Physics 129 LECTURE 5 January 21, 2014

- GR passes very precise tests, including binary pulsars
- GR is essential for the GPS system - so, for our everyday lives!
- Black Holes in the Universe
  - Two types of BHs definitely exist:
    - Black Holes from Stellar Collapse
    - Supermassive Black Holes at the Centers of Galaxies
  - Intermediate Mass Black Holes?
  - Hawking Radiation from Small Black Holes (a quantum effect)
  - Evaporating Black Holes?
- Particle Physics Symmetries (Perkins Chapter 3)
  - Lagrangian Deductions
  - Rotations
  - Parity

Schwarzschild Metric

Einstein derived the precession of Mercury and the deflection of light near the sun by perturbing around flat space. But a few months after Einstein invented GR, Karl Schwarzschild discovered the exact solution of Einstein's equations around a massive object:

$$ds^2 = \left[1 - \frac{2GM}{(rc^2)}\right] c^2 dt^2 - \left[1 - \frac{2GM}{(rc^2)}\right]^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) \tag{2.21}$$

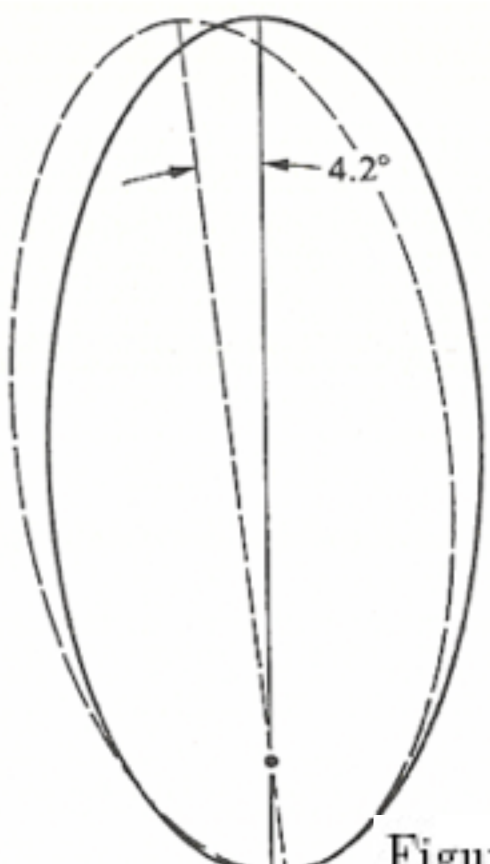
If we set dr = dθ = dφ = 0, the proper time interval dτ is given by

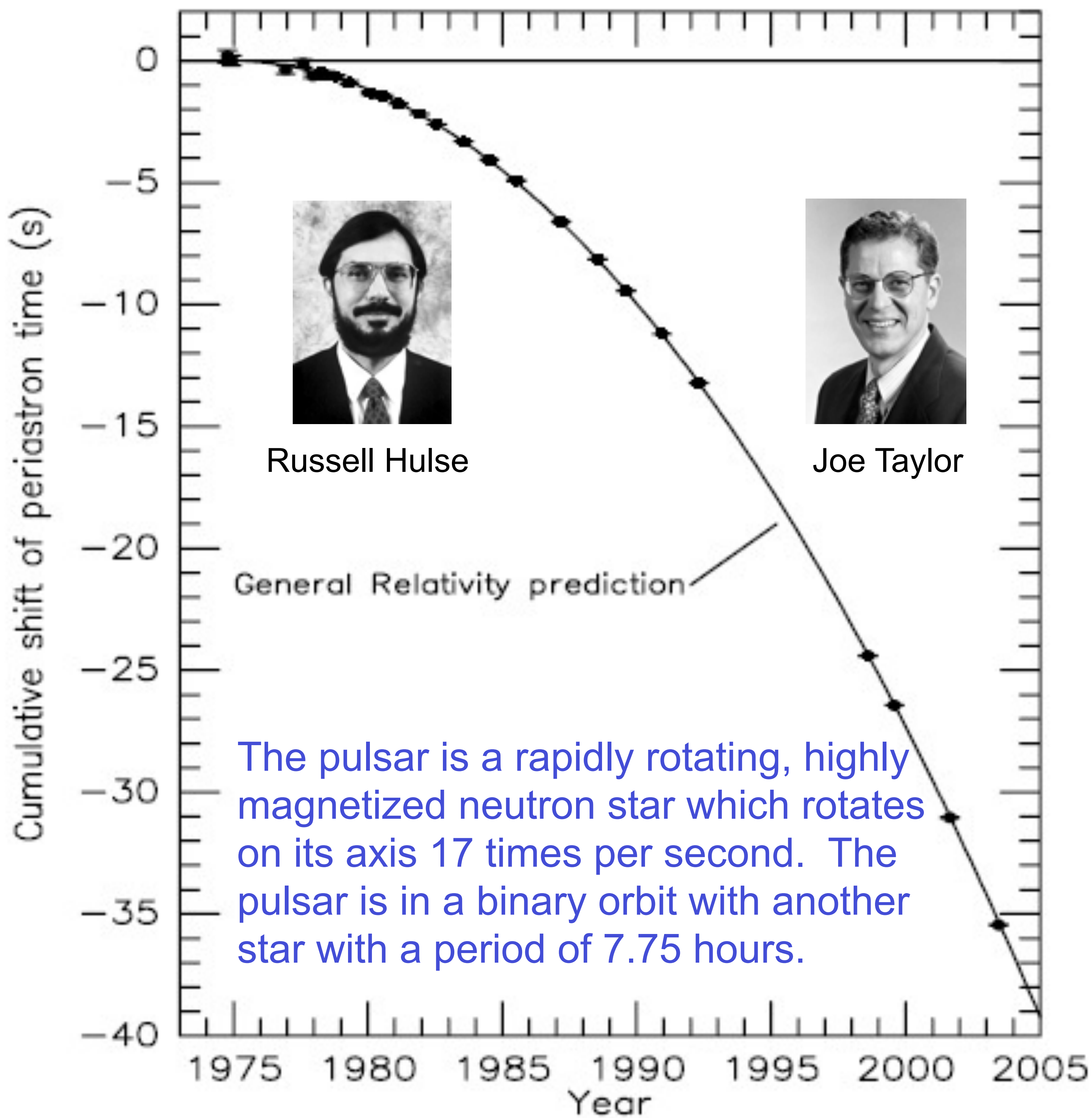
$$d\tau = \sqrt{\left[1 - \frac{2GM}{(rc^2)}\right]} \cdot dt \qquad \text{or,} \qquad d\tau = (1 - r_S/r)^{1/2}\, dt\,.$$

The quantity $r_S = 2GM/c^2 = 2.95$ km $(M/M_\odot)$ is called the Schwarzschild radius. Perkins uses the Schwarzschild metric to discuss precession, light deflection, and the Shapiro time delay.

BINARY PULSAR test of gravitational radiation

In 1993, the Nobel Prize in Physics was awarded to Russell Hulse and Joseph Taylor of Princeton University for their 1974 discovery of a pulsar, designated PSR 1913+16, in orbit with another star. The system is radiating gravity waves just as GR predicts, causing the shift of the periastron (closest approach) shown by the figure at the right.





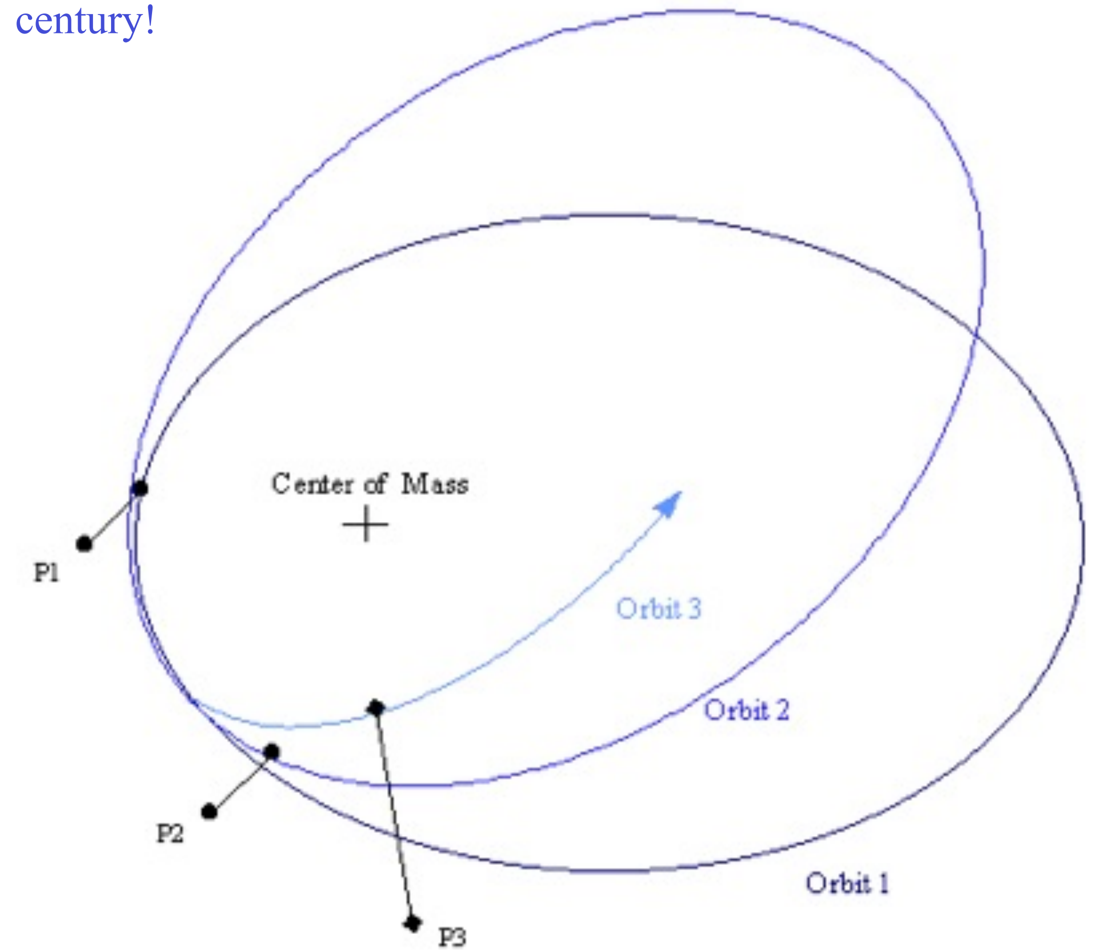
The pulsar is a rapidly rotating, highly magnetized neutron star which rotates on its axis 17 times per second. The pulsar is in a binary orbit with another star with a period of 7.75 hours.

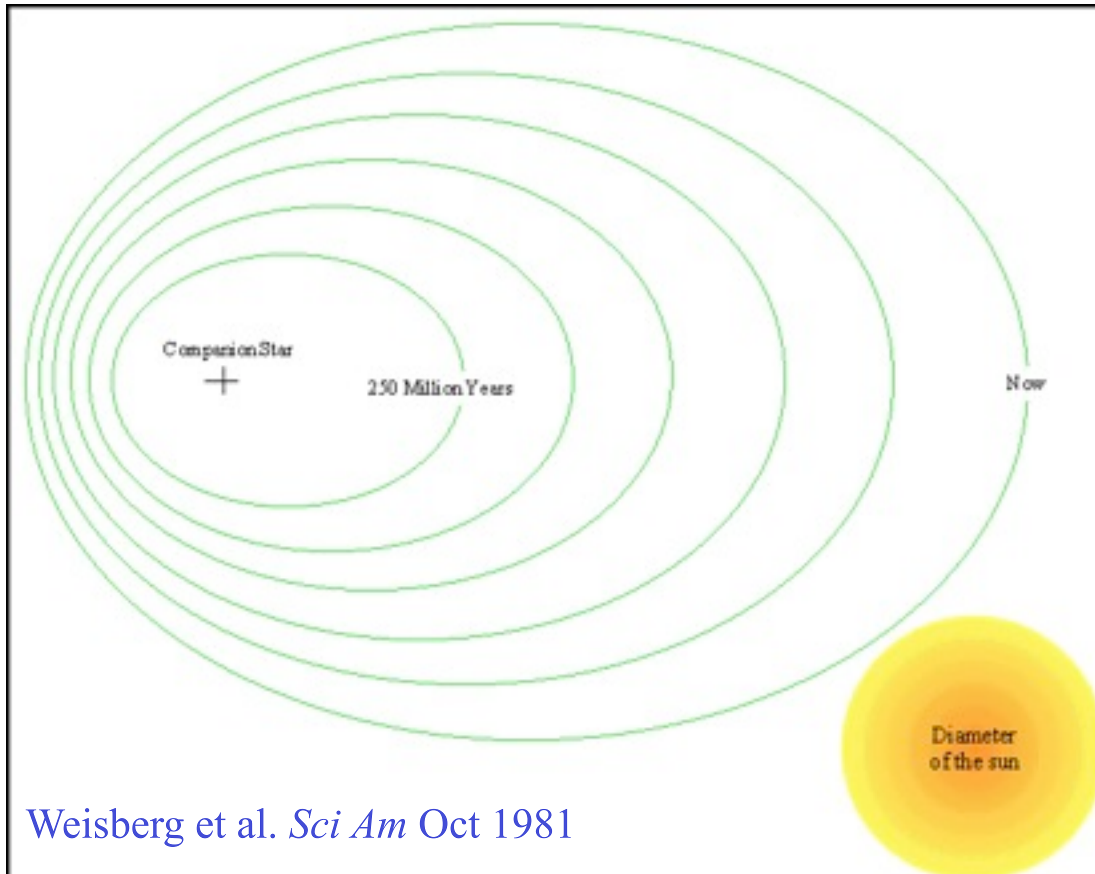
Figure 7: *Plot of the cumulative shift of the periastron time from 1975–2005. The points are data, the curve is the GR prediction. The gap during the middle 1990s was caused by a closure of Arecibo for upgrading [243].*

Data on the PSR B1913+16 system:

Right ascension	19h13m12.4655s
Declination	+16°01'08.189"
Distance	21,000 light years
Mass of detected pulsar	1.441 M_{Sun}
Mass of companion	1.387 M_{Sun}
Rotational period of detected pulsar	59.02999792988 sec
Diameter of each neutron star	20 km
Orbital period	7.751939106 hr
Eccentricity	0.617131
Semimajor axis	1,950,100 km
Periastron separation	746,600 km
Apastron separation	3,153,600 km
Orbital velocity of stars at periastron	300 km/sec
Orbital velocity of stars at apastron	75 km/sec
Rate of decrease of orbital period	0.0000765 sec per year
Rate of decrease of semimajor axis	3.5 meters per year
Calculated lifetime (to final inspiral)	300,000,000 years

Periastron advance per day = Mercury perihelion advance per century!

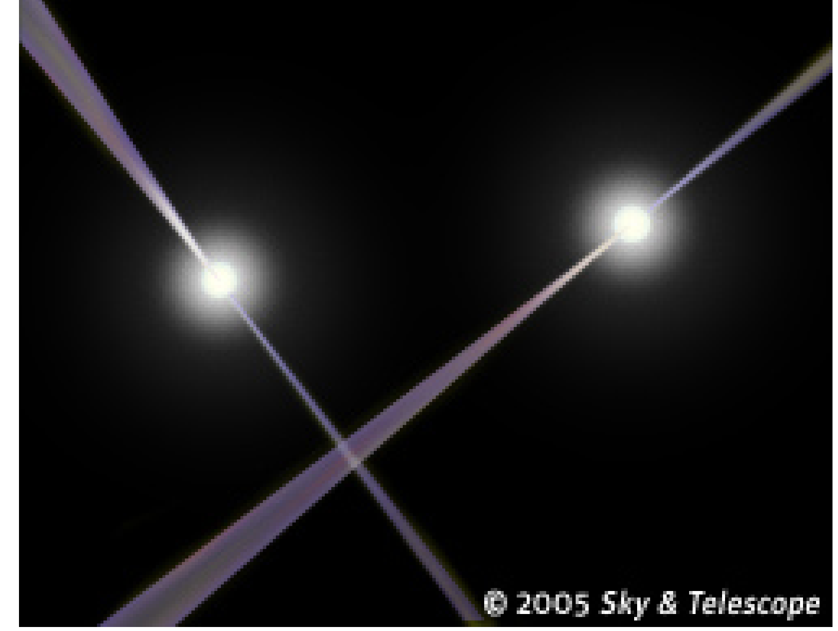




Weisberg et al. *Sci Am* Oct 1981

Einstein Passes New Tests

Sky & Telescope, March 3, 2005, by Robert Naeye



A binary pulsar system provides an excellent laboratory for testing some of the most bizarre predictions of general relativity. The two pulsars in the J0737-3039 system are actually very far apart compared to their sizes. In a true scale model, if the pulsars were the sizes of marbles, they would be about 750 feet (225 meters) apart.

Albert Einstein's 90-year-old general theory of relativity has just been put through a series of some of its most stringent tests yet, and it has passed each one with flying colors. Radio observations show that a recently discovered binary pulsar is behaving in lockstep accordance with Einstein's theory of gravity in at least four different ways, including the emission of gravitational waves and bizarre effects that occur when massive objects slow down the passage of time.

An international team led by Marta Burgay (University of Bologna, Italy) discovered the binary pulsar, known as J0737–3039 for its celestial coordinates, in late 2003 using the 64-meter Parkes radio telescope in Australia. Astronomers instantly recognized the importance of this system, because the two neutron stars are separated by only 800,000 kilometers (500,000 miles), which is only about twice the Earth–Moon distance. At that small distance, the two 1.3-solar-mass objects whirl around each other at a breakneck 300 kilometers per second (670,000 miles per hour), completing an orbit every 2.4 hours.

General relativity predicts that two stars orbiting so closely will throw off gravitational waves — ripples in the fabric of space-time generated by the motions of massive objects. By doing so, they will lose orbital energy and inch closer together. Radio observations from Australia, Germany, England, and the United States show that the system is doing exactly what Einstein's theory predicts. "The orbit shrinks by 7 millimeters per day, which is in accordance with general relativity," says Michael Kramer (University of Manchester, England), a member of the observing team.

Chin. J. Astron. Astrophys. Vol. 6 (2006), **Suppl. 2**, 162–168

A Review of The Double Pulsar - PSR J0737−3039

A. G. Lyne ⋆

Table 1 Basic Observed Parameters of PSRs J0737−3039A and B

Pulsar	PSR J0737−3039A	PSR J0737−3039B
Pulse period P	22.7 ms	2.77 s
Period derivative $\dot{P}$	1.7×10^{-18}	0.88×10^{-15}
Orbital period $P_{\rm b}$	2.45 hours	
Eccentricity e	0.088	
Orbital inclination	~88 deg	
Projected semi-major axis x	1.42 sec	1.51 sec
Stellar mass M	1.337(5) $M_\odot$	1.250(5) $M_\odot$
Mean orbital velocity $V_{\rm orb}$	301 km $\rm s^{-1}$	323 km $\rm s^{-1}$
Characteristic age τ	210 Myr	50 Myr
Magnetic field at surface B	6.3×10^9 G	1.2×10^{12} G
Radius of Light cylinder $R_{\rm LC}$	1080 km	132 000 km
Spin-down luminosity $\dot{E}$	6000×10^{30} erg $\rm s^{-1}$	1.6×10^{30} erg $\rm s^{-1}$

2 TESTS OF GRAVITATIONAL THEORY USING BINARY SYSTEMS

Non-relativistic binary systems are usually precisely described by the five Keplerian parameters, $P_{\rm orb}$, $a\sin i$, e, ω and T_o and these are all accurately measured when one object is a pulsar. However, a number of general relativistic corrections to this classical description of the orbit - the so-called post-Keplerian (PK) parameters - are needed if the gravitational fields are sufficiently strong. In only a few months, using the Parkes Telescope, the Lovell Telescope at Jodrell Bank and the Green Bank Telescope, it was possible to measure several general relativistic effects in 6 months that took years to measure with the Hulse-Taylor binary pulsar, PSR B1913+16.

The following five PK relativistic parameters have already been measured in A, all causing small, but highly significant, modifications to the arrival times of the pulsars' radio pulses:

Relativistic periastron advance, $\dot{\omega}$. This is the rotation of the line connecting the two pulsars at their closest approach to one another. It arises from the distortion of space-time caused by the two stars, but can also be understood as the result of the finite time needed for the gravitational influence of one star to travel to another. This causes a time delay, during which the stars move so that the attractive force is no longer radial.

Gravitational redshift and time dilation, γ. The redshift results in clocks appearing to run slowly in a gravitational potential well and time dilation is the special relativistic effect which results in moving clocks appearing to run slowly. Both effects cause clocks close to a neutron star to tick more slowly than those further away. In other words the apparent pulse rate for A will slow down when it is close to B, and vice versa.

Shapiro delay, r and s. Radiation passing close to a massive body is delayed because its path length is increased by the curvature of space-time, an effect that Einstein overlooked but that was discovered in 1964 by Irwin Shapiro (Shapiro 1964) from radar measurements in the Solar System. Signals from A are measured after they have passed through the distorted space-time of B (in principle the effect could also be measured for the signals from B but its pulses are much broader and do not provide sufficient temporal resolution). The signal delay is essentially a function of two parameters: s, the shape, and r, the range, of the delay experienced by the pulses (with s being dependent on the inclination of the orbital plane and r on the mass of B).

Gravitational radiation and orbital decay, dP_b/dt**.** Almost every theory of gravitation predicts that the movement of massive bodies around one another in a binary system will result in the emission of gravitational waves. This emission causes the bodies to lose energy and hence to spiral into one another, so that they will eventually merge, creating a burst of gravitational waves when they do so. The rate of decrease of the orbital period, dP_b/dt, indicates that orbits of the pulsars are currently shrinking by about 7 mm per day.

5 CONCLUSION

Several fortunate circumstances have come together to make these studies possible. Not only is this a double-neutron-star system, but

- It has a very compact orbit, giving rise to intense gravitational fields and accelerations and hence abundant post-Keplerian gravitational effects
- One pulsar is a millisecond pulsar which enables these effects to be measured with high precision
- Both neutron stars are visible, allowing the mass-ratio to be determined
- Both pulsars have large flux densities, giving high-precision measurements
- The orbit is nearly edge on, so that the Shapiro delay can be measured with high precision.

All these properties enhance the quality and speed of the tests of gravitation theories in the strong-field regime. Furthermore, the last three also enable the investigations of the interactions between the stars and the probing of the magnetospheric properties.

Future observations of binary systems like PSR J0737−3039 promise to greatly increase our knowledge of strong-field gravity, but finding these systems will be a challenge. This is because double pulsars are extremely rare and, more importantly, because the Doppler effect causes their pulse periods to vary rapidly even during a short observation. It therefore becomes more difficult to detect the pulsars' periodicity using normal Fourier techniques and more sophisticated and computationally challenging search algorithms will have to be employed to uncover them.

Conclusions

We find that general relativity has held up under extensive experimental scrutiny. The question then arises, why bother to continue to test it? One reason is that gravity is a fundamental interaction of nature, and as such requires the most solid empirical underpinning we can provide. Another is that all attempts to quantize gravity and to unify it with the other forces suggest that the standard general relativity of Einstein is not likely to be the last word. Furthermore, the predictions of general relativity are fixed; the theory contains no adjustable constants so nothing can be changed. Thus every test of the theory is either a potentially deadly test or a possible probe for new physics. Although it is remarkable that this theory, born 90 years ago out of almost pure thought, has managed to survive every test, the possibility of finding a discrepancy will continue to drive experiments for years to come.

Clifford Will, Living Reviews, 2006

How the GPS System Works

There are ~30 GPS satellites in orbits such that 4 or more are visible at any time from almost any point on earth. Each GPS satellite carries very accurate atomic clocks, and it corrects the time it transmits so that it is what a stationary clock would read. A GPS receiver uses the times it receives to correct its own much less accurate clock. It knows the location of each GPS satellite, and it then triangulates to determine its position: i.e. it solves the (Euclidean space) equations $c^2(t - t_j)^2 = |\boldsymbol{r} - \boldsymbol{r}_j|^2$. In 3D there is only one solution, which determines the location of the GPS receiver. Since $c \approx 30$ cm/ns, it is essential to have the GPS receiver clock correct to ~ 1 ns to get accurate locations. If the GPS satellites didn't correct their time signals for SR and GR effects, the errors would be huge!

General Relativity

CURVED SPACE TELLS MATTER HOW TO MOVE

$$\frac{du^{\mu}}{ds} + \Gamma^{\mu}{}_{\alpha\beta}\, u^{\alpha} u^{\beta} = 0$$

MATTER TELLS SPACE HOW TO CURVE

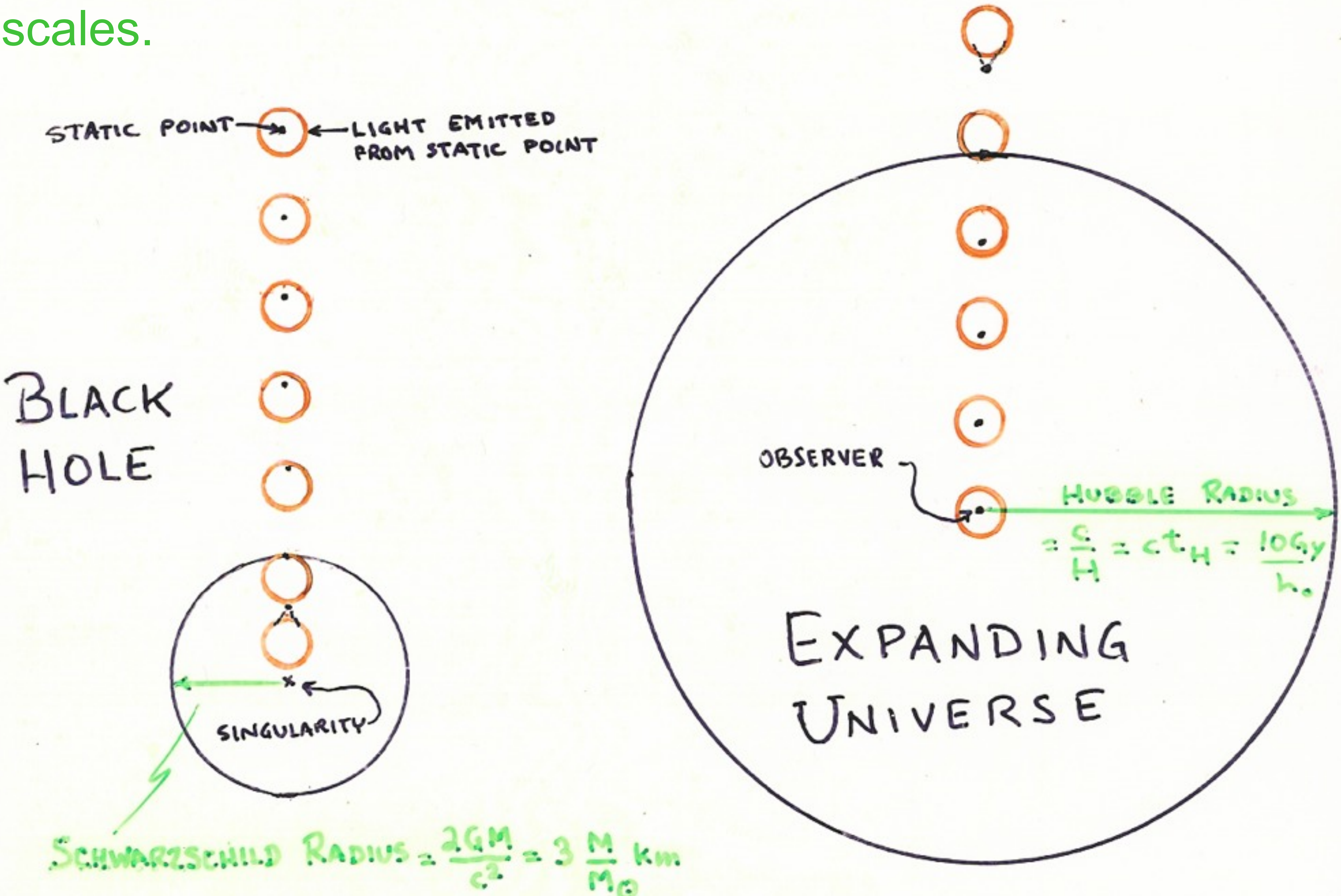
Einstein Field Equations

$$G^{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} = -\,8\pi G T^{\mu\nu} - \Lambda g^{\mu\nu}$$

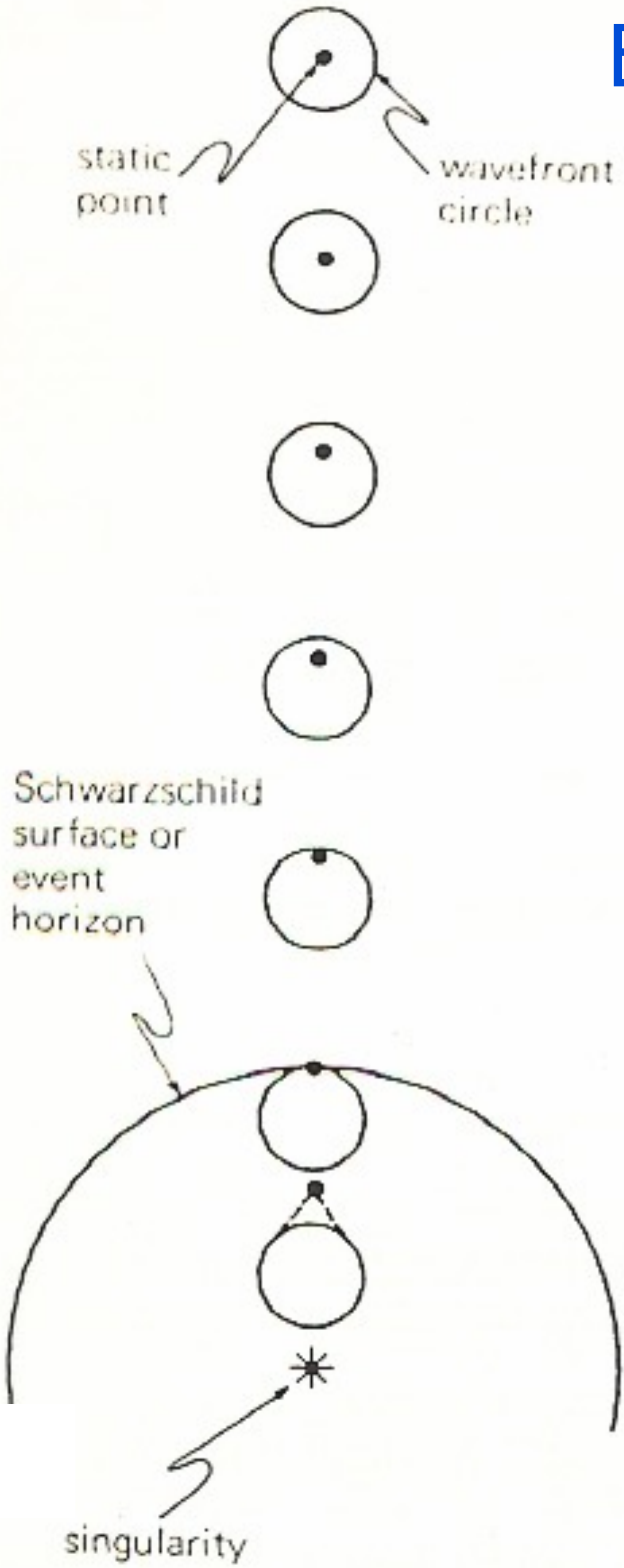
Curved spacetime is not just an arena within which things happen, spacetime is dynamic. Curvature can even cause horizons, beyond which information cannot be sent.

There are event horizons around black holes, and we are surrounded by both particle and event horizons on cosmic scales.

There are event horizons around black holes, and we are surrounded by both particle and event horizons on cosmic scales.



EFFECTS OF CURVATURE NEAR A BLACK HOLE



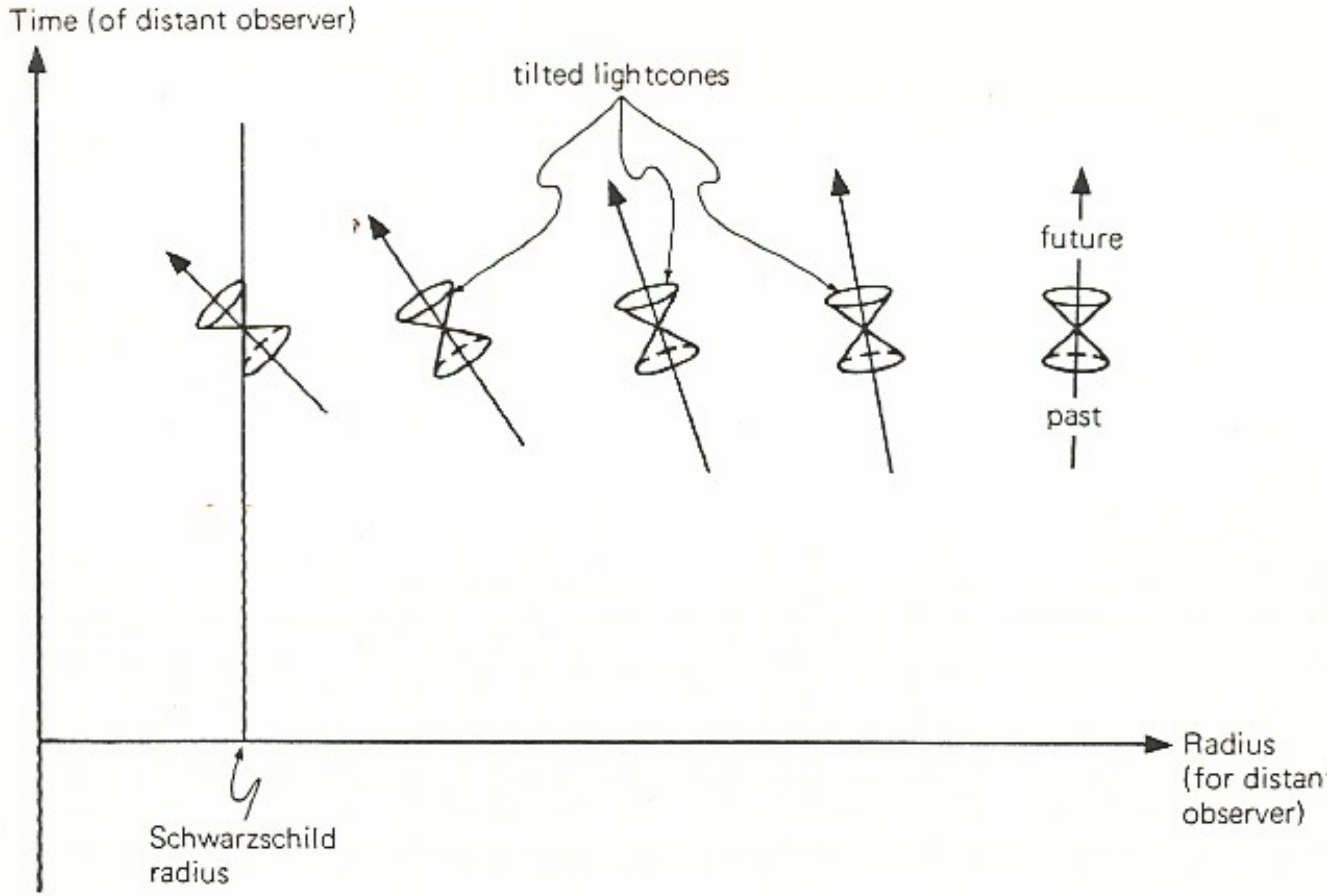
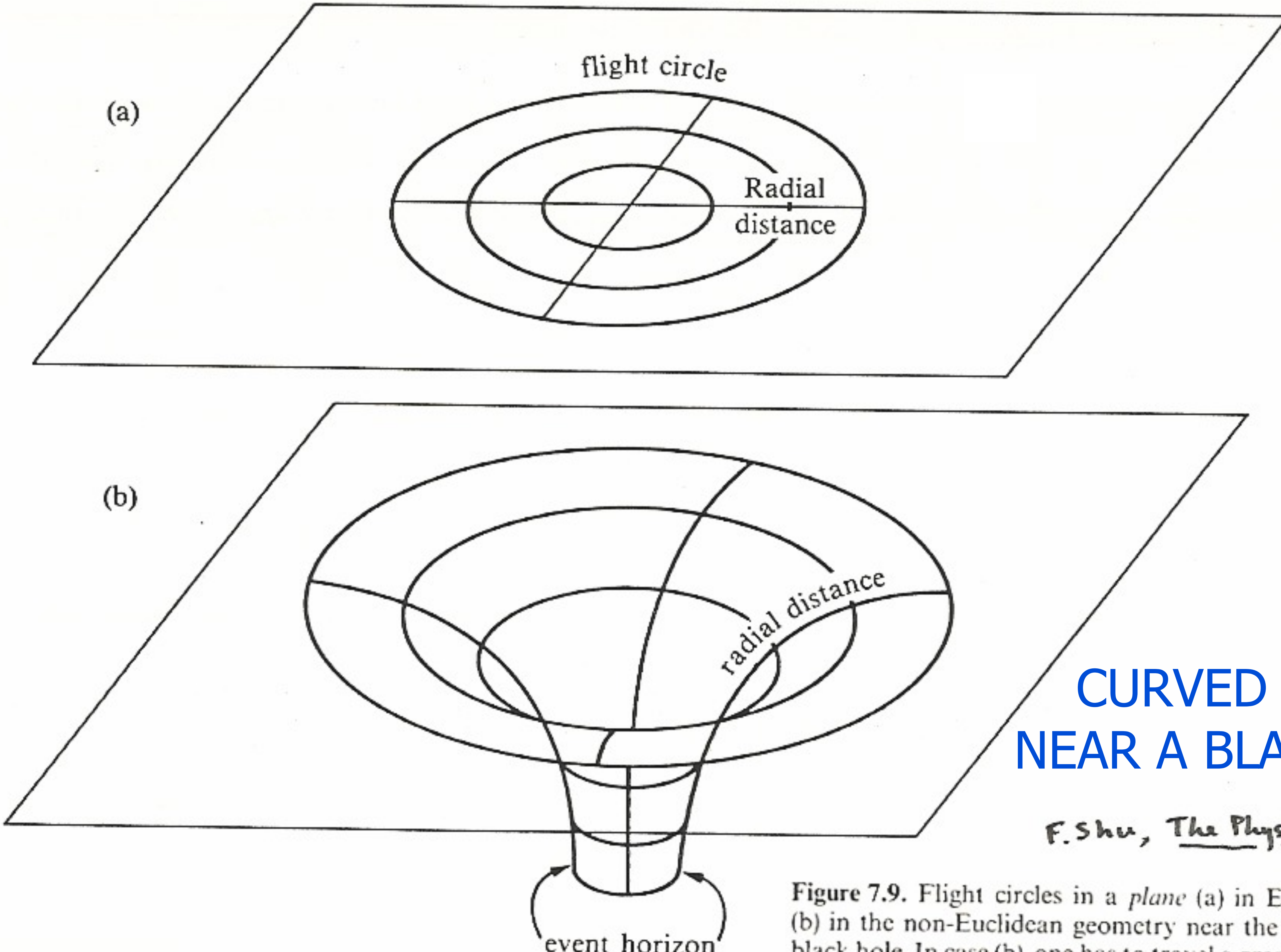


Figure13.6. *The effect of spacetime curvature near a black hole. Lightcones are tilted in such a way that the future-pointing lightcone tips toward the black hole and the past-pointing lightcone tips away from the black hole. At the surface of the black hole (the Schwarzschild surface), all rays emitted in the future direction fall into the black hole, and no rays from the past are received from the black hole. A person passing into a black hole therefore receives no information of what lies ahead.*

E. Harrison, Cosmology

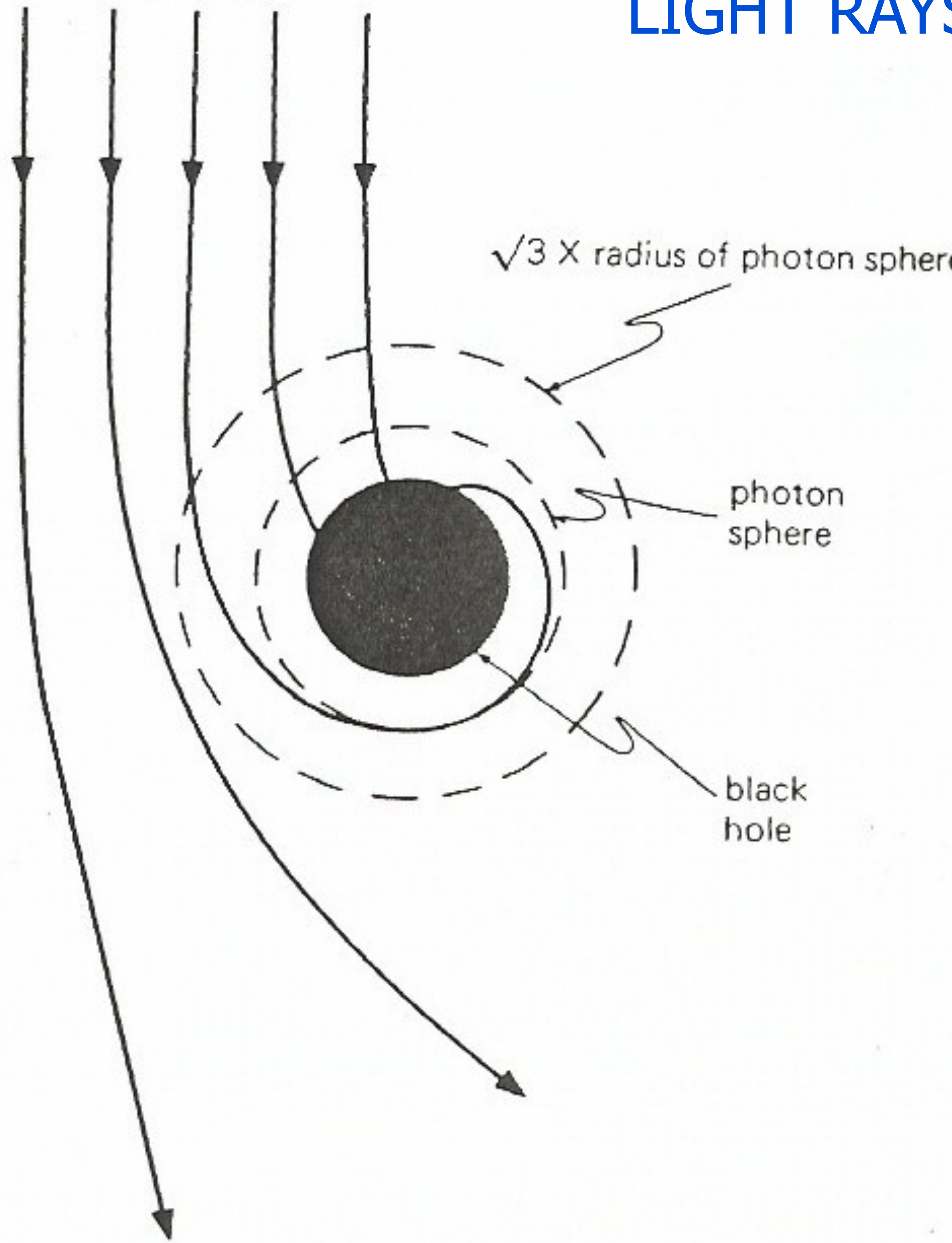


CURVED SPACE NEAR A BLACK HOLE

F. Shu, The Physical Universe

Figure 7.9. Flight circles in a *plane* (a) in Euclidean geometry, (b) in the non-Euclidean geometry near the event horizon of a black hole. In case (b), one has to travel a greater distance inward than in case (a) to have a flight circle of given smaller circumference. The radial direction in both cases is as indicated. At great distances from the event horizon (not drawn), the "curvature" of our embedding diagram becomes negligibly small, and the flight circles of case (b) have nearly the same geometry as case (a).

LIGHT RAYS NEAR A BLACK HOLE



. *Deflection of lightrays by a black hole. Rays approaching closer than √3 times the radius of the photon sphere are captured.*

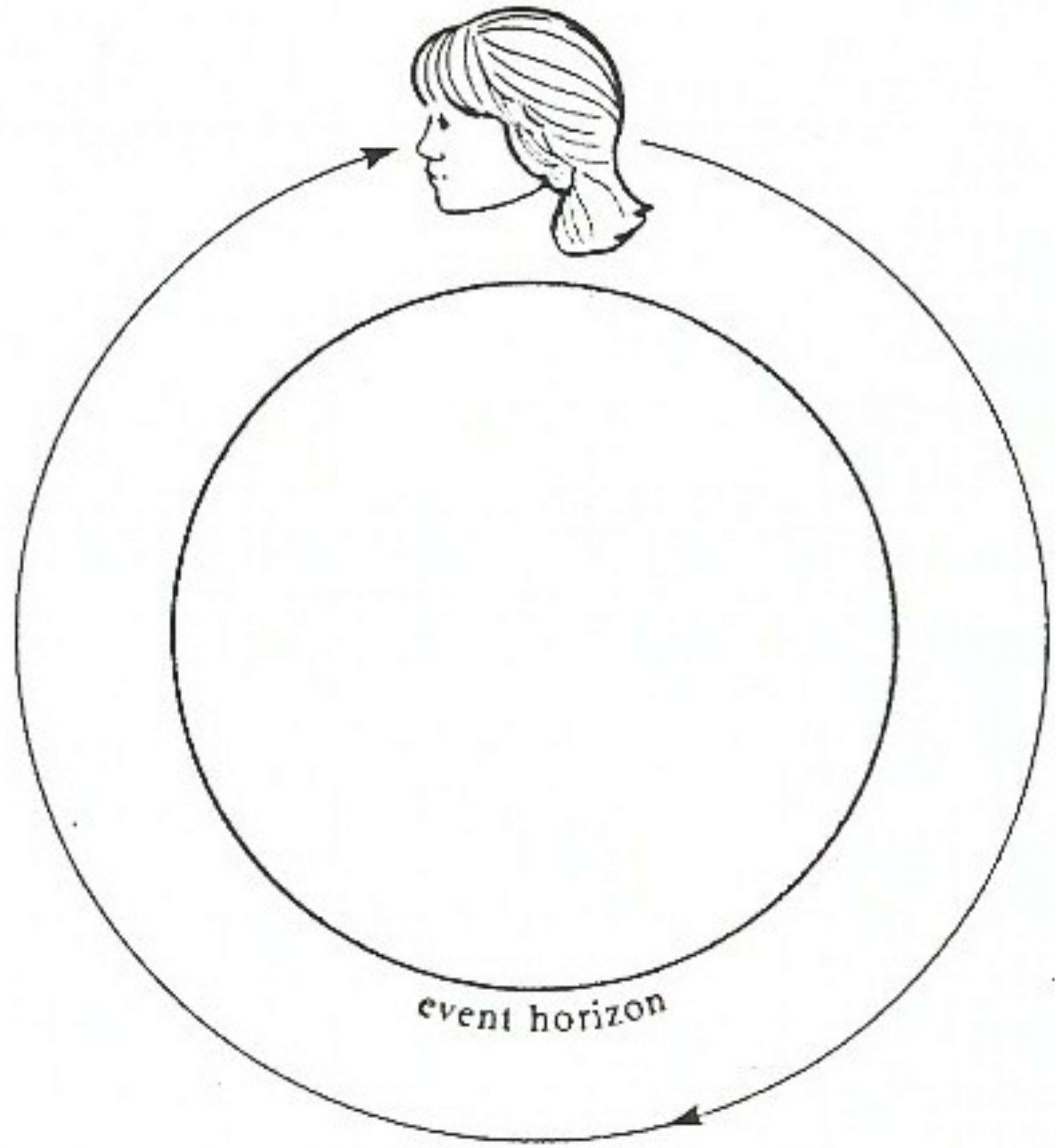
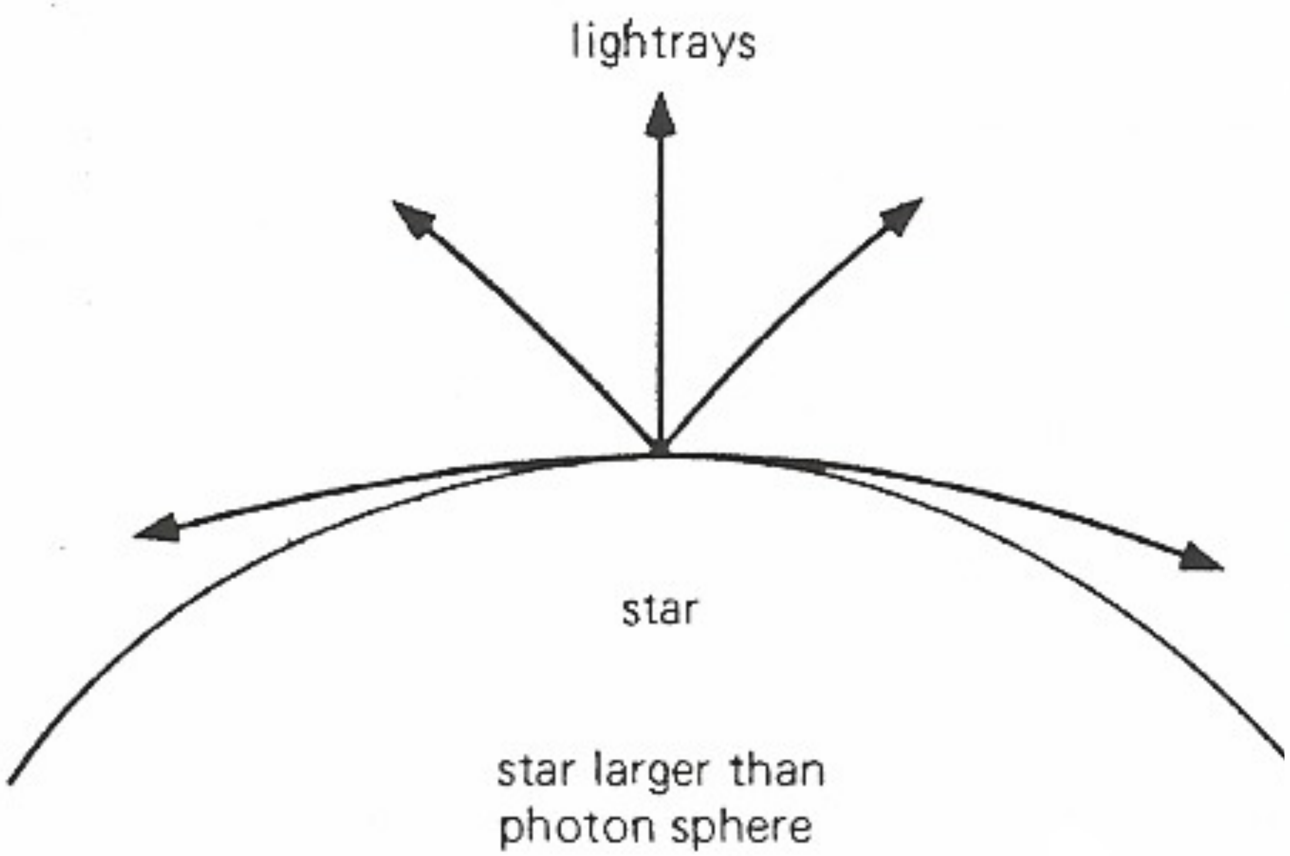
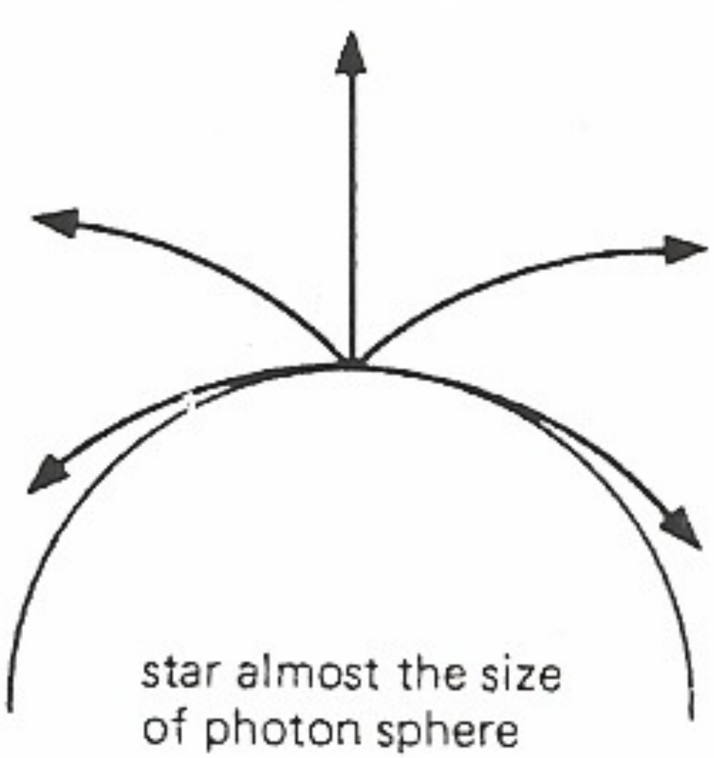
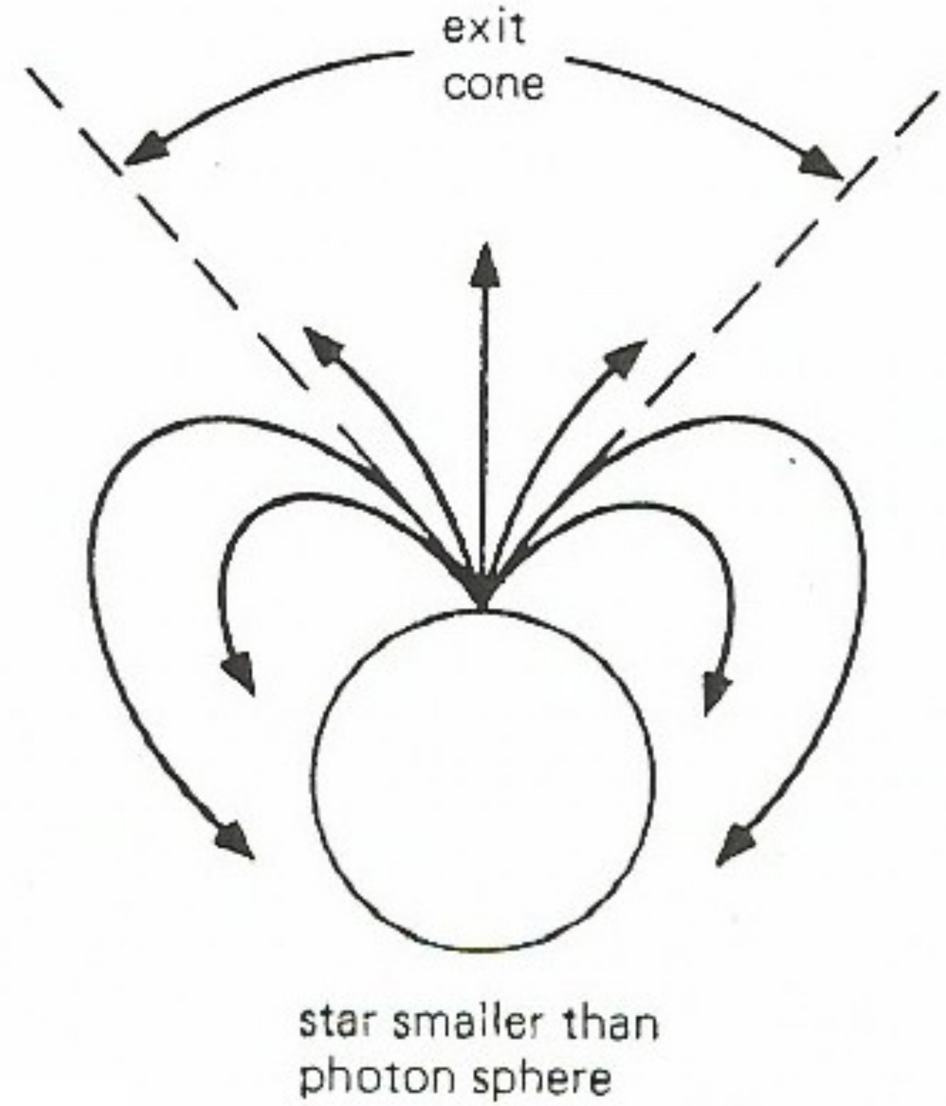


Figure 7.10. When at a circumference equal to 1.5 times the circumference of the event horizon of a black hole, a suitably suspended astronaut can see the back of her own head without the benefit of any mirrors.

E. R. Harrison, *Cosmology*







Lightrays leaving a gravitating body are curved as shown. As the body shrinks in size the rays become more curved. When the radius is less than 1.5 times the Schwarzschild radius, which is the radius of the photon sphere, the exit cone begins to close. Rays emitted within the exit cone escape, but those outside are trapped and fall back.

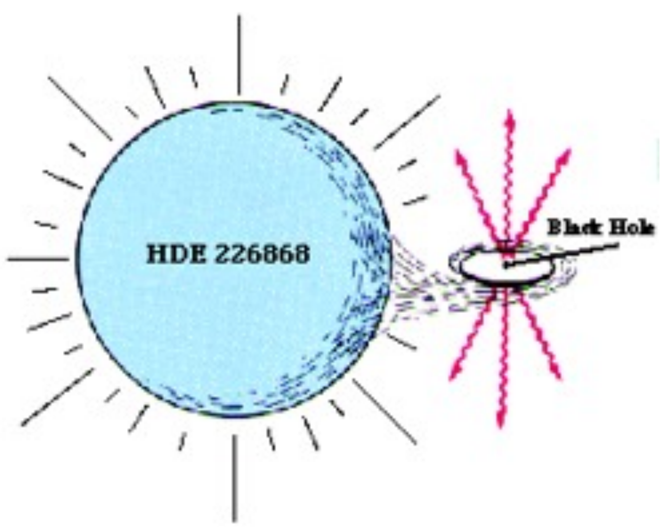
E. R. Harrison, *Cosmology*

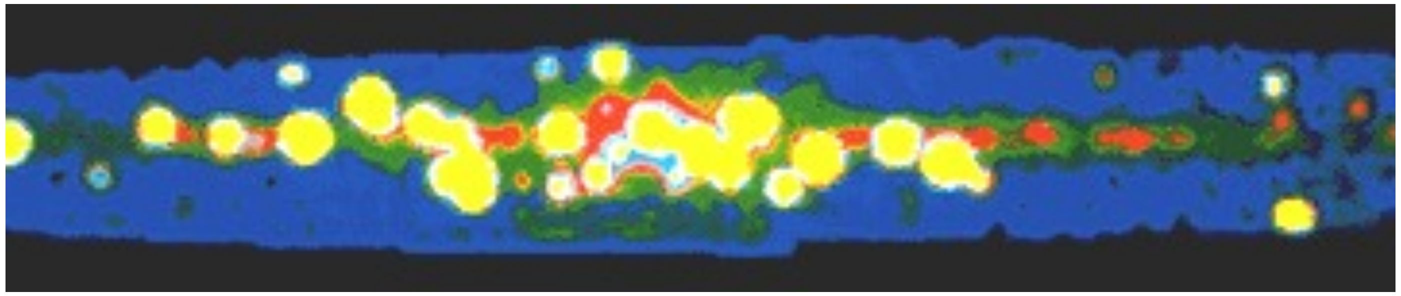
More About Black Holes

- Black Holes
  - Schwarzschild Radius, Photon Sphere, etc.
- Two types of BHs definitely exist:
  - Black Holes from Stellar Collapse
  - Supermassive Black Holes at the Centers of Galaxies
- Intermediate Mass Black Holes?
- Hawking Radiation from Black Holes (a quantum effect)
- Evaporating Black Holes?

BLACK HOLES FROM STELLAR COLLAPSE

If a large number of stars form, about 10% of the mass turns into a small number of stars more massive than 8 solar masses. Such high mass stars are rare, only about 0.2% of all stars. But they are at least 100,000 times as bright as the sun, and they fuse all the available fuel in their centers within a few million years. They then collapse into either neutron stars or black holes. If the black hole is a member of a binary star system, we can see its effects. Sometimes the black hole attracts matter to it from the other star, and as this matter (mostly hydrogen) falls into the black hole's event horizon it is heated tremendously and it radiates X-rays, which we can detect. One of these binary systems, called GRO J1655-40, was discovered in 2002 to be moving roughly toward us at about 110 kilometers per second. Such accreting black holes acquire angular momentum from the accreted material, so they should be spinning. Evidence that some black holes spin has been found in "quasi-periodic oscillations" of their X-rays at frequencies too high to come from non-spinning black holes.



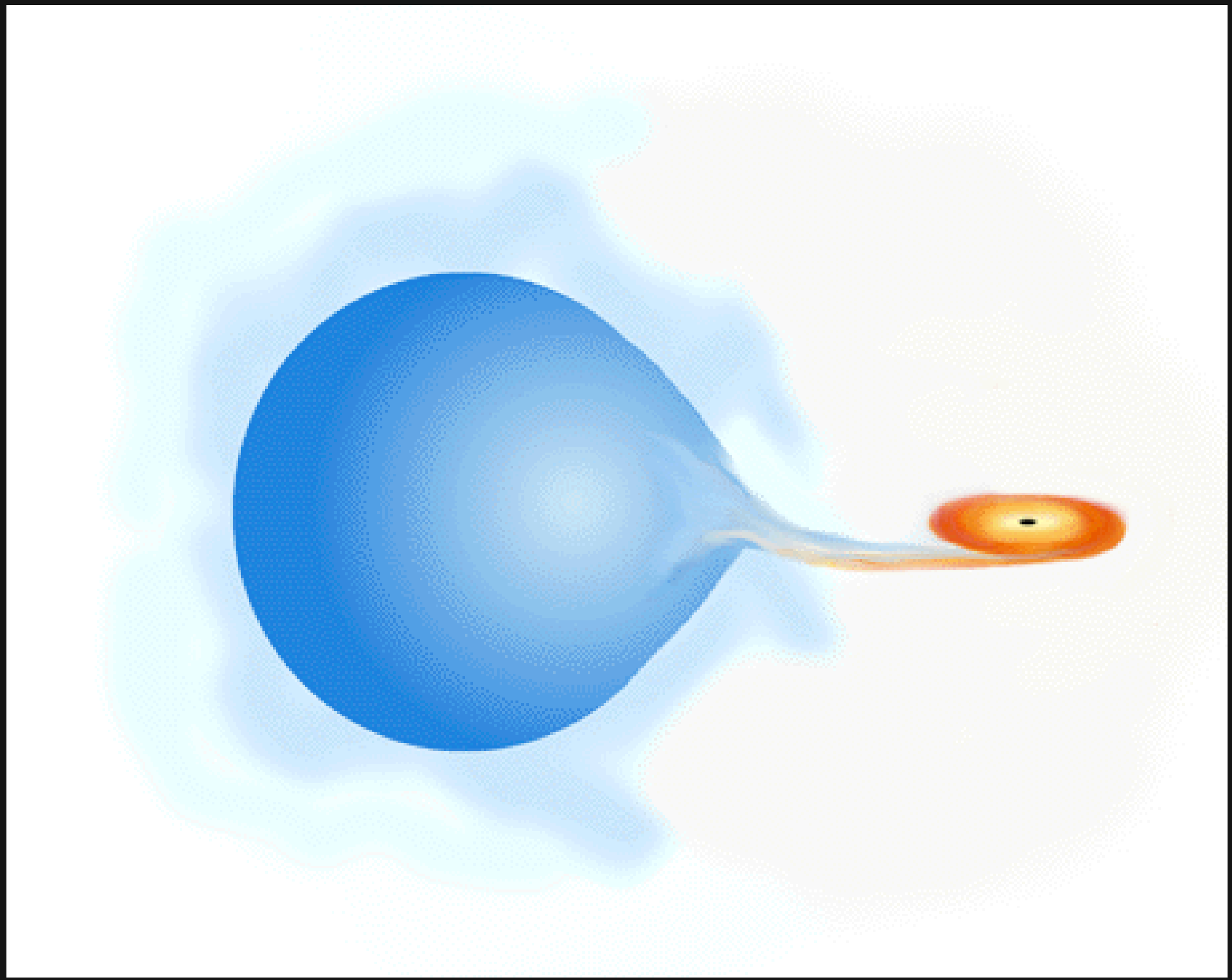


X-ray Binaries (in yellow) near the Galactic Center

Interesting links: Fantasy trip to and around a black hole - http://antwrp.gsfc.nasa.gov/htmltest/rjn_bht.html
See also http://ircamera.as.arizona.edu/NatSci102/lectures/blackhole.htm

Black Holes in Binary Star Systems

- Black holes are often part of a binary star system - two stars revolving around each other.
- What we see from Earth is a visible star orbiting around what appears to be nothing.
- We can infer the mass of the black hole by the way the visible star is orbiting around it.
- The larger the black hole, the greater the gravitational pull, and the greater the effect on the visible star.

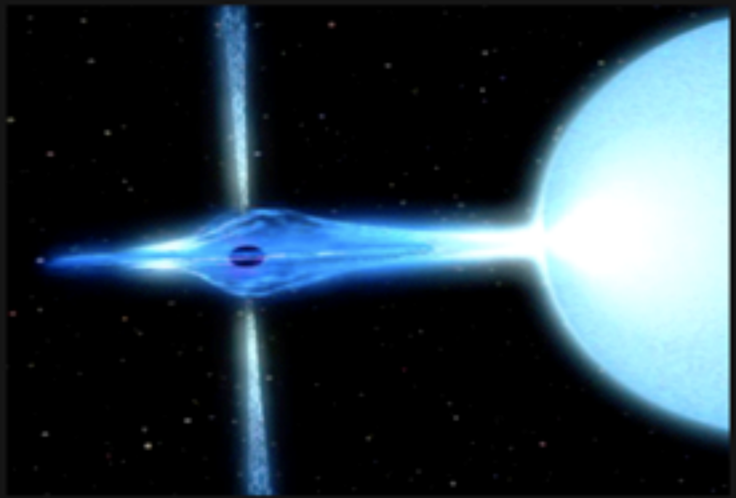


Chandra illustration

BLACK HOLES FROM STELLAR COLLAPSE

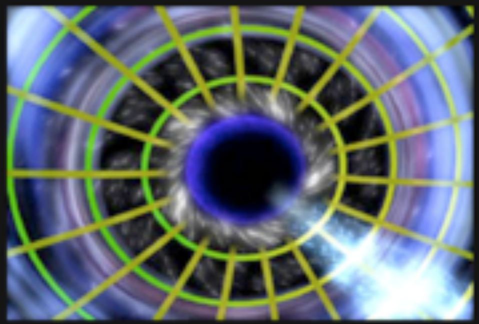
X-rays from Black Holes

In close binary systems, material flows from normal star to black hole. X-rays are emitted from disk of hot gas swirling around the black hole.

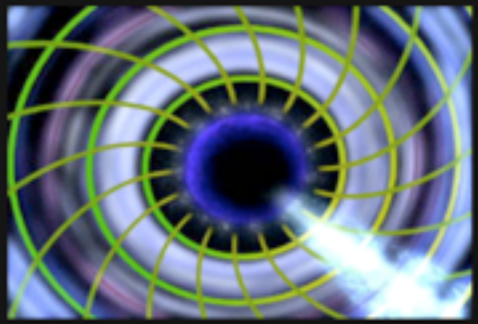


X-ray: A Rotating Black Hole

We expect everything in the Universe to rotate. Non-rotating black holes are different from rotating ones.



Non-rotating black hole

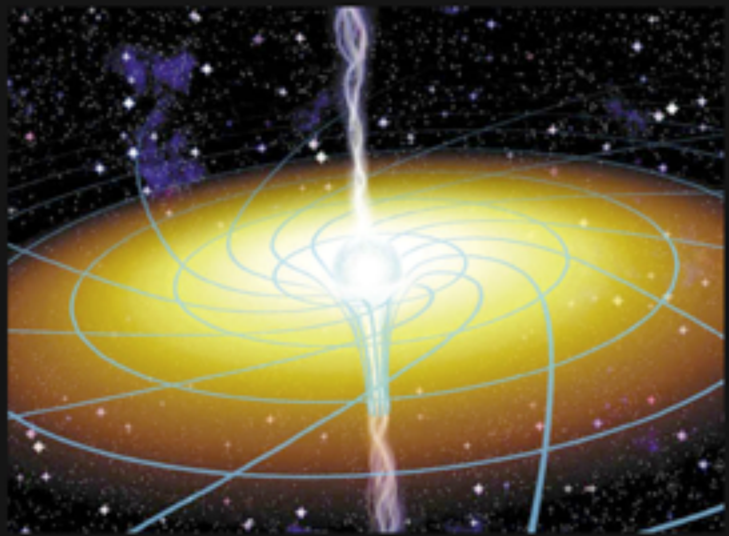


Rotating black hole

In GRO J1655-40, a 2.2 ms period was discovered. This implies an orbit that is too small to be around a non-rotating black hole. This means the black hole is rotating.

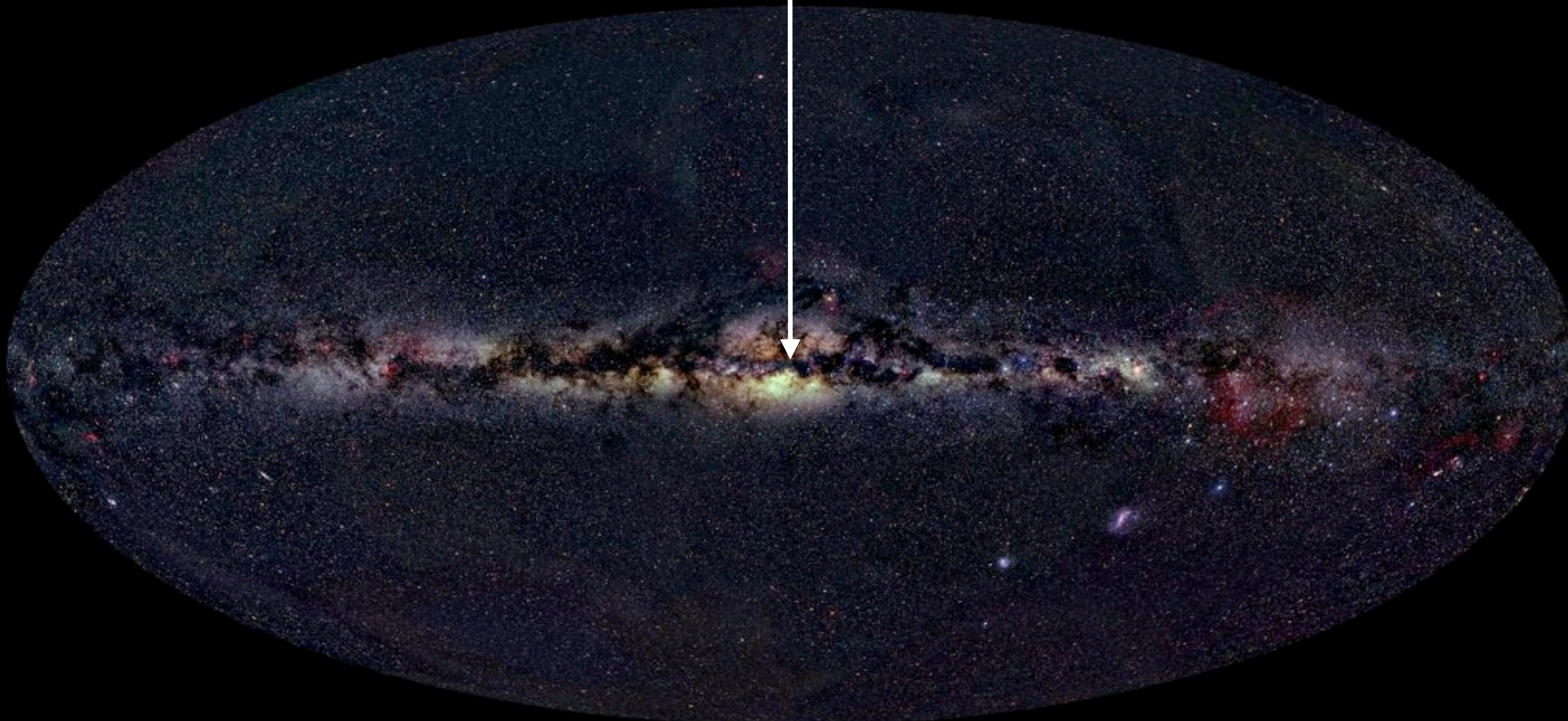
X-ray: Frame Dragging

- Detection of a period in GRO J1655-40 due to precession of the disk.
- This precession period matches that expected for frame dragging of space-time around the black hole.



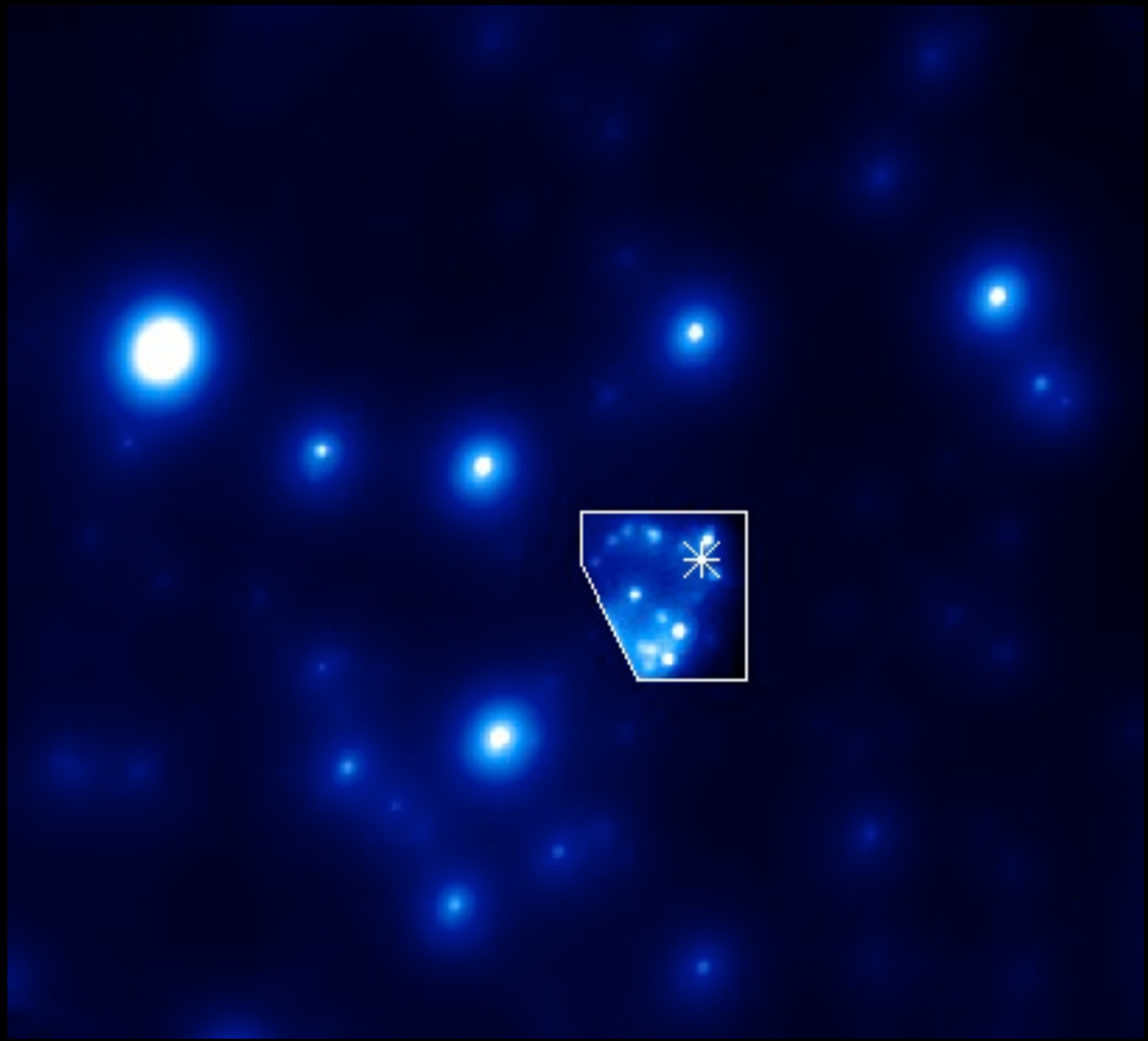
Credit: J. Bergeron, Sky & Telescope Magazine

The center of our galaxy



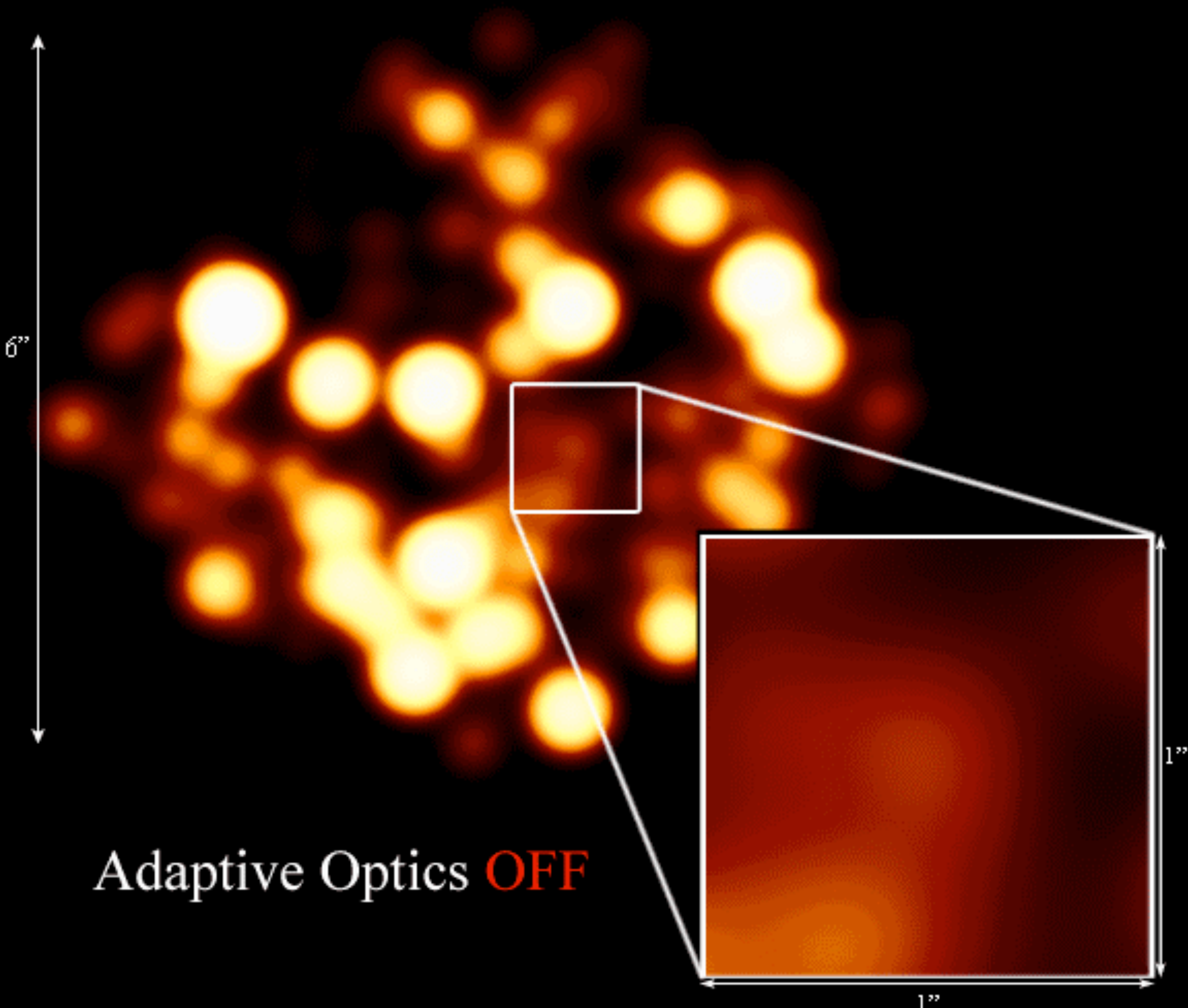
There's a supermassive black hole at the center of our galaxy...

- Modern large telescopes can track individual stars at galactic center
- Need infrared (to penetrate dust)
- Need very good resolution -- use adaptive optics



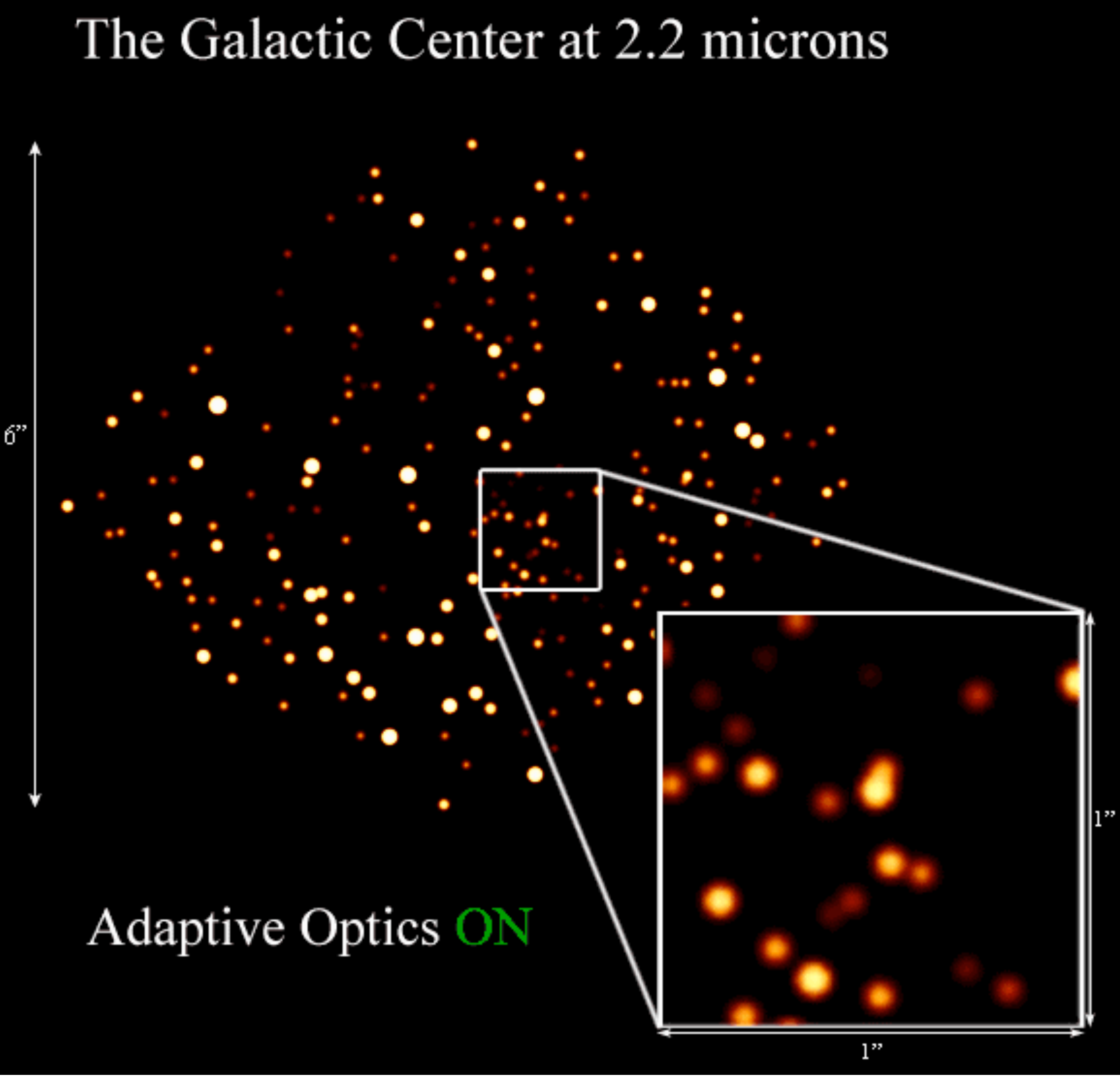
Keck, 2 μm *Ghez, et al.*

The Galactic Center at 2.2 microns

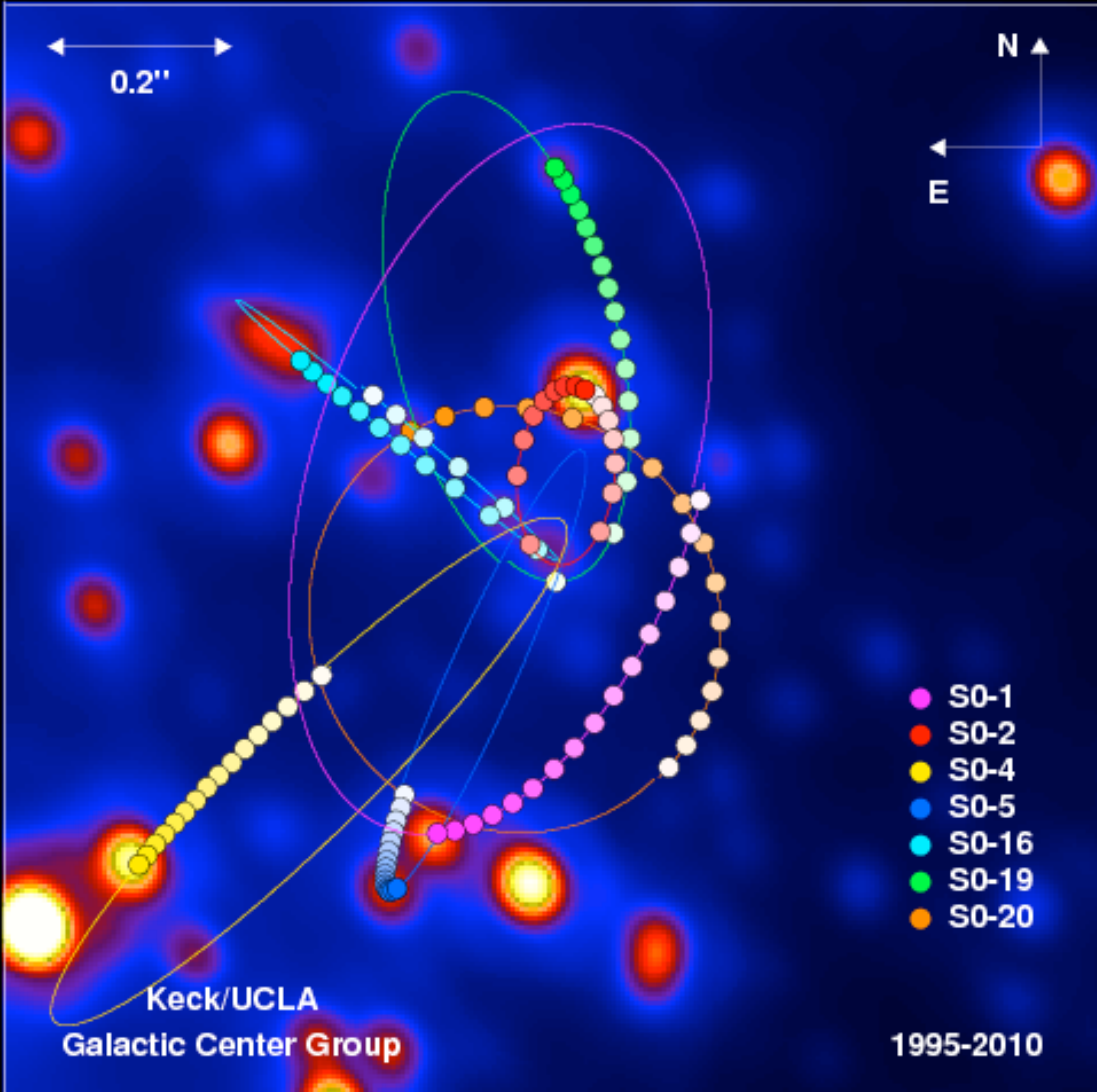


Adaptive Optics OFF

Ghez, et al.

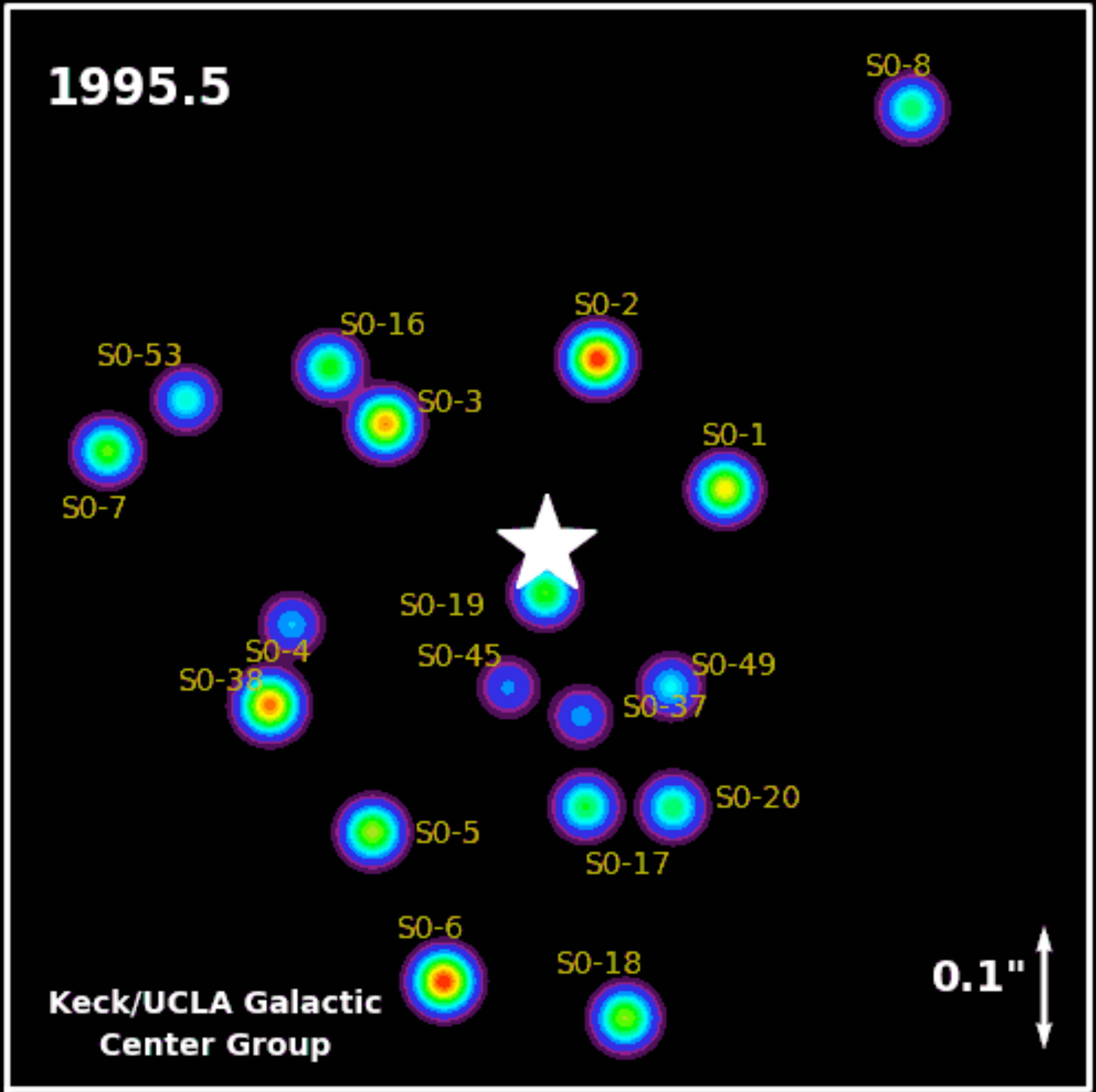


Ghez, et al.

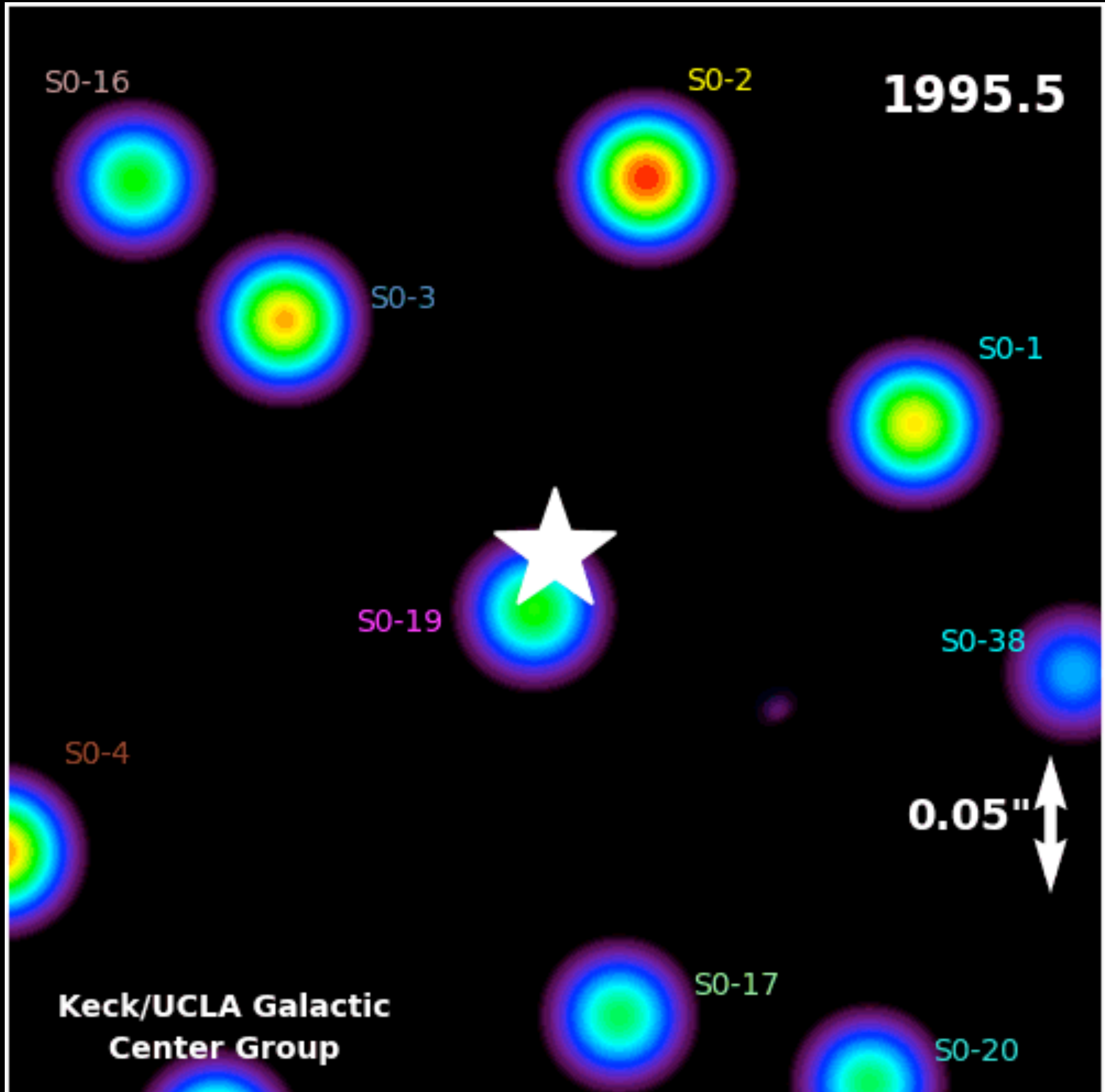


Motions of stars consistent with large, dark mass located at Sgr A...*

Ghez, et al.

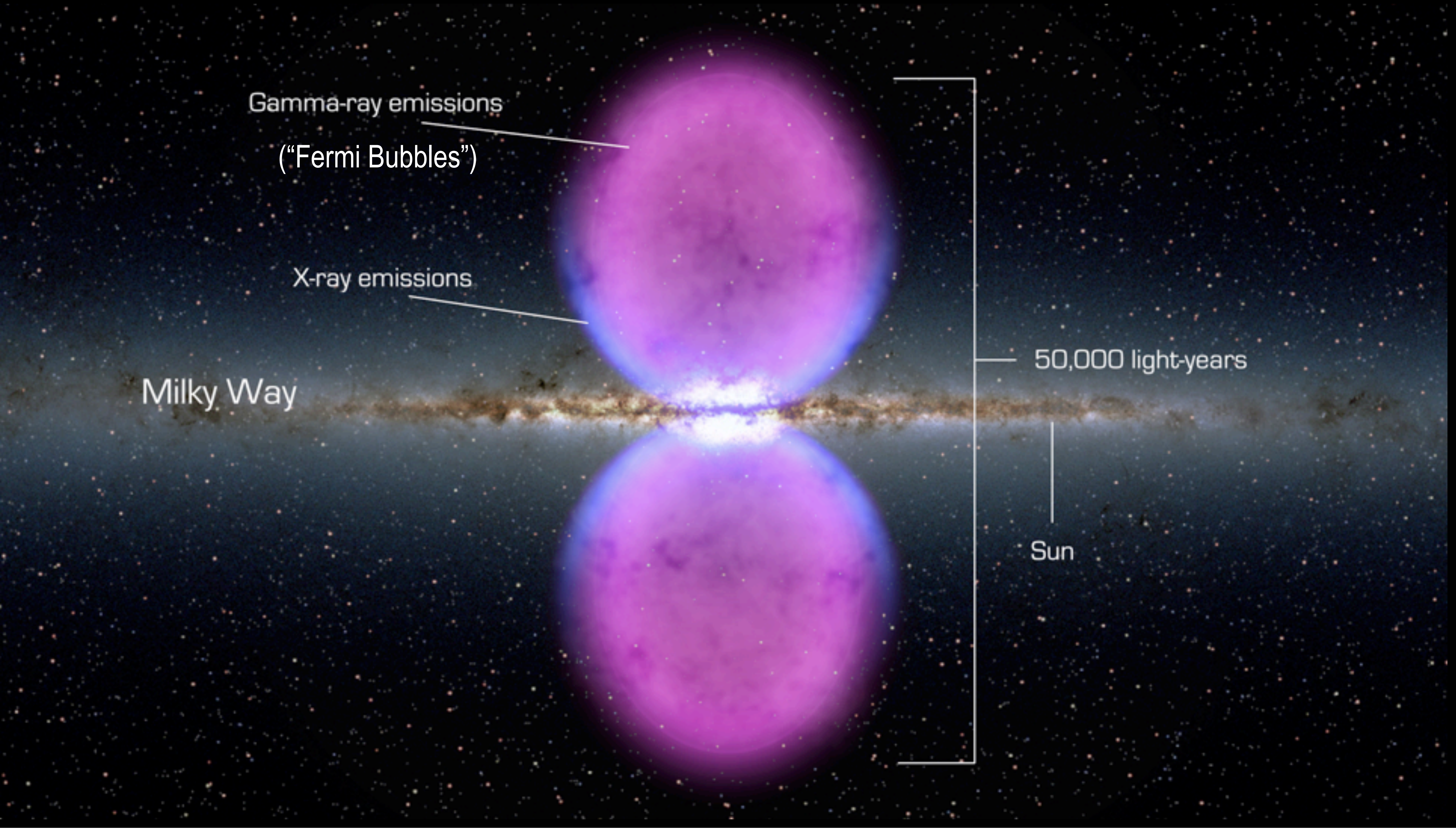


A 2.2 micron animation of the stellar orbits in the central parsec. Images taken from the years 1995 through 2011 are used to track specific stars orbiting the proposed black hole at the center of the Galaxy. These orbits, and a simple application of Kepler's Laws, provide the best evidence yet for a supermassive black hole, which has a mass of 4 million times the mass of the Sun. Especially important are the stars S0-2, which has an orbital period of only 15.78 years, and S0-16, which comes a mere 90 astronomical units from the black hole. -- UCLA Prof. Andrea Ghez

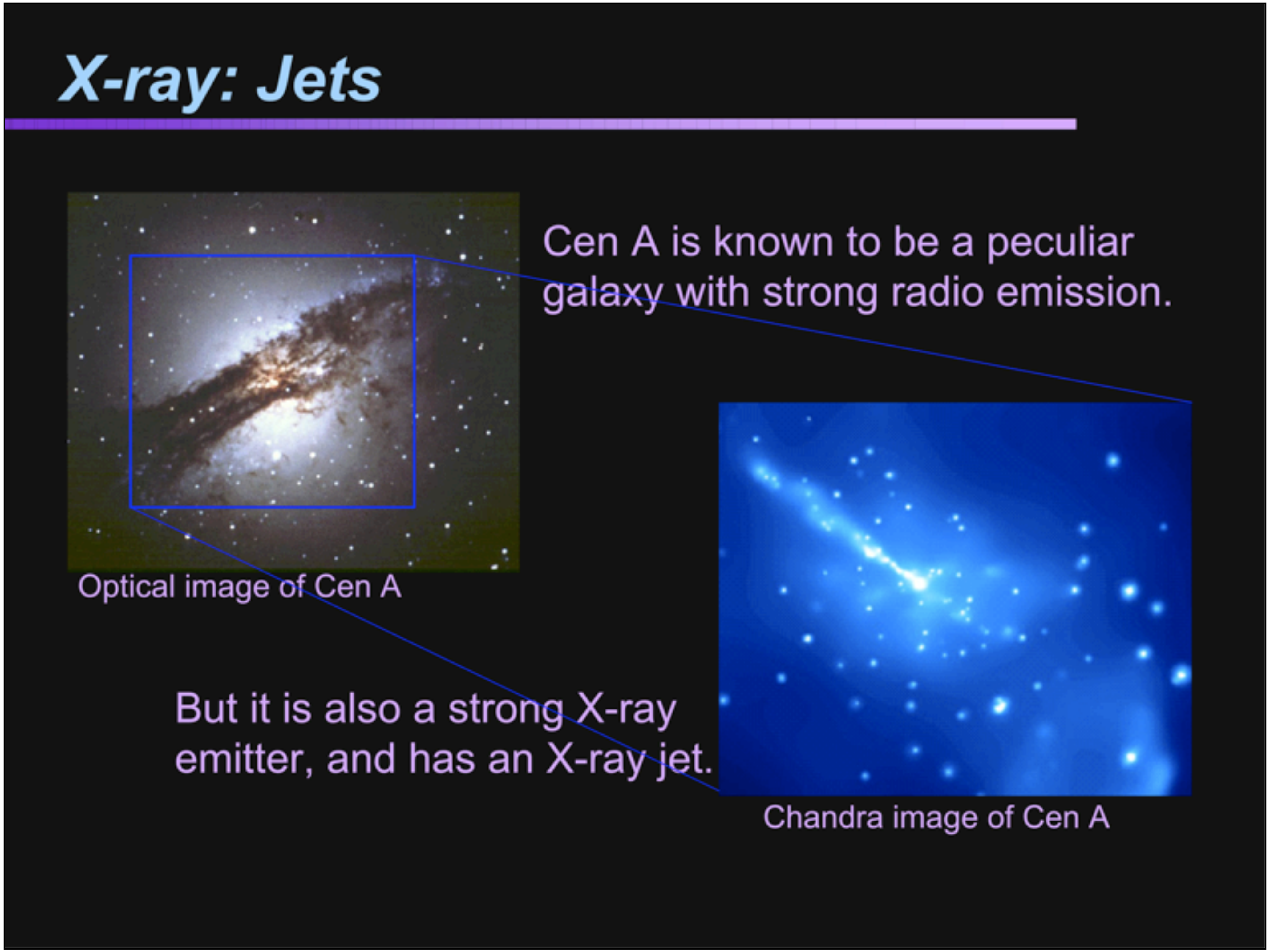


S0-16
S0-2
1995.5
S0-3
S0-1
S0-19
S0-38
S0-4
0.05"
S0-17
Keck/UCLA Galactic Center Group
S0-20

- **The central object at the center of the Milky Way is…**
  - Very massive (~4 million solar masses).
  - Must be very compact (star S0-2 gets within 17 light hours of the center).
  - Now seen to flare in X-rays and IR, in the past in Gamma rays.

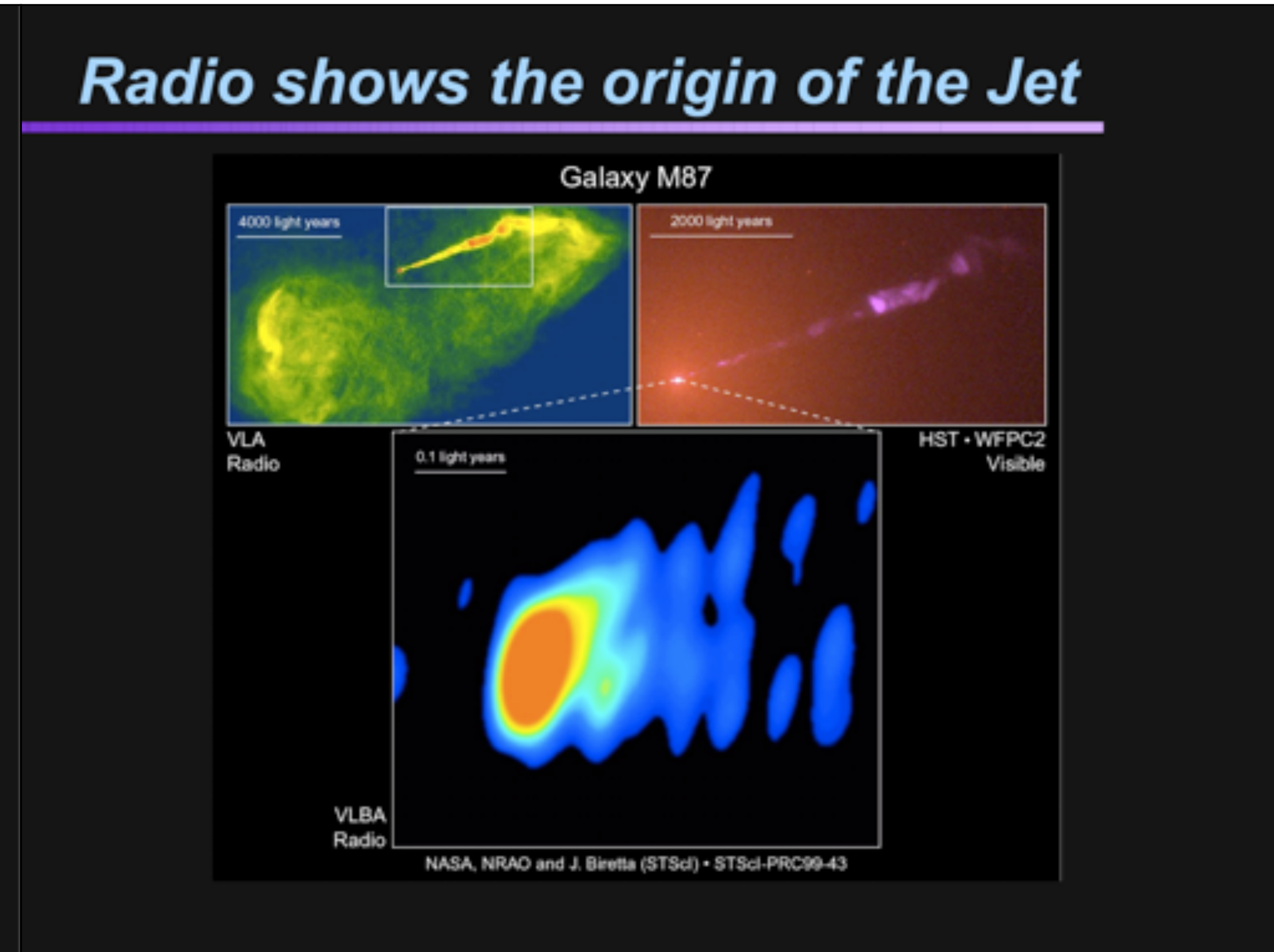


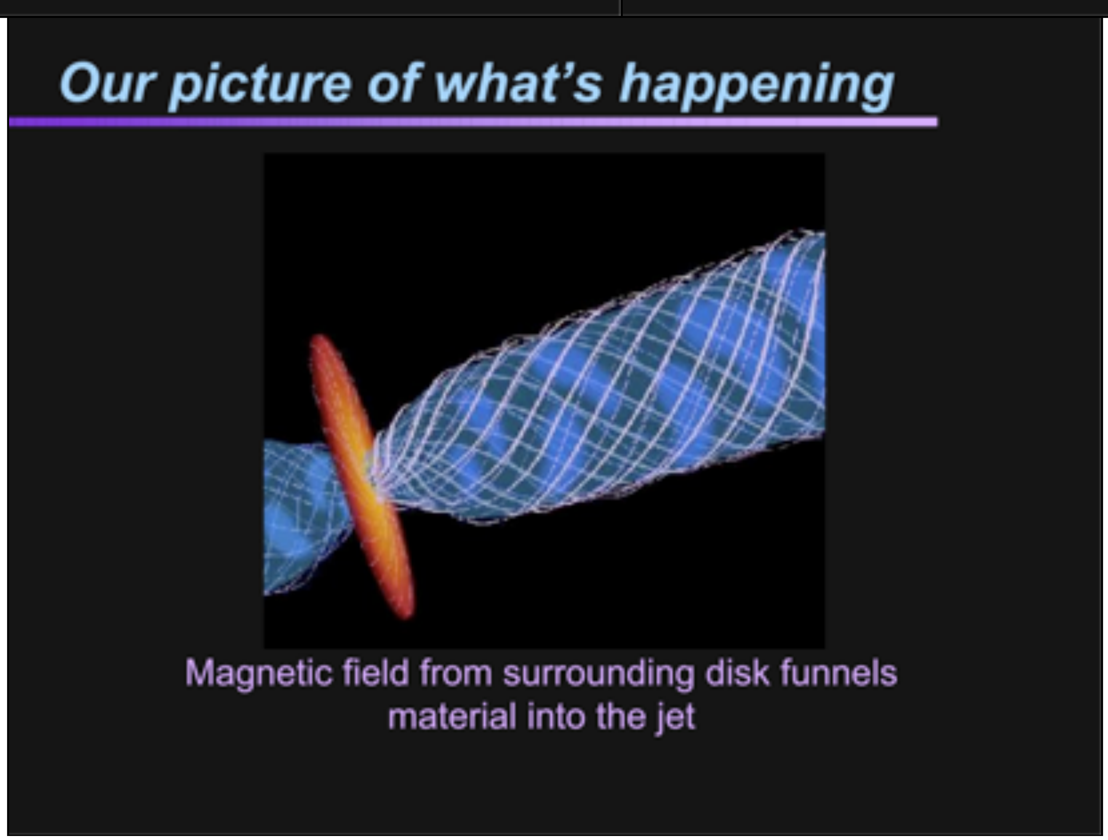
BLACK HOLES AT CENTERS OF GALAXIES



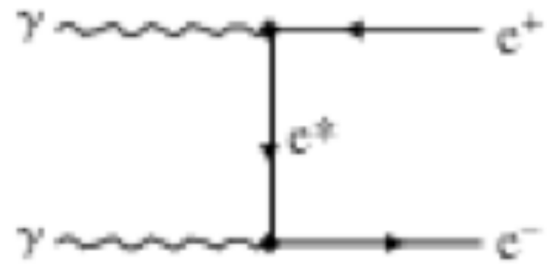
The mass of the black holes in galaxy centers is about 1/1000 the mass of the central spheroid of stars.

BLACK HOLES AT CENTERS OF GALAXIES



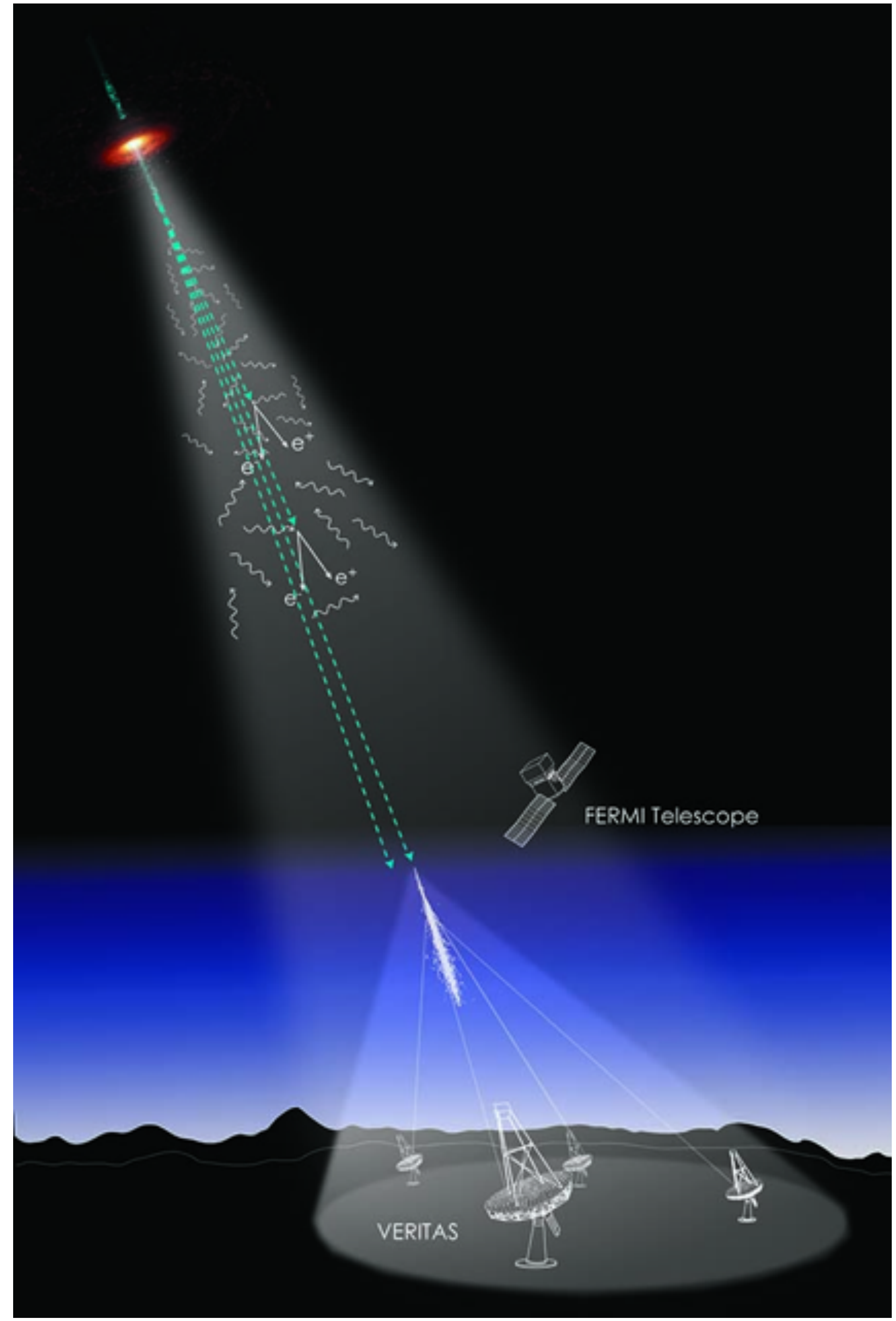


$\gamma\gamma \longrightarrow e^+e^-$



$$\sigma(\gamma\gamma \to e^+e^-) = \left(\frac{4\pi\alpha^2}{s}\right)\left[\ln\left(\frac{s}{m^2}\right) - 1\right]$$

The energy threshold for this process is $s_{th} = 4m_e^2$, and $\sigma \sim \beta\alpha^2/s$, where $\beta = (1 - 4m_e^2/s)^{1/2}$ is the CM velocity of the produced electron and positron. This process is the main way that high energy gamma rays from blazars are removed on their way to us, by interacting with low-energy photons ("extragalactic background light," EBL) radiated as starlight or as radiation from cool dust, and producing e^+e^- pairs.

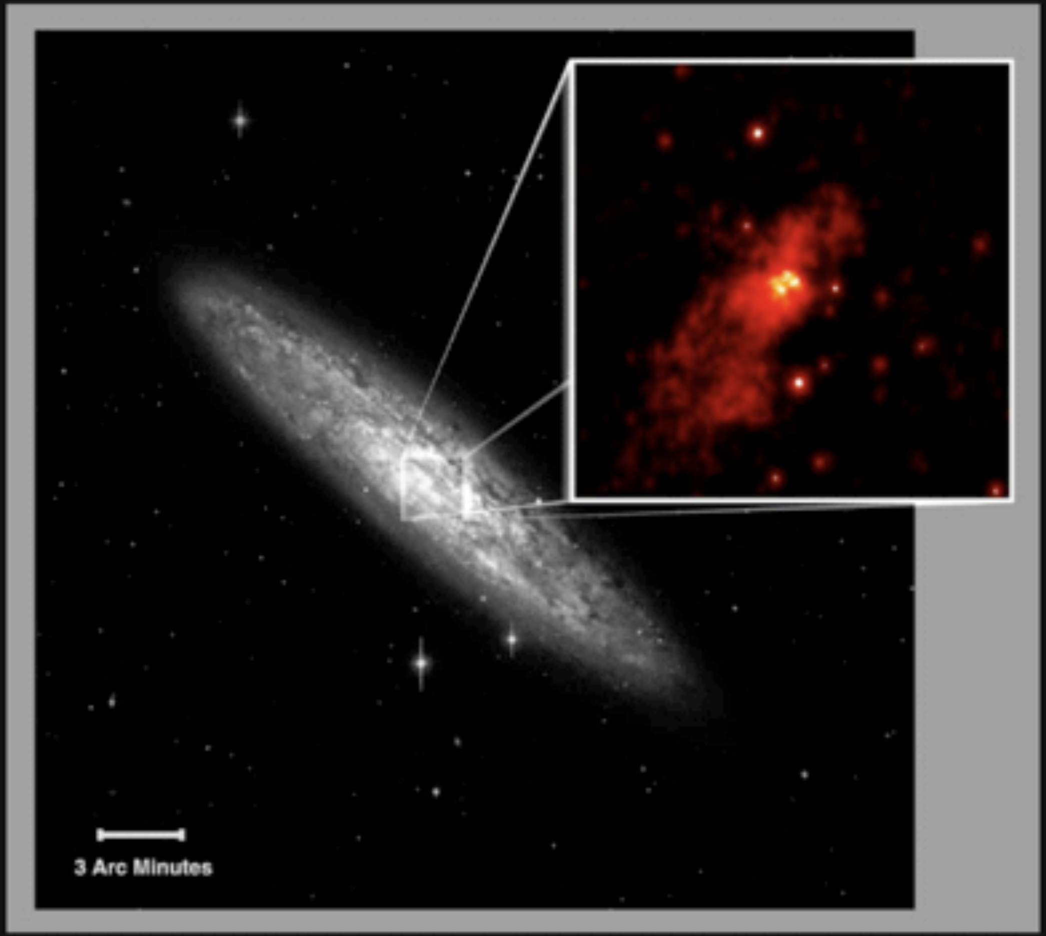


Energetic gamma rays (dashed lines) from a distant blazar strike photons of extragalactic background light (wavy lines) in intergalactic space, annihilating both gamma ray and photon. Different energies of EBL photons waylay different energies of gamma rays, so comparing the attenuation of gamma rays at different energies from different spacecraft and ground-based instruments indirectly measures the spectrum of EBL photons.

INTERMEDIATE MASS BLACK HOLES

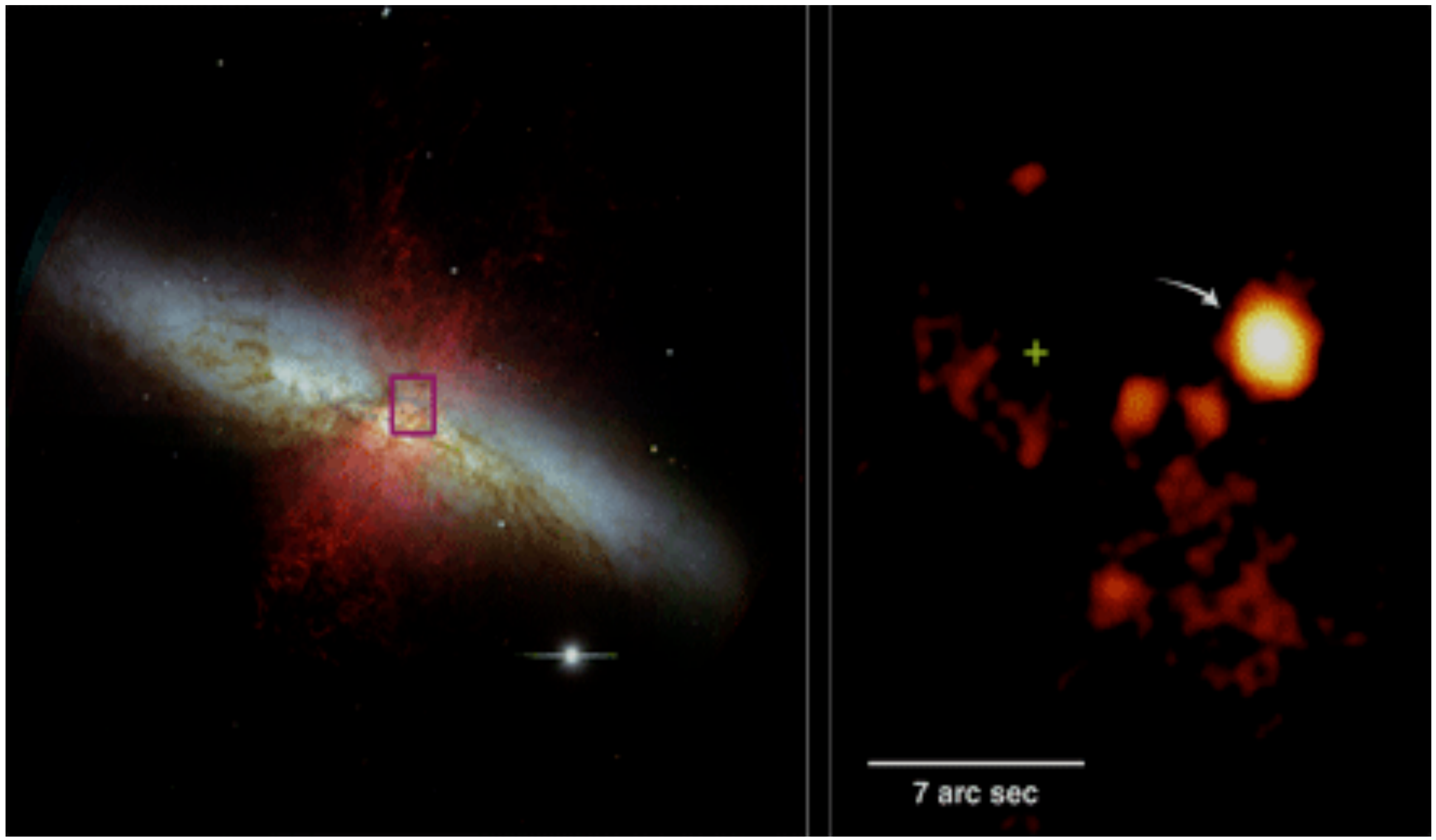
X-ray: Mid mass black holes

- Black Holes with masses a few hundred to a few thousand times the mass of the sun have been found outside the central regions of a number of galaxies.
- Often found in Starburst galaxies.
- May be precursors to Active Galaxies.



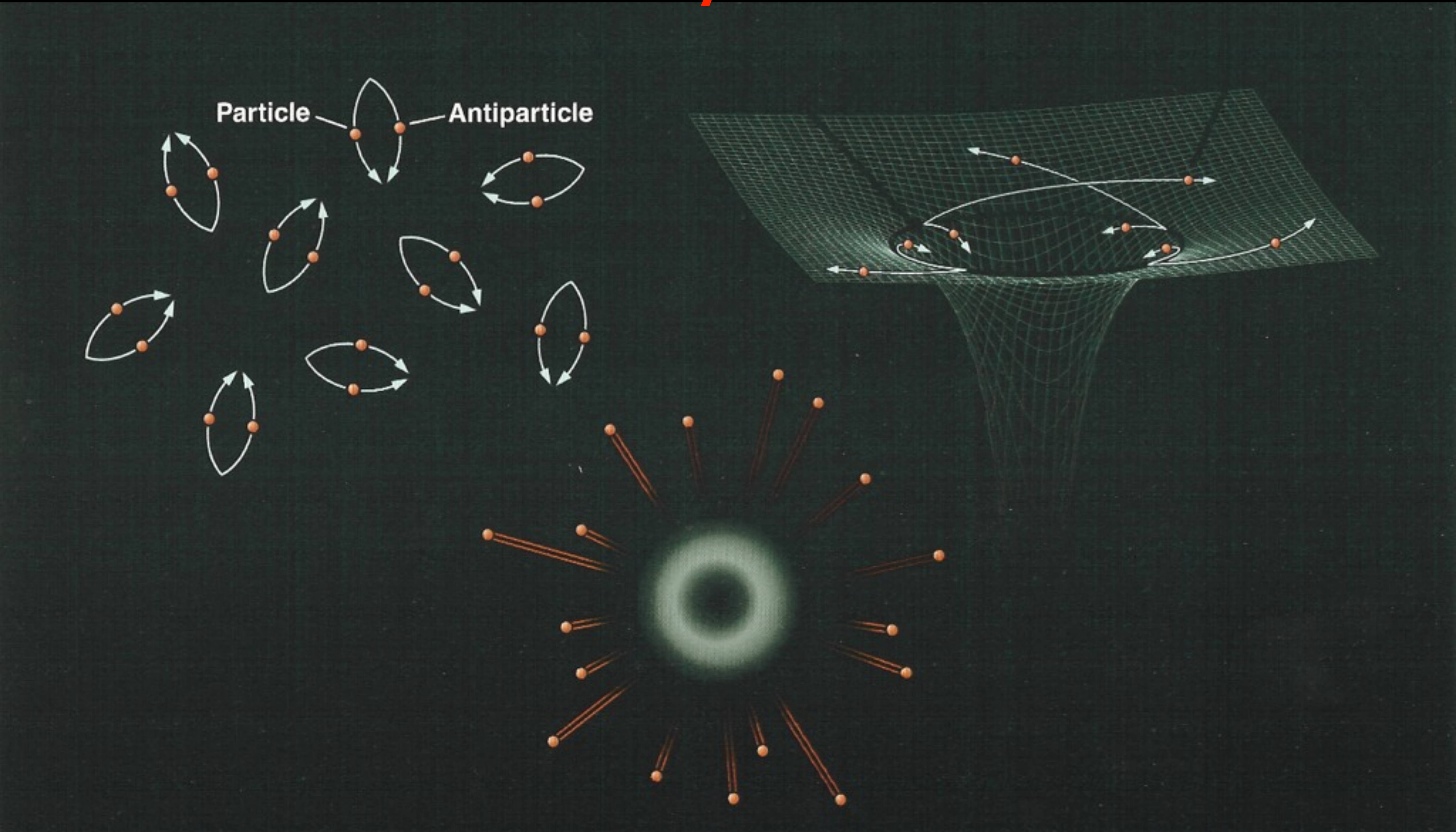
Optical and X-ray images of NGC 253

INTERMEDIATE MASS BLACK HOLES



The best candidate for an intermediate-mass black hole. Optical (**left**) and Chandra x-ray (**right**) images of the M82 galaxy. The arrow points to the location of the ultraluminous x-ray source that is likely to be an intermediate-mass black hole. The area covered by the right image lies within the rectangle at the center of the left image. The green cross (right image) is the galaxy nucleus. CREDIT: LEFT PANEL: SUBARU TELESCOPE, NAO JAPAN; RIGHT PANEL: NASA/SAO/CXC

HAWKING RADIATION from tiny black holes



HAWKING RADIATION

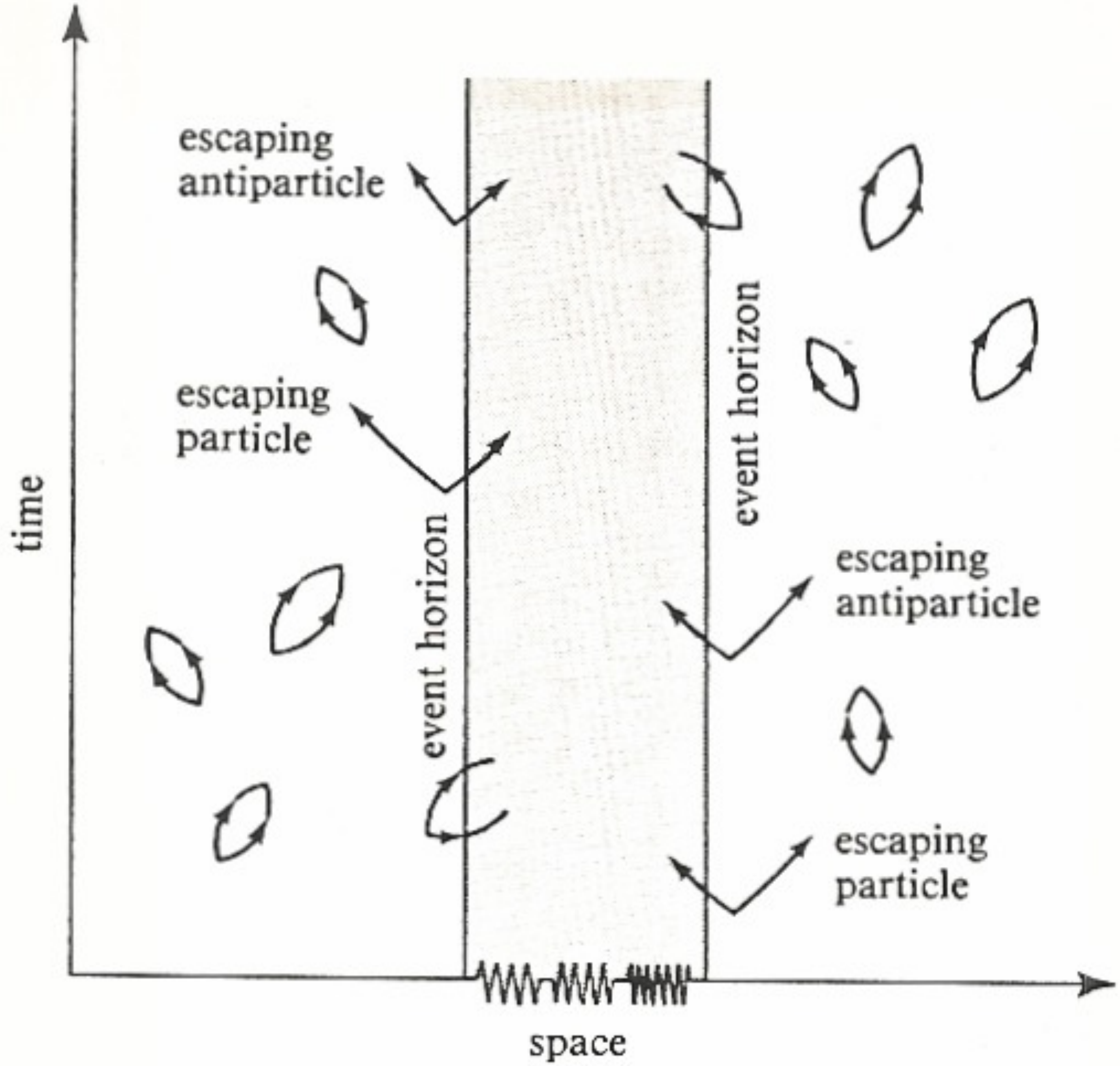
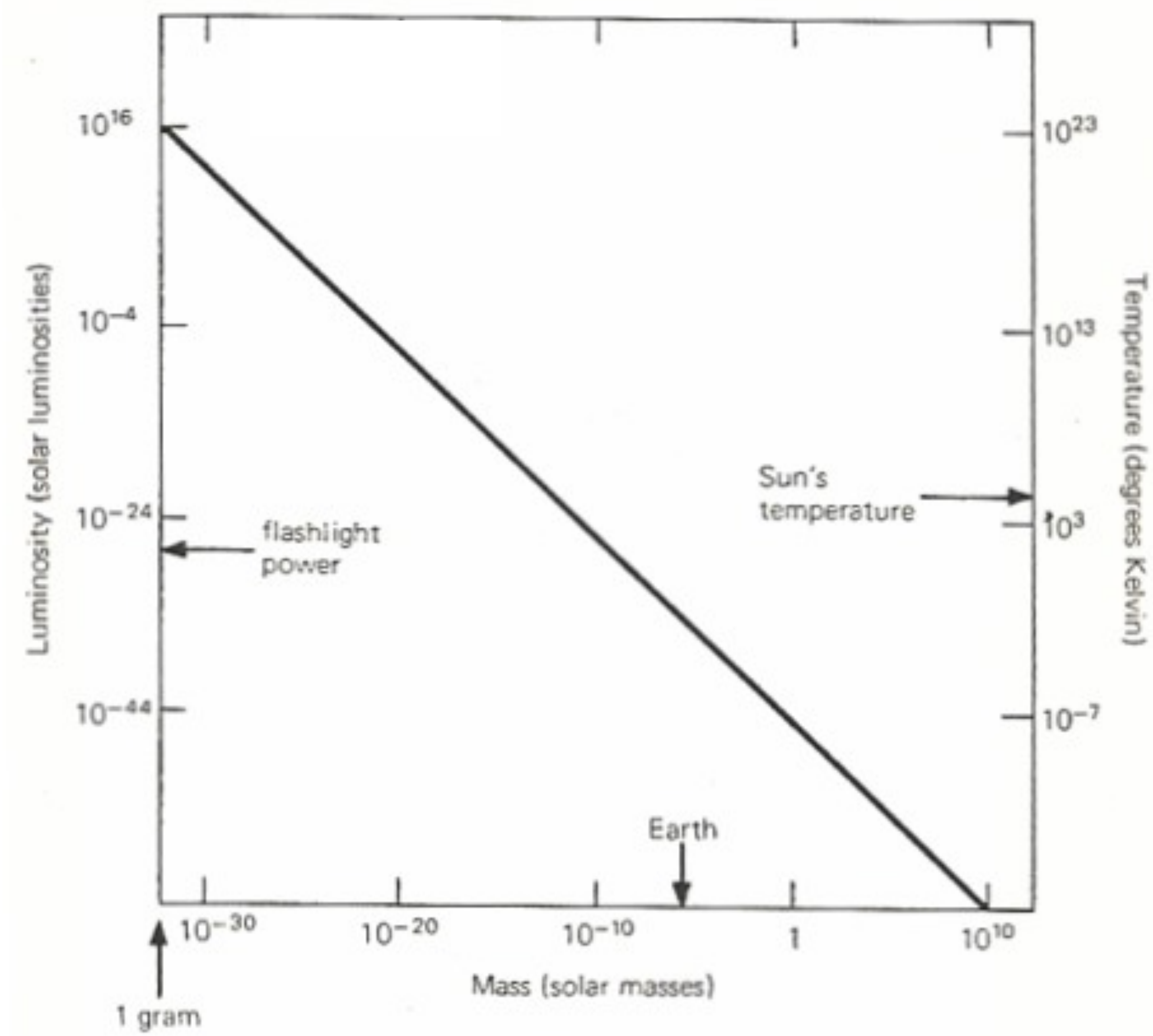


Figure 7.13. Hawking process near the event horizon of a black hole. The creation of particle-antiparticle pairs usually ends with their destruction a short time later. Occasionally, however, the tidal forces of the black hole can cause one of the pair to fall into the black hole, enabling its liberated partner to "tunnel" safely to infinity and escape. Notice the assumption that the black hole is (quasi) stationary implies that the two sides of the event horizon remain fixed in space. Actually, this diagram is even more schematic than usual, because the actual spacetime near a black hole is highly curved.

F. Shu, *The Physical Universe*



Luminosity and temperature of a black hole of given mass.

E. R. Harrison, *Cosmology*

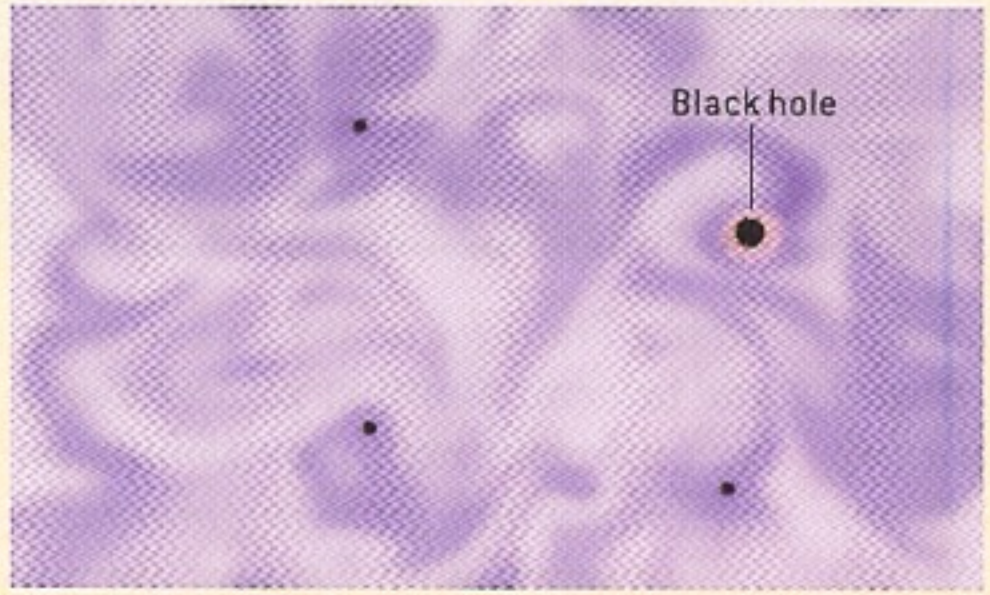
$$T_{BH} = 10^{-7}\ M_{sun}/M$$

$$T_{evap} = 10^{62}\ (M_{sun}/M)^3\ \text{yr}$$

Making Evaporating Black Holes?

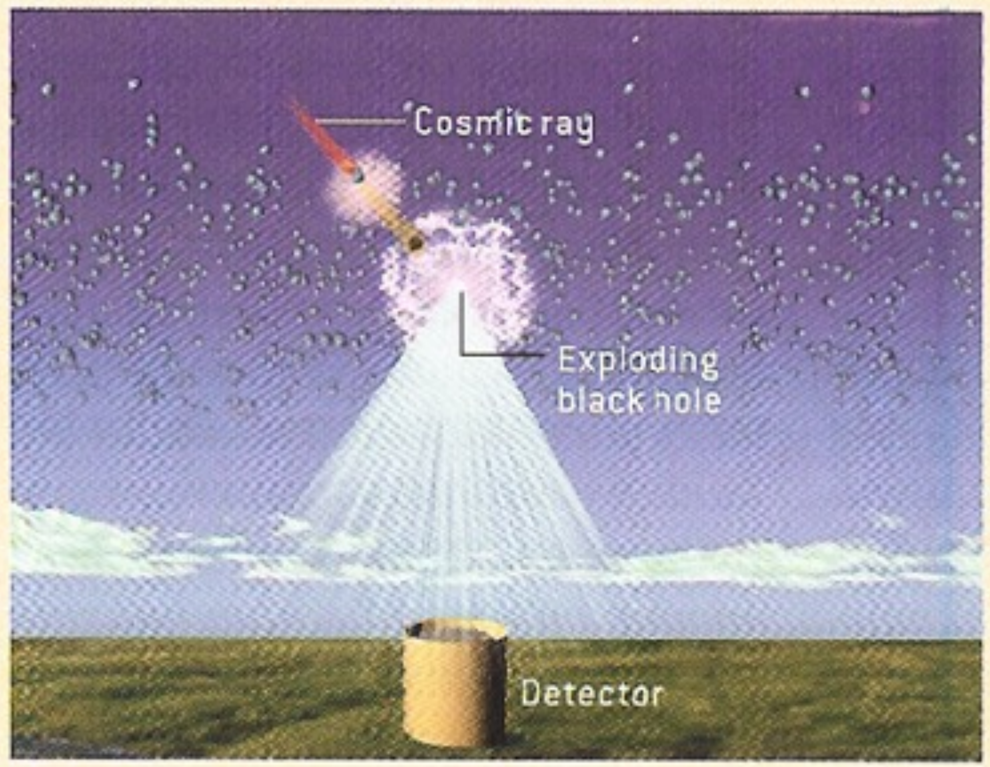
See “Quantum Black Holes” by Steve Giddings and Bernard Carr in the May 2005 *Scientific American*

WAYS TO MAKE A MINI BLACK HOLE



PRIMORDIAL DENSITY FLUCTUATIONS

Early in the history of our universe, space was filled with hot, dense plasma. The density varied from place to place, and in locations where the relative density was sufficiently high, the plasma could collapse into a black hole.



COSMIC-RAY COLLISIONS

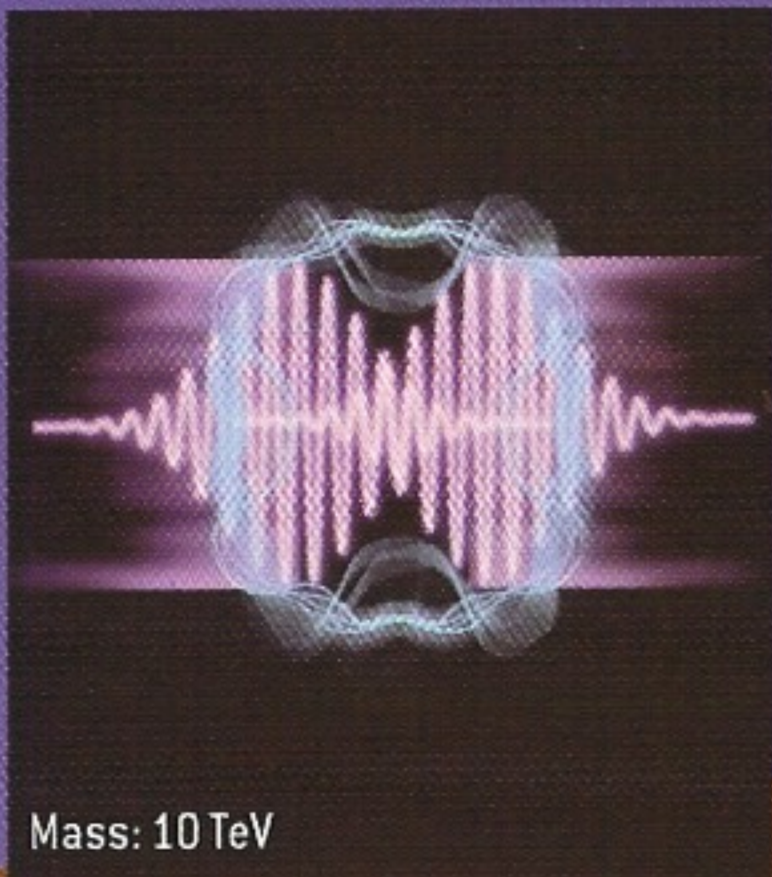
Cosmic rays—highly energetic particles from celestial sources—could smack into Earth's atmosphere and form black holes. They would explode in a shower of radiation and secondary particles that could be detected on the ground.

PARTICLE ACCELERATOR

An accelerator such as the LHC could crash two particles together at such an energy that they would collapse into a black hole. Detectors would register the subsequent decay of the hole.

THE RISE AND DEMISE OF A QUANTUM BLACK HOLE

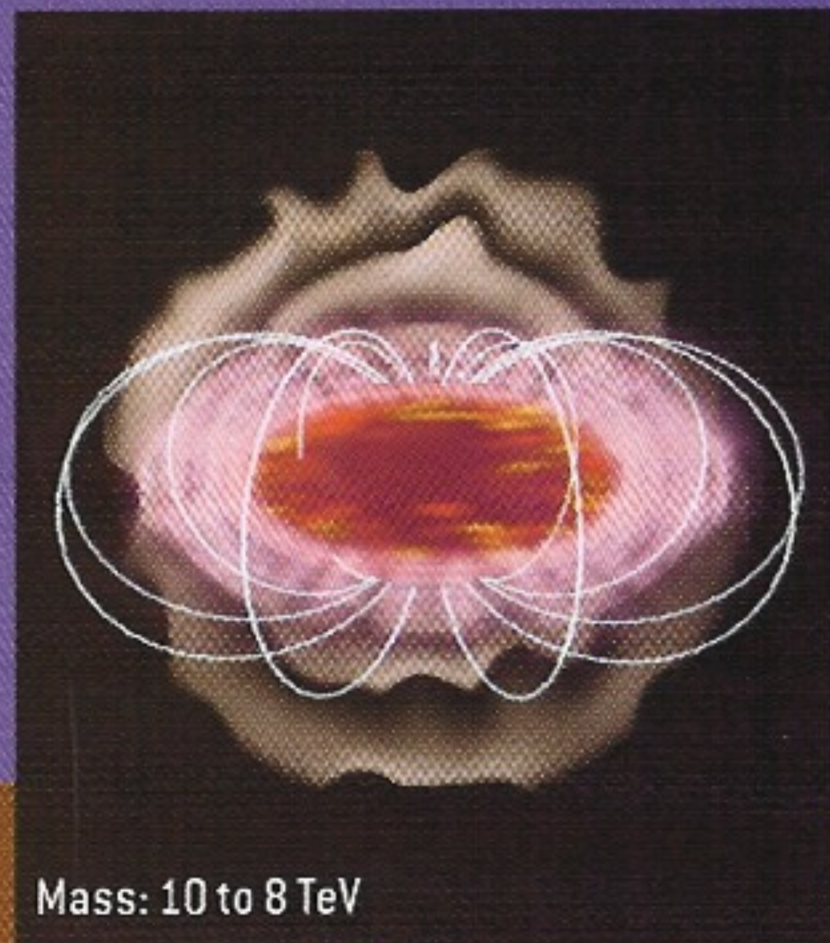
BIRTH



TIME 0

If conditions are right, two particles (shown here as wave packets) can collide to create a black hole. The newborn hole is asymmetrical. It can be rotating, vibrating and electrically charged. (Times and masses are approximate; 1 TeV is the energy equivalent of about 10^{-24} kilogram.)

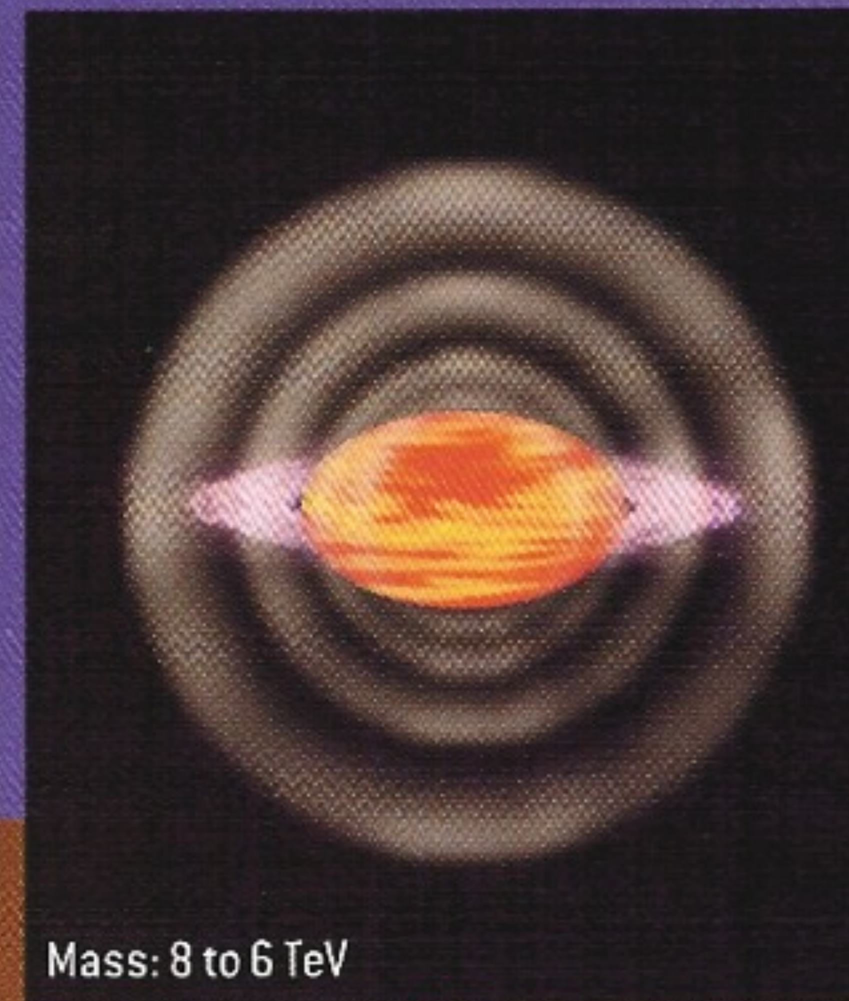
BALDING PHASE



0 to 1×10^{-27} second

As it settles down, the black hole emits gravitational and electromagnetic waves. To paraphrase physicist John A. Wheeler, the hole loses its hair—it becomes an almost featureless body, characterized solely by charge, spin and mass. Even the charge quickly leaks away as the hole gives off charged particles.

SPIN-DOWN PHASE



1 to 3×10^{-27} second

The black hole is no longer black: it radiates. At first, the emission comes at the expense of spin, so the hole slows down and relaxes into a spherical shape. The radiation emerges mainly along the equatorial plane of the black hole.

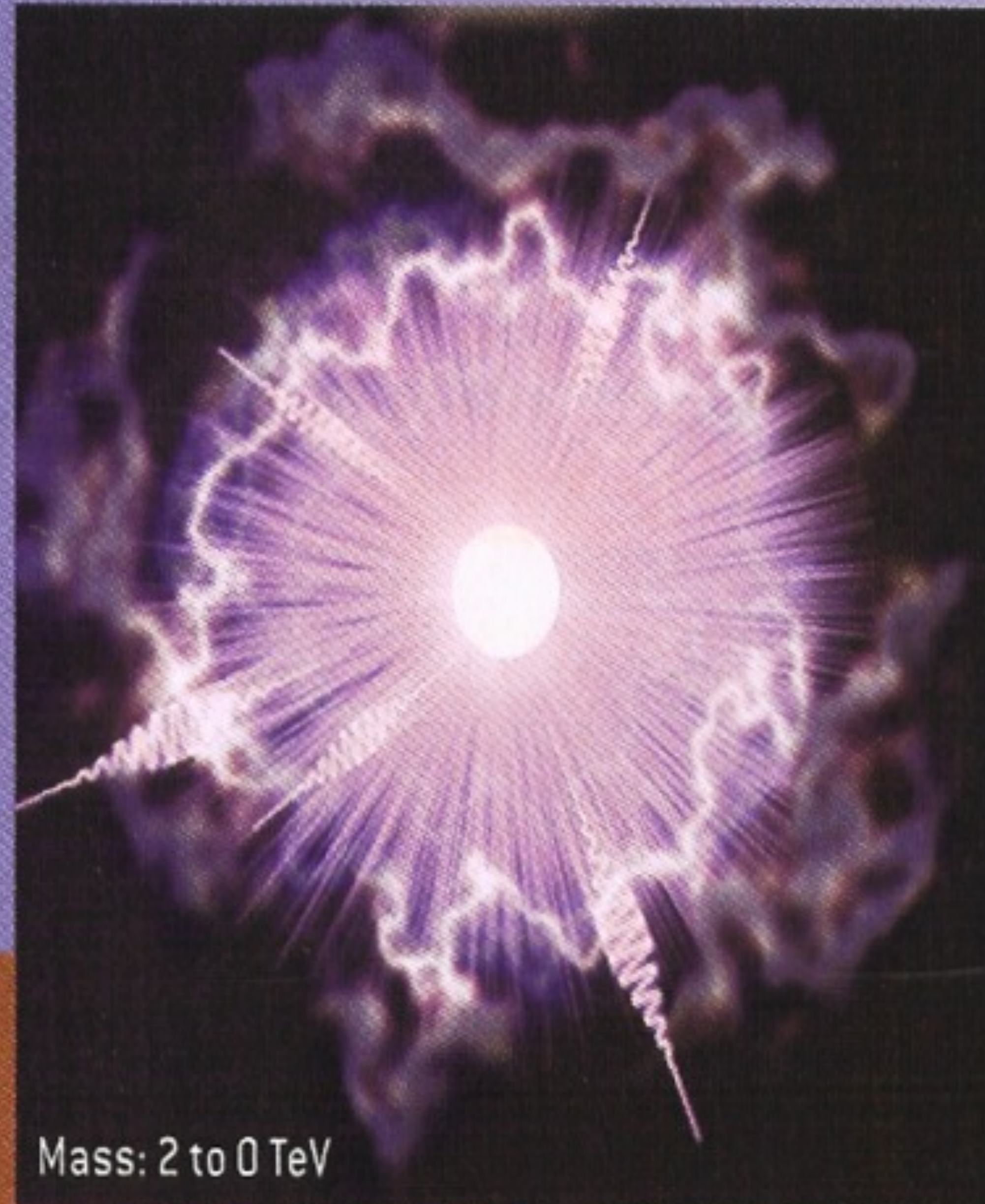
SCHWARZSCHILD PHASE

Mass: 6 to 2 TeV

3 to 20 × 10^{-27} second

Having lost its spin, the black hole is now an even simpler body than before, characterized solely by mass. Even the mass leaks away in the form of radiation and massive particles, which emerge in every direction.

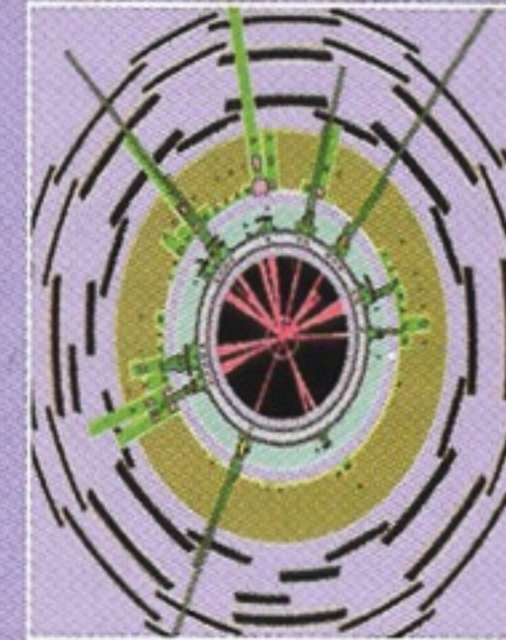
PLANCK PHASE



Mass: 2 to 0 TeV

20 to 22 × 10^{-27} second

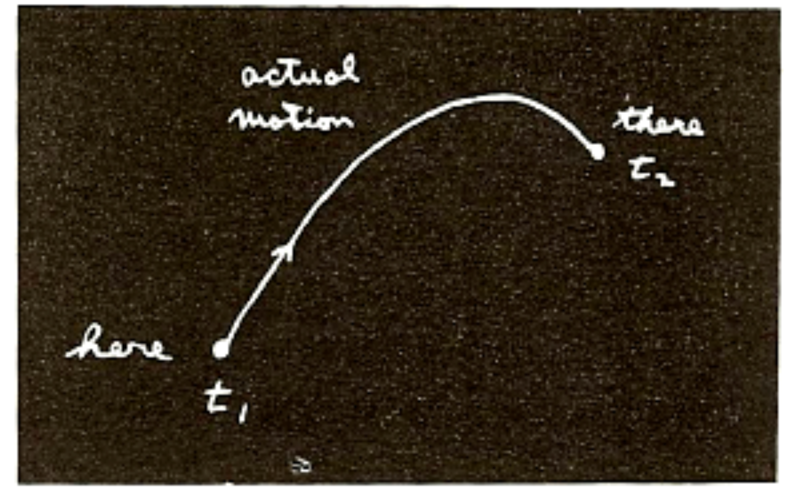
The hole approaches the Planck mass—the lowest mass possible for a hole, according to present theory—and winks into nothingness. String theory suggests that the hole would begin to emit strings, the most fundamental units of matter.

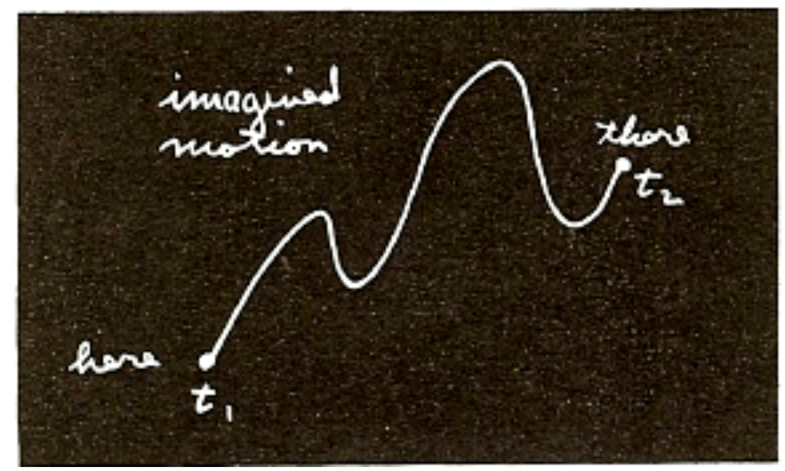


SIMULATED DECAY of a black hole shows a particle accelerator and detector in cross section. From the center of the accelerator pipe (*black circle*) emerge particles (*spokes*) registered by layers of detectors (*concentric colored rings*).

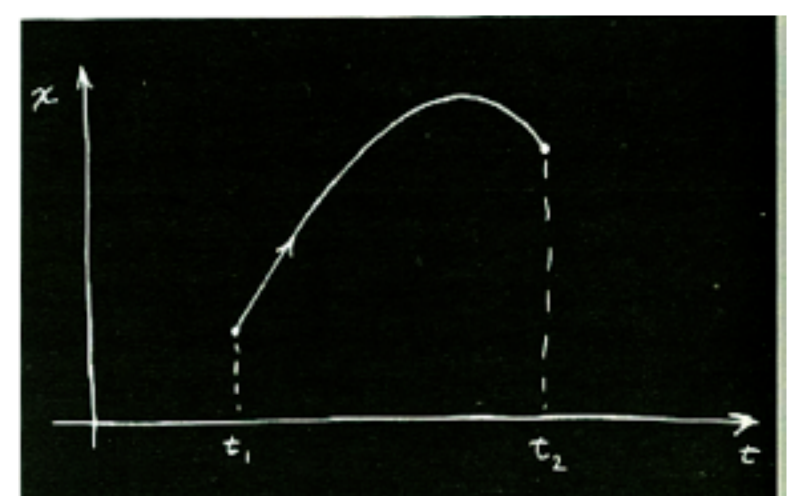
Symmetries in Particle Physics

I next explain how the laws of classical mechanics follow from extremizing the Action, and I briefly mention how this generalizes to quantum mechanics: the path integral formulation of quantum theory. I follow the *Feynman Lectures on Physics*, Vol. 2, Chapter 19: "The Principle of Least Action."





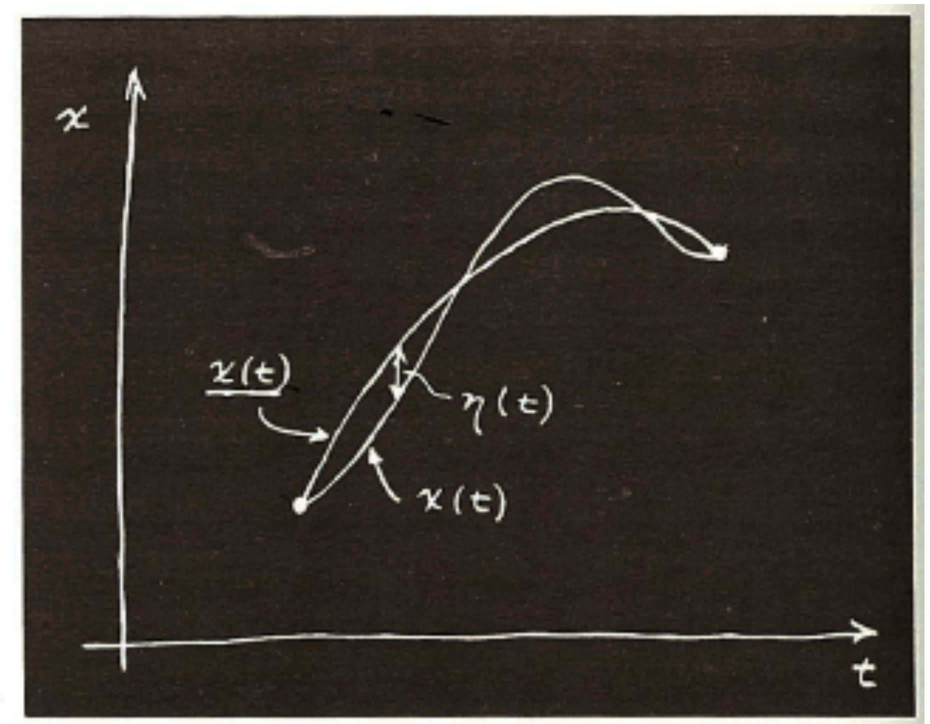
In studying classical mechanics, you have thus far used Newton's 2nd Law, F = ma, as the basic principle. However, there is a completely different mathematical approach which leads to the same equations and the same solutions, but which looks completely different. It turns out that to find the path that a particle travels from some initial starting point (t_1, x_1) to some final point (t_2, x_2), we can minimize a quantity called the Action. (Actually, what we want is the extremum of the Action -- either the minimum or the maximum.)



The Action is the integral of the Lagrangian L = kinetic minus potential energy, Action = ∫ L dt = ∫(KE - PE) dt. To be specific, for a particle moving along the x axis from point (t_1, $\mathbf{x}_1$) to point (t_2, $\mathbf{x}_2$) under gravity, the integral is

$$\text{Action} = S = \int_{t_1}^{t_2} (\text{KE} - \text{PE})\, dt = \int_{t_1}^{t_2} \left[\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 - mgx\right] dt.$$

This is a problem in the branch of mathmatics called "calculus of variations." There are fancy ways of dealing with this, but we will try to do it in a simple way.



Let's imagine that there is a true path $\underline{x}(t)$, and consider some other path $\underline{x}(t) + \eta(t)$ that differs by a small amount $\eta(t)$, where $\eta(t_1) = \eta(t_2) = 0$ since the path must start and end at the fixed points (t_1, x_1), (t_2, x_2). A really hard way to find the true path would be to calculate the integral for all possible paths. Fortunately, there is an easier way.

If the Action is minimized, the first derivative vanishes. This means that any small variation will be not first order, but second order. That is, if the path $\underline{x}(t)$ is the true path, the Action is minimized, so for any small variation around this path, to first order there will be no difference in the Action -- any difference will arise at second order in η. (If there were a difference at first order, then in one direction the sign would be positive and in the other direction negative -- and we could therefore get the action to decrease, contrary to the assumption that it is minimum. A similar argument holds if instead the Action is maximized.)

If we consider a path $\underline{x}(t) + \eta(t)$, then the Action integral becomes

$$S = \int_{t_1}^{t_2} \left[\frac{m}{2} \left(\frac{d\underline{x}}{dt} + \frac{d\eta}{dt} \right)^2 - V(\underline{x} + \eta) \right] dt.$$

Let's write out the squared term:

$$\left(\frac{d\underline{x}}{dt}\right)^2 + 2\,\frac{d\underline{x}}{dt}\,\frac{d\eta}{dt} + \left(\frac{d\eta}{dt}\right)^2 \quad \text{so} \quad \frac{m}{2}\left(\frac{d\underline{x}}{dt}\right)^2 + m\,\frac{d\underline{x}}{dt}\,\frac{d\eta}{dt} + \text{(second and higher order)}.$$

For the potential energy, we expand in a Taylor series:

$$V(\underline{x} + \eta) = V(\underline{x}) + \eta V'(\underline{x}) + \frac{\eta^2}{2} V''(\underline{x}) + \cdots$$

Thus the Action becomes

$$S = \int_{t_1}^{t_2} \left[\frac{m}{2} \left(\frac{d\underline{x}}{dt} \right)^2 - V(\underline{x}) + m \frac{d\underline{x}}{dt} \frac{d\eta}{dt} - \eta V'(\underline{x}) + \text{(second and higher order)} \right] dt\,.$$

Leaving out the "second and higher order" terms, we find that the change in the Action is

$$\delta S = \int_{t_1}^{t_2} \left[m \frac{d\underline{x}}{dt} \frac{d\eta}{dt} - \eta V'(\underline{x}) \right] dt.$$

We now use "integration by parts" to simplify this further. Recall that, for any functions η and f,

$$\frac{d}{dt}(\eta f) = \eta \frac{df}{dt} + f \frac{d\eta}{dt}\,.$$

Integrating, we get the standard "integration by parts" result

$$\int f \frac{d\eta}{dt}\, dt = \eta f - \int \eta \frac{df}{dt}\, dt.$$

Applying this to our case,

$$\delta S = m \frac{d\underline{x}}{dt} \eta(t) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(m \frac{d\underline{x}}{dt} \right) \eta(t)\, dt - \int_{t_1}^{t_2} V'(\underline{x})\, \eta(t)\, dt.$$

The first term vanishes because $\eta(t_1) = \eta(t_2) = 0$, since the path must start and end at the fixed points (t_1, x_1), (t_2, x_2).

The change in the Action then becomes

$$\delta S = \int_{t_1}^{t_2} \left[-m \frac{d^2 \underline{x}}{dt^2} - V'(\underline{x}) \right] \eta(t)\, dt.$$

But we know that if the Action was really extremized for path $\underline{x}(t)$, then the change in the Action must vanish. Since it must vanish for any possible value of η(t), it must be true that

$$\left[-m \frac{d^2 \underline{x}}{dt^2} - V'(\underline{x}) \right] = 0.$$

Since V'(x) = - F(x), this is just our old friend F = ma.

So the assumption that the Action is extremized is just another way to do Newtonian mechanics! We can readily generalize this calculation to three dimensions, where η(t) will now have x, y, and z components, and the kinetic energy will be

$$\mathrm{KE} = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right].$$

We can also generalize the argument to any number of particles: we just add the Actions for each particle, and we get Newton's 2nd law in three dimensions for any number of particles. There is also a straightforward generalization to special relativity.

The laws of optics can be derived from the "principle of least time". It turns out that **all the fundamental laws of physics that have yet been discovered can be written in the form of an Action-extremization principle**, so this approach is very powerful. It is also very useful, since it is often easier to solve problems this way -- and the method gives a very interesting way to go from classical to quantum mechanics.

There is quite a difference between a law that says that every particle just follows its nose, that is it just inches along determining the change in velocity from the force at each point divided by the mass, versus a law that says the entire path is determined by extremizing a certain integral. And yet these two different-sounding formulations are mathematically equivalent. So the lesson is: don't become too wedded to any particular formulation of a physical law!

But how does the particle know which path extremizes the Action? Does it feel out other paths in order to choose the right one? Amazingly, in quantum mechanics we learn that this is exactly what happens!

The probability that a particle starting at point x_1 and time t_1 will arrive at point x_2 at time t_2 is the square $|\Psi|^2$ of a probability amplitude Ψ. The total amplitude is the sum of the amplitudes for each possible path. The amplitude Ψ for each path is proportional to $\exp(iS/\hbar)$ where "h-bar" $\hbar = h/2\pi = 1.054 \times 10^{-34}$ m^2 kg/s, and Planck's constant $h = 6.626 \times 10^{-34}$ m^2 kg/s. The Action S has dimensions of energy x time, and h has the same dimensions.

The size of the Action compared to $\hbar$ determines how important quantum effects are. If for all paths, S is large compared to $\hbar$, then the complex phase will be wildly different for all paths except for those that are extremely close to the true path, the one for which the Action is extremized, since any small variation about that path will not change the Action to first order. Then all the other paths will cancel out because of the wild change in phase, and the path that will be taken will be the classical path, the one for which the Action is extremized. But if the Action is comparable in size to $\hbar$, other paths besides the classical one can be important.

The fact that quantum mechanics can be formulated this way was discovered by Richard Feynman in 1942, when he was still a graduate student, based on earlier work by Dirac.

Symmetries in Particle Physics

The Action S is the integral of the Lagrangian L = kinetic minus potential energy, Action = $\int L\, dt = \int (KE - PE)\, dt$. In classical mechanics, it is standard to write the Lagrangian as a function of position variables q_i and their first derivatives with respect to time $L = L(q_i, \dot{q}_i)$, and to define the canonical momentum p_i corresponding to position variable q_i by $p_i \equiv \partial L/\partial \dot{q}_i$. For the example of a particle moving in one dimension, we can take $q = x$, then the Lagrangian is $L = \frac{1}{2} m \dot{q}^2 - V(q)$, and $p = m\dot{q}$.

We follow the same argument that we went through to deduce the equations of motion from the requirement that the Action is extremized integrating from $t = t_1$ to $t = t_2$

$$\delta S = \int \left[\left(\frac{\partial L}{\partial q} \right) \delta q + \left(\frac{\partial L}{\partial \dot{q}} \right) \delta \dot{q} \right] \mathrm{d}t = 0$$

In each product we sum over the variables, for example $\left(\frac{\partial L}{\partial q} \right) \delta q = \sum_i \left(\frac{\partial L}{\partial q_i} \right) \delta q_i$.

We integrated by parts

$$\int \left(\frac{\partial L}{\partial \dot{q}} \right) \delta \dot{q}\, \mathrm{d}t = \left[\delta q \left(\frac{\partial L}{\partial \dot{q}} \right) \right] - \int \delta q \left[\frac{\mathrm{d}\,(\partial L/\partial \dot{q})}{\mathrm{d}t} \right] \mathrm{d}t$$

The first term vanishes because $\delta q = 0$ at $t = t_1$ and $t = t_2$, but δq is otherwise arbitrary.

Then $\partial S = \int \left\{ \left(\frac{\partial L}{\partial q} \right) - \frac{\mathrm{d}\,(\partial L/\partial \dot{q})}{\mathrm{d}t} \right\} \delta q\, \mathrm{d}t = 0$ which implies $\boxed{\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0}$

The equations of motion that we have just derived by extremizing the Action $\delta S = 0$ are known as the Euler-Lagrange equations:

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0}$$

If the Lagrangian is independent of positions q_i, i.e. $\partial L/\partial q_i = 0$, it follows that the second term vanishes. Recalling that momentum $p_i \equiv \partial L/\partial \dot{q}_i$, this implies that $dp_i/dt = 0$. Thus invariance of the Lagrangian under space translations implies that momentum is conserved.

Thus far we have been discussing particle mechanics. In field theory, the Lagrangian density $\mathscr{L}$ is the difference between kinetic and potential energy densities. The variables are the values of the fields in space and time, and the position and time coordinates play a role similar to the indexes *i* in particle physics. If $\mathscr{L}$ is invariant under space and time translations, it follows from Noether's Theorem, a generalization of the above argument, that the 4-momentum $P = (E, \boldsymbol{p})$ is conserved. More generally, if $\mathscr{L}$ is invariant under any continuous symmetry, there is a corresponding conserved quantity. For example, conservation of angular momentum follows from invariance under rotations, and conservation of electric charge follows from invariance under gauge transformations.

Note that Perkins meant to write (for rotations by angle $\delta\varphi$ about the z axis, where $\delta x = -y\,\delta\varphi$ and $\delta y = x\,\delta\varphi$)

$$\psi(x + \delta x, y + \delta y, z) = \psi(x, y, z) + \delta x\left(\frac{\partial \psi}{\partial x}\right) + \delta y\left(\frac{\partial \psi}{\partial x}\right) =$$

$$= \left[1 + \delta\varphi\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]\psi$$

Parity in Particle Physics

The parity operation is the inversion of coordinates: $\boldsymbol{r} \longrightarrow -\boldsymbol{r}$. The parity operator P applied to the wavefunction Ψ gives $P\,\Psi(\boldsymbol{r}) = \Psi(-\boldsymbol{r})$. Since $P^2\Psi(\boldsymbol{r}) = \Psi(\boldsymbol{r})$, the eigenvalues of P are ±1. These are referred to as the "parity" of the system described by the wavefunction. Thus if $\Psi(x) = \cos x$ the parity is +1, while if $\Psi(x) = \sin x$ the parity is −1. On the other hand, if $\Psi(x) = \cos x + \sin x$, then $\Psi(x)$ is not an eigenfunction of parity. Parity is useful in particle physics since the particles have well-defined parities and their interactions often have well-defined properties under the parity operation. In particular, under space inversion $\boldsymbol{r} \longrightarrow -\boldsymbol{r}$, $\theta \longrightarrow \pi-\theta$, $\varphi \longrightarrow \pi+\varphi$, and the spherical harmonics have the parity eigenvalues $(-1)^{\ell}$:

$$P\,Y_l^m(\theta,\phi) = Y_l^m(\pi-\theta,\pi+\phi) = (-1)^l\,Y_l^m(\theta,\phi)$$

The symmetry of a pair of identical nonrelativistic particles under interchange can be written as a product of space and spin functions:

$$\psi = \chi\,(\text{space})\,\alpha\,(\text{spin})$$

Consider two identical fermions of spin-1/2, and use up and down arrows to indicate the spin orientation along the z-axis. The four possible spin states are

$$\left.\begin{array}{l}\alpha(1,+1) = \uparrow\uparrow \\ \alpha(1,-1) = \downarrow\downarrow \\ \alpha(1,0) = (\uparrow\downarrow + \downarrow\uparrow)/\sqrt{2}\end{array}\right\} \quad S = 1,\ \text{symmetric}$$

$$\alpha(0,0) = (\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2} \qquad S = 0,\ \text{antisymmetric}$$

The first three are symmetric under interchange, the last anti-symmetric; thus the eigenvalue under interchange is $(-1)^{S+1}$. If the orbital angular momentum eigenvalue is ℓ then the complete wavefunction has parity $(-1)^{\ell+S+1}$.

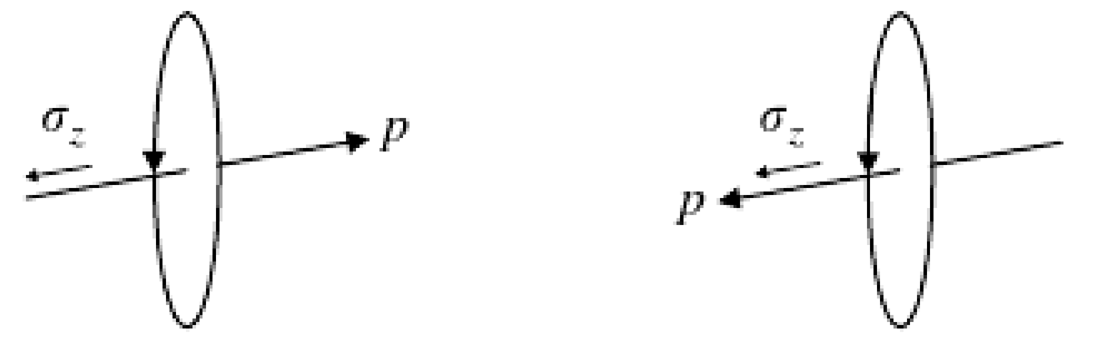
Note that parity is multiplicative: the parity of a product state is the product of the parities.

Perkins section 3.4 shows that the pion must have negative intrinsic parity $P_\pi = -1$ using the fact that parity is conserved in the Strong interactions. (Parity is also conserved in Electromagnetic interactions.)

To show this, it applies the result that the complete wavefunction has parity $(-1)^{\ell+S+1}$ to the process $\pi^- + d \longrightarrow n + n$ in which the initial state is known to be an S state, i.e. with orbital angular momentum $\ell = 0$. The deuteron d (heavy hydrogen nucleus, with one proton and one neutron) has spin S=1, so the total angular momentum J of the initial state and therefore also the final state is J =1. Since the neutrons are identical particles, the final state must be antisymmetric in their interchange, i.e. $(-1)^{\ell+S+1} = -1$, so $\ell + S$ is even, and thus $\ell = S = 1$ is the only possibility. Thus the final state neutrons are in a state of parity $(-1)^{\ell} = -1$. Thus the pion must be assigned intrinsic parity $P_\pi = -1$.

The nucleons are by convention assigned parity = +1, and since there are two nucleons on both sides of $\pi^- + d \longrightarrow n + n$ the nucleon parity cancels out in the above case. In the Dirac theory of spin-1/2 particles, anti-particles have the opposite parity of particles. Perkins describes an experiment which confirmed this.

Parity is not conserved in the Weak interactions; on the contrary, the $W^{\pm}$ interacts only with “left-handed” particles. In the Standard Model, neutrinos are massless, which means that they are eigenstates of helicity $H = \boldsymbol{\sigma} \cdot \mathbf{p}/|p|$ with $H = \pm 1$. While neutrinos are not actually massless, their masses, $\lesssim 0.1$ eV, are so small that they can be neglected in high-energy processes. Then we can treat neutrinos as having helicity −1 (“left handed”) and anti-neutrinos as having helicity +1 (“right handed”), as shown at right.



Parity can be important in determining the values of cross sections. Perkins shows, for example, that although the two processes shown in the diagrams at right are very similar, the cross section for e+e− annihilation to neutrinos is only one-sixth that of the electron-neutrino scattering process. The e+e− annihilation to neutrinos is extremely important in core-collapse supernovae and also in the early universe.