HW #2 (Solutions)

Problem 1

(a) $p = \begin{pmatrix} E/c \\ \hat{z}E \end{pmatrix}$

(b) $\vec{p} = \gamma m\vec{v}$; $E = \gamma mc^2$ & $\frac{\vec{p}}{E} = \frac{v}{c^2} \Rightarrow \boxed{\vec{v} = \frac{\vec{p}}{E}}$

(c) $p_\pi = p_\mu + p_\nu$

$\Rightarrow \quad p_\nu = p_\pi - p_\mu$

Now taking a square of both side

$$p_\nu^2 = p_\pi^2 + p_\mu^2 - 2\, p_\pi \cdot p_\mu$$

in the CM-frame $p_\pi = \begin{pmatrix} m_\pi \\ 0 \end{pmatrix}$. ~~there~~ & also

$p_\nu^2 = 0$, $p_\pi^2 = m_\pi^2$ & $p_\mu^2 = m_\mu^2$

~~where~~ Which gives me,

$$0 = m_\pi^2 + m_\mu^2 - 2 m_\pi E_\mu$$

$$\Rightarrow \quad E_\mu = \frac{m_\pi^2 + m_\mu^2}{2 m_\pi} \quad \text{-------} \quad (1)$$

But ~~E_μ~~ $m_\pi = E_\mu + E_\nu$ (E conservation)

& $\vec{p}_\mu + \vec{p}_\nu = 0$ (Momentum conservation)

& $\vec{p}_\nu = \hat{z} E_\nu$ ... 6

then,

$$|\vec{p}_\mu| = |\vec{p}_\nu| = E_\nu = m_\pi - E_\mu = \frac{m_\pi^2 - m_\mu^2}{2 m_\pi}$$

So $$v = \frac{|\vec{p}_\mu|}{E_\mu} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}$$

$$\Rightarrow \boxed{v = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} c}$$

$v = 0.998c$

(d) $d_{non\text{-}rel} = v\tau$

$= 0.998 \times 3 \times 10^8 \times 2.2 \times 10^{-6}$ m

$= 658.68$ m

$d_{Relativistic} = \gamma v \tau$

≈ 10.4 km

$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

≈ 15.8

Problem 2 The quickest way to solve this one is in CM-frame $p_{Tot} = p_1 + p_2 = \begin{pmatrix} E+E \\ (E-E)\hat{z} \end{pmatrix} = \begin{pmatrix} 2E \\ 0 \end{pmatrix}$

in rest-frame,

$$p'_{tot} = \begin{pmatrix} m \\ 0 \end{pmatrix} + \begin{pmatrix} E' \\ \vec{p}' \end{pmatrix} = \begin{pmatrix} E'+m \\ \vec{p}' \end{pmatrix}$$

then, invariance of the square of four momentum tells us.

$$p_{tot}^2 = p_{tot}'^2$$

$$\Rightarrow \quad 4E^2 = (E'+m)^2 - |\vec{p}'|^2 \quad \cdots \quad (1)$$

Now $|\vec{p}'|^2 = E'^2 - m^2$

Thus (1) $\Rightarrow$

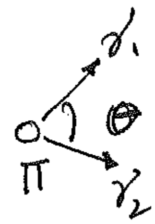
$$4E^2 = E'^2 + m^2 + 2E'm - E'^2 + m^2$$

$$\Rightarrow \quad 4E^2 = 2m(E'+m)$$

Now using $T = E - m$ & $T' = E' - m$ we get

$\boxed{T' = 4T\left(1 + \frac{T}{2m}\right)}$ for $T \ll m \to T' \simeq 4T$ as expected.

Problem 2.3



$$P_\pi = P_1 + P_2$$

$$\Rightarrow \quad P_\pi^2 = P_1^2 + P_2^2 + 2P_1 \cdot P_2$$

$$\Rightarrow \quad m_\pi^2 = 0 + 0 + 2\left(E_1E_2 - E_1E_2\cos\theta\right)$$

$$= 2E_1E_2\left(1-\cos\theta\right)$$

$$\Rightarrow \quad \boxed{E_1E_2 = \frac{m_\pi^2}{2(1-\cos\theta)}}$$

Note the above relationship is valid for any referance frame.

the happens when $\theta = \pi$ then.

$$\sqrt{E_1E_2} = \frac{m_\pi}{2}$$

$E_\gamma = m_\pi/2$ is only valid in ~~rest~~ CM frame.

Problem 2.4

$$\frac{\lambda_o}{\lambda_e} = z+1;$$

λ_o ← observed, λ_e ← emitted.

In this problem the redshift is occuring due to the gravitational effect.

So

$$\frac{\lambda_o}{\lambda_e} = z+1 = \left(1 - \frac{2GM}{r}\right)^{-1/2}$$

Now on the surface of the Sun the "observed" wavelength is λ_s then.

$$\frac{\lambda_s}{\lambda} = \left(1 - \frac{2GM_\odot}{R_\odot}\right)^{-1/2}$$

Similarly on the ~~E~~earth's surface

$$\frac{\lambda_E}{\lambda} = \left(1 - \frac{2GM_E}{R_E}\right)^{-1/2}$$

then.

$$\frac{\lambda_s}{\lambda_E} = \frac{\lambda_s/\lambda}{\lambda_E/\lambda} = \left(\frac{1 - \frac{2GM_E}{R_E}}{1 - \frac{2GM_\odot}{R_\odot}}\right)^{1/2}$$

Now remembering to put the c^2 in the correct place

$$\frac{\lambda_E}{\lambda_s} = \left(\frac{1 - \frac{2GM_\odot}{c^2 R_\odot}}{1 - \frac{2GM_E}{c^2 R_E}} \right)^{1/2}$$

then

$$\frac{\Delta\lambda}{\lambda_s} = 1 - \frac{\lambda_E}{\lambda_s} \simeq 2.12 \times 10^{-6}$$

Problem 2.5 See ~~book~~ back of the book.

~~2.6~~

Problem 2.6 The radius of the ~~sattel~~ satellite orbit is r then

$T = 12h = 12 \times 3600\,s$

$G = 6.673 \times 10^{-11} m^3 kg^{-1} s^{-2}$

$M_E = 5.98 \times 10^{24}\,kg$

$$T^2 = \frac{4\pi^2}{GM_E} r^3$$

$$\Rightarrow \quad r = \left(\frac{GM_E}{4\pi^2}\right)^{1/3} T^{2/3}$$

$$= 2.8 \times 10^7\,m$$

then $v = \frac{2\pi r}{T} \approx 4072.43$ m/s

(a) Now clock drifts due to time dilation

$$\Delta t' = \gamma \Delta t_{GPS}$$

$$\Rightarrow \Delta t_{GPS} = \Delta t'/\gamma = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \Delta t' \approx \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)\Delta t'$$

then the amount of shift due to SR is

$$\boxed{-\frac{v^2}{2c^2} \approx -0.92 \times 10^{-10} \text{ s/s}}$$

(b) $\cancel{\Delta t' = \sqrt{\left(1 + \frac{2\Delta\Phi}{c^2}\right)}\, \Delta t_{GPS}}$

$$\Delta t' = \sqrt{\left(1 + \frac{2\Delta\Phi}{c^2}\right)}\, \Delta t_{GPS}$$

$$\Rightarrow \Delta t_{GPS} \approx \left(1 - \frac{\Delta\Phi}{c^2}\right)\Delta t'$$

Here $\Delta\Phi = \Phi(R_e) - \Phi(r)$ & $\Phi(a) = -\frac{GM_e}{a}$

then time shift

$$-\left\{-\frac{GM_e}{c^2 R_e} + \frac{GM_e}{c^2 r}\right\} = +5.4 \times 10^{-10} \text{ s/s}$$

(c) Net shift is 4.48×10^{-10} s/s $= 3.87 \times 10^{-5}$ s/day

~~1 day ⇒~~

Then the net error in distance is

$$3.8\cancel{7} \times 10^{-5} \times 3 \times 10^{8} \text{ m/day} = 11.6 \times 10^{3} \text{ m/day}.$$