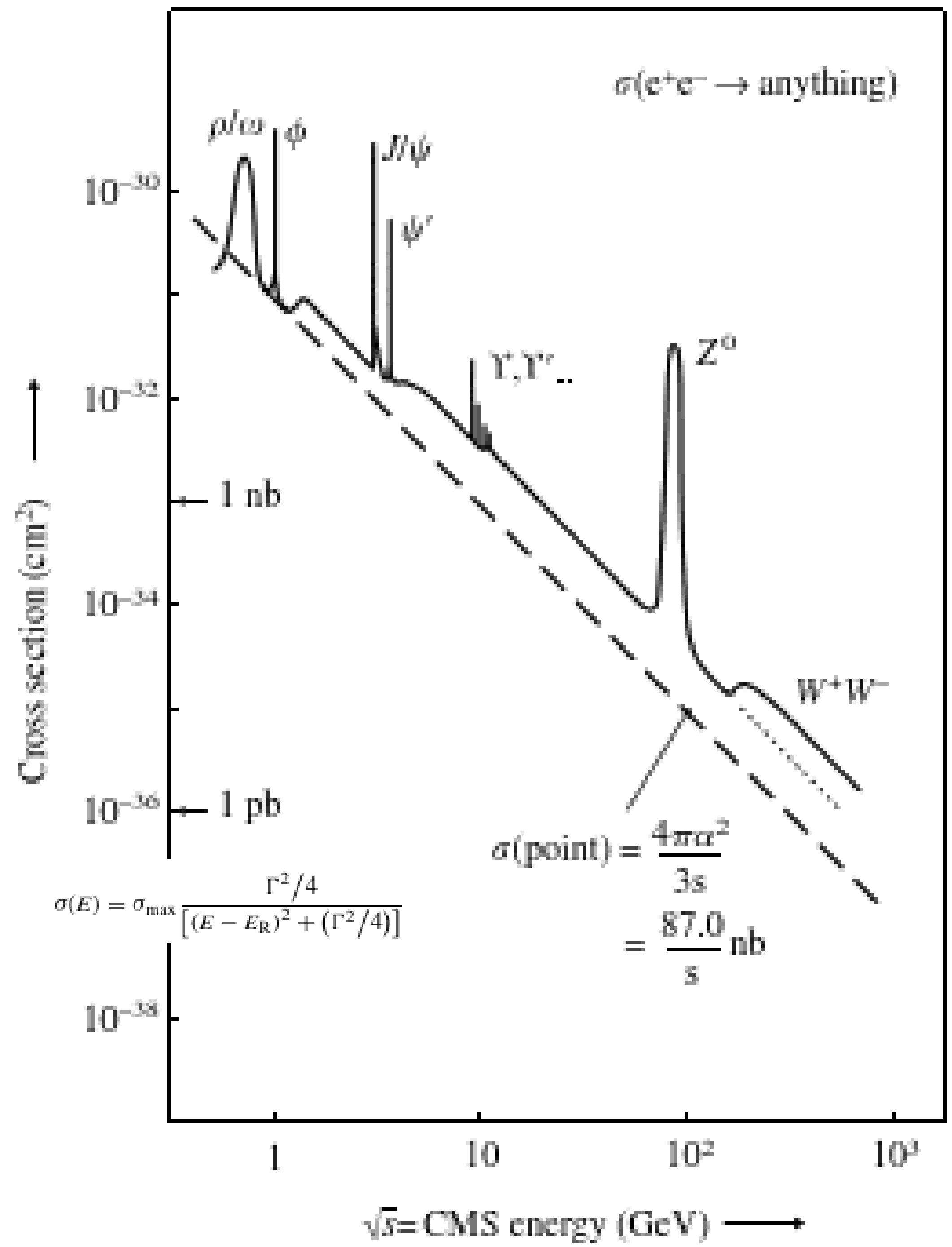
Resonances

Fourier transforming from time to energy, the "Breit-Wigner" formula for the cross section in the vicinity of a resonance is

$$\sigma(E) = \sigma_{\max} \frac{\Gamma^2/4}{\left[(E - E_R)^2 + (\Gamma^2/4)\right]}$$

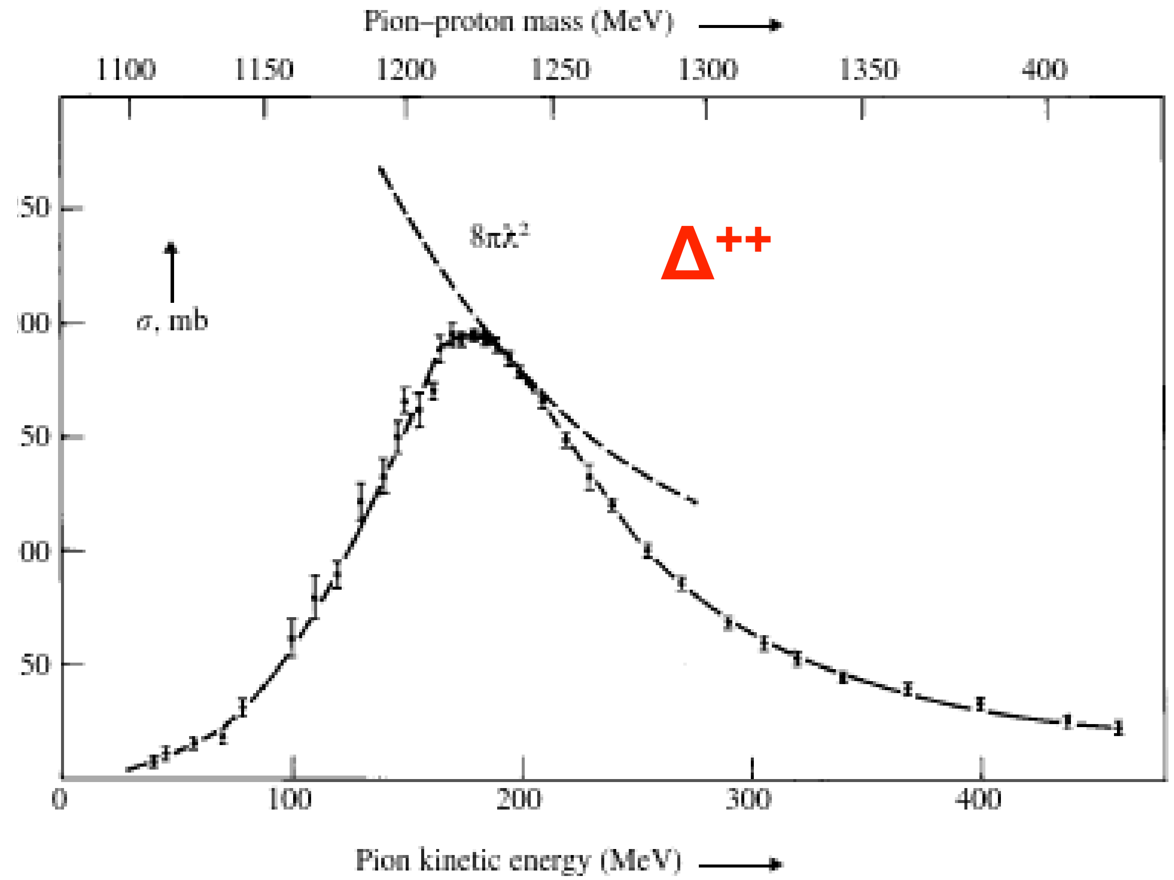
represented by the curve below



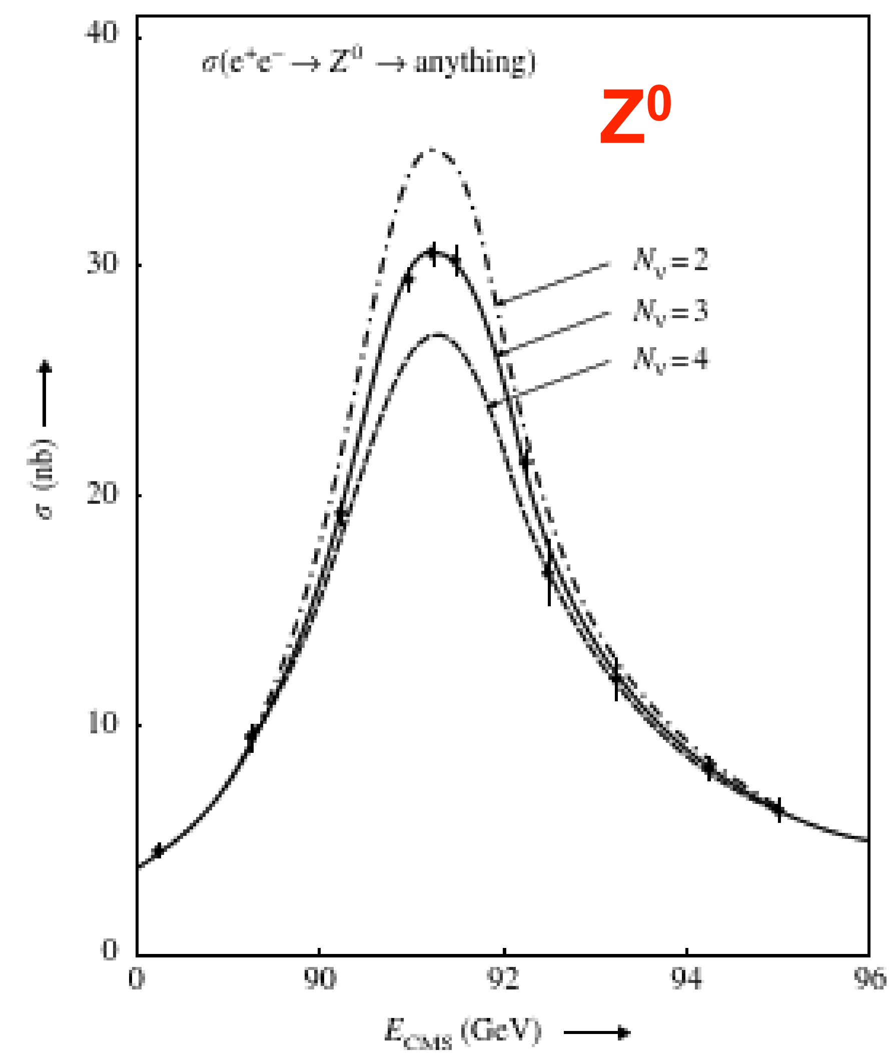


Note that the cross section falls to half its value at the peak $E = E_R$ when $E = E_R \pm \Gamma/2$. Resonant particles P have specific spin angular momenta, and taking angular momentum into account leads to more complicated formulas given in Perkins.

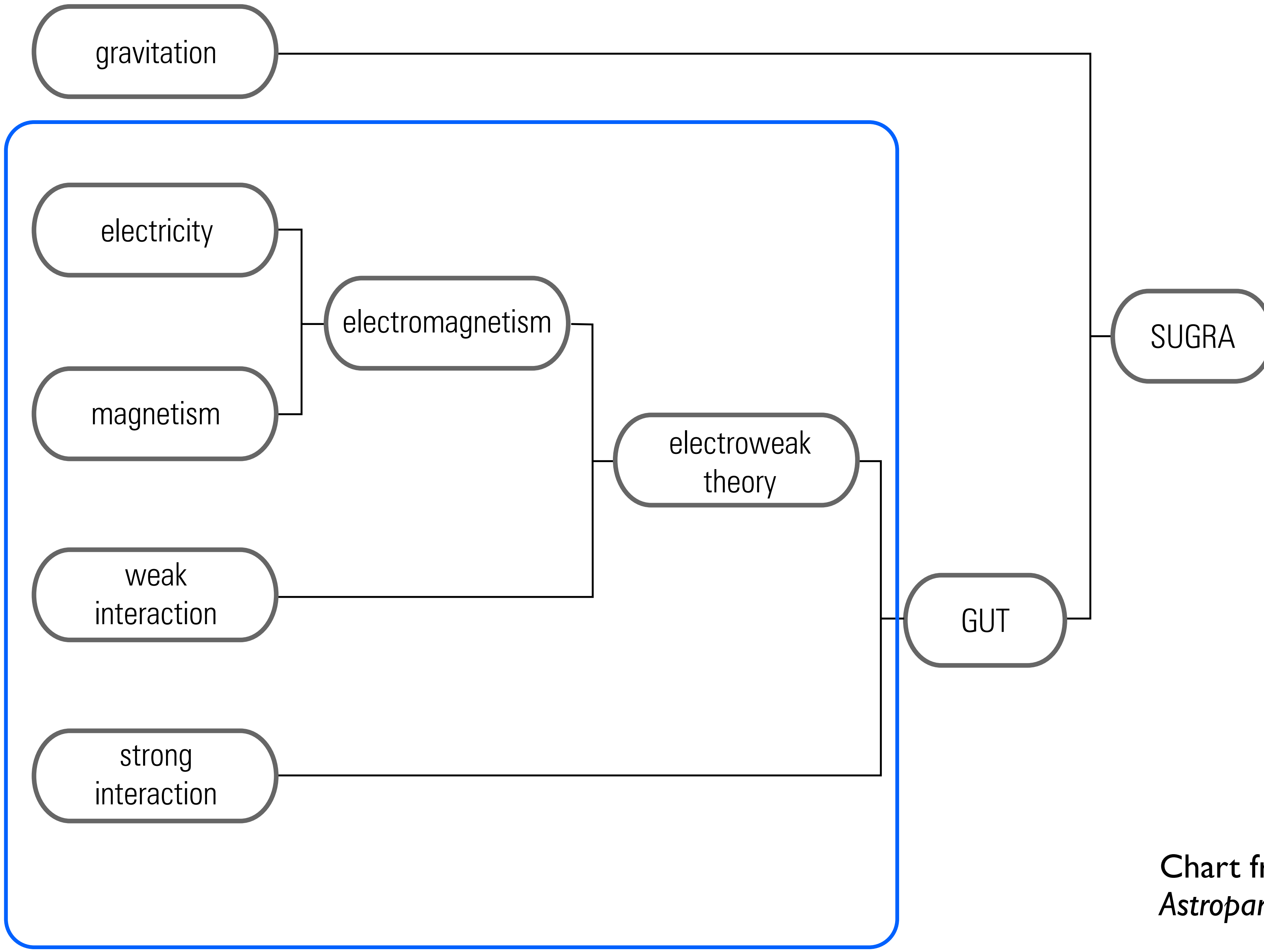
The first high-energy physics resonance ever discovered is the Δ^{++} in $p + \pi^+ \longrightarrow \Delta^{++} \longrightarrow p + \pi^+$ scattering at 1232 MeV CM energy. The cross section is shown in the figure at the right.



The Z^0 resonance in e^+e^- scattering is shown in the solid curve in the lower figure at the right. The data are clearly consistent with the Z^0 decaying into $N_\nu = 3$ types of neutrinos, namely the electron, muon, and tau types. Even before the discovery of the Z^0 boson, nucleosynthesis of the light elements in the Big Bang had already indicated that $N_\nu \approx 3$.



Possible Unifications of the Fundamental Interactions



Standard Model of Particle Physics

Chart from Grupen, *Astroparticle Physics*