Homework Set 1 with Solutions

1. For a flat universe with $\Omega_{m,0} < 1$ and positive cosmological constant $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$, the density contributions of the matter and cosmological constant are equal when the scale factor has the value $a_{m\Lambda} = (\Omega_{m,0}/\Omega_{\Lambda,0})^{1/3}$. This equals 0.75 for the Benchmark Model: $\Omega_{m,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$. Show that for this case the Friedmann equation can be integrated to give the expression

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln[y^{3/2} + \sqrt{1+y^3}],$$

where $y \equiv a/a_{m\Lambda}$. Show that for $a \ll a_{m\Lambda}$, this reduces to

$$a(t) \approx (\frac{3}{2}\sqrt{\Omega_{m,0}}H_0 t)^{2/3},$$

and for $a \gg a_{m\Lambda}$, it reduces to

$$a(t) \approx a_{m\Lambda} \exp(\sqrt{1-\Omega_{m,o}}H_0 t).$$

Show finally that the age of the universe today in this case is

$$t_0 = \frac{2}{3H_0\sqrt{1-\Omega_{m,o}}} \ln\left[\frac{\sqrt{1-\Omega_{m,o}}+1}{\sqrt{\Omega_{m,0}}}\right],$$

and that for the Benchmark Model this is $t_0 = 0.964 H_0^{-1}$.

<u>Solution</u>

Assuming $\Omega_m < 1, \Omega_\Lambda = 1 - \Omega_m, a_{m\Lambda} = (\Omega_m/\Omega_\Lambda)^{1/3}$:

$$\frac{H^2}{H_0^2} = \Omega_m a^{-3} + \Omega_\Lambda$$

$$\frac{da}{dt}\frac{1}{a} = H_0\sqrt{\Omega_m a^{-3} + \Omega_\Lambda}$$

$$\int_a^1 \frac{da}{\sqrt{\Omega_m a^{-3} + \Omega_\Lambda}} = \int H_0\, dt$$

$$H_0 t = \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(2(\sqrt{\Omega_\Lambda\Omega_m + \Omega_\Lambda^2 a^3} + \Omega_\Lambda a^{3/2})\right) + C$$

$$= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(2\sqrt{\Omega_\Lambda\Omega_m}(\sqrt{1+y^3} + y^{3/2})\right) + C$$

$$= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(\sqrt{1+y^3} + y^{3/2}\right) + \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(2\sqrt{\Omega_\Lambda\Omega_m}\right) + C$$

$$t = 0 \implies a = y = 0 \implies C = -\frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(2\sqrt{\Omega_\Lambda\Omega_m}\right)$$

$$H_0 t = \frac{2}{3\sqrt{\Omega_\Lambda}} \ln\left(\sqrt{1+y^3} + y^{3/2}\right)$$

For $a \ll a_{m\Lambda}, y \to 0$:

$$H_0 t = \frac{2}{3\sqrt{\Omega_\Lambda}} \ln \left(\sqrt{1 + y^3} + y^{3/2} \right)$$

$$= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln \left(1 + y^{3/2} \right)$$

$$\approx \frac{2}{3\sqrt{\Omega_\Lambda}} \left(y^{3/2} \right)$$

$$a^{3/2} = a_{m\Lambda}^{3/2} \frac{3\sqrt{\Omega_\Lambda} H_0 t}{2}$$

$$a = \left(\frac{3}{2} \sqrt{\Omega_m} H_0 t \right)^{2/3}$$

For $a \gg a_{m\Lambda}, y \to \infty$:

$$H_0 t = \frac{2}{3\sqrt{\Omega_\Lambda}} \ln \left(\sqrt{1 + y^3} + y^{3/2} \right)$$

$$\approx \frac{2}{3\sqrt{\Omega_\Lambda}} \ln y^{3/2} + y^{3/2}$$

$$2y^{3/2} = \exp \left(\frac{3 H_0 t \sqrt{\Omega_\Lambda}}{2} \right)$$

$$a \approx a_{m\Lambda} \exp(\sqrt{\Omega_\Lambda} H_0 t)$$

For the present time, $a = 1, y = \frac{1}{a_{m\Lambda}}$:

$$H_0 t = \frac{2}{3\sqrt{\Omega_\Lambda}} \ln \left(\sqrt{1 + \left(\frac{1}{a_{m\Lambda}} \right)^3} + \left(\frac{1}{a_{m\Lambda}} \right)^{3/2} \right)$$

$$= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln \left(\sqrt{\frac{\Omega_\Lambda + \Omega_m}{\Omega_m}} + \sqrt{\frac{\Omega_\Lambda}{\Omega_m}} \right)$$

$$= \frac{2}{3\sqrt{\Omega_\Lambda}} \ln \left(\frac{\sqrt{\Omega_\Lambda} + 1}{\sqrt{\Omega_m}} \right)$$

For the Benchmark model, $\Omega_m = 0.3, \Omega_\Lambda = 0.7 \implies t_0 = 0.964099$.

2. Geometry. (a) Show that if $k = 0$ and the scale factor a grows as $t^{2/3}$, the apparent angular sizes of distant objects of the same linear size have a minimum at $z = 1.25$. (b) Consider a galaxy of physical (visible) size 5 kpc. What angle would this galaxy subtend if situated at redshift 0.1? 1? 5? Do the calculation in a flat universe, first with zero cosmological constant, and then in the Benchmark Model with $\Omega_{m,0} = 0.3$. You are welcome to use Ned Wright's Cosmology Calculator, at http://www.astro.ucla.edu/~wright/CosmoCalc.html

Solution

(a) In this Einstein-de Sitter cosmology, the proper distance to an object at scale factor a_e is

$$r_e = \int_0^{r_e} dr = \int_{t_e}^{t_0} \frac{dt}{a} = \int_{a_e}^1 \frac{da}{a^2 H}$$

Here $H = H_0 a^{-3/2}$ so

$$r_e = \int_{r_e}^1 \frac{da}{H_0 a^{1/2}} = \frac{2}{H_0}(1 - a_e^{1/2}).$$

(as derived in class). The angle θ subtended by length L at r_e is

$$\theta = \frac{L}{a_e r_e} = \frac{L H_0}{2} \frac{1}{a_e - a_e^{3/2}}.$$

Then $d\theta/da = 0$ implies $0 = 1 - (3/2)a^{1/2}$, $a^{1/2} = 2/3$, $a = 4/9 = (1+z)^{-1}$, so $1+z = 9/4$, $z = 5/4 = 1.25$, as claimed.

(b) Using Ned Wright's CosmoCalc (θ in arc seconds)

Redshift	θ for $\Omega_m = 1$	θ for Benchmark Model
0.1	2.85"	2.71"
1	0.82"	0.62"
5	1.22"	0.80"

The basic pattern is that as z increases the angular size first decreases, and then it grows larger. These numbers were calculated for Hubble parameter $h=0.70$.

3. Short problems:

(a) If a neutrino has mass m_ν and decouples at $T_{\nu d} \sim 1$ MeV, show that the contribution of this neutrino and its antiparticle to the cosmic density today is (Dodelson Eq. 2.80)

$$\Omega_\nu = \frac{m_\nu}{94h^2 \text{eV}} .$$

Solution

The neutrino number density can be found from the photon number density as:

$$n_\nu = \frac{3}{11} n_\gamma$$

Given the current temperature of the universe, this works out to ≈ 112 cm^{-3}. At late times the energy density of massive neutrinos ρ_ν reduces to $m_\nu n_\nu$, giving the following equation for neutrino density:

$$\Omega_\nu = \frac{\rho_\nu}{\rho_{cr}} = m_\nu \frac{8\pi G n_\nu}{3H_0^2}$$

$$= m_\nu \frac{8\pi G n_\nu}{3} \left(\frac{h_P/2\pi}{2.133 \times 10^{-33} h \text{ eV}}\right)^2$$

$$= m_\nu \frac{2 G n_\nu h_P^2}{3\pi} \frac{1}{(2.133 \times 10^{-33} h \text{ eV})^2}$$

$$\approx \frac{m_\nu}{94h^2 \text{ eV}}$$

(b) Verify that $\eta_b \equiv n_b/n_\gamma$ is given by (Dodelson Eq. 3.11; Weinberg, *Cosmology*, pp. 168-169)

$$\eta_b = 5.5 \times 10^{-10} \left(\frac{\Omega_b h^2}{0.020}\right) .$$

Solution   As in the previous problem, we recall that $n_\gamma(T) = 2\zeta(3)\pi^{-2}T^3$. Since the total number of baryons and photons is more or less conserved, we can evaluate this for the present day temperature of the CMB and compare to the baryon fraction. The baryon number density must be related to the cosmic baryon fraction via

$$\Omega_b = \frac{\rho_b}{\rho_{cr}} = \frac{m_b n_b}{8.098 \times 10^{-11} h^2 \text{eV}^4}$$

Adopting the proton mass as the typical mass of a baryon, we obtain

$$\eta_b = \frac{n_b}{n_\gamma} = \frac{8.098 \times 10^{-11} h^2 \text{eV}^4 \Omega_b/m_p}{2\zeta(3)\pi^{-2}(2.725K)^3}\left(\frac{11605K}{1eV}\right)^3 \left(\frac{m_p}{938 \times 10^6 eV}\right) = 2.74 \times 10^{-8}\Omega_b h^2$$

so factoring out two percent from the baryon energy density,

$$\eta_b = 5.48 \times 10^{-10}\left(\frac{\Omega_b h^2}{.02}\right)$$

(c) Verify the time-temperature relation (Dodelson Eq. 3.30)

$$t = 132 \sec (0.1\text{MeV}/T)^2 \quad .$$

Solution

The time-temperature relation is found from:

$$\frac{1}{T}\frac{dT}{dt} = -\sqrt{\frac{8\pi G\rho}{3}}$$

When decays become important and e^+e^- annihilation is complete, the energy density becomes:

$$\rho = 3.36\frac{\pi^2}{30}T^4$$

Combining these two and integrating yields:

$$\frac{1}{T}\frac{dT}{dt} = -T^2\sqrt{\frac{8\pi G(3.36\pi^2/30)}{3}}$$

$$t = \frac{1}{2\pi T^2}\sqrt{\frac{90}{26.88\pi G}}$$

$$\approx 132 \text{ s} \left(\frac{0.1 \text{ MeV}}{T}\right)^2$$

4. Suppose that the neutron decay time were $\tau_n = 89$ s instead of $\tau_n = 890$ s, with all other physical parameters unchanged. Estimate Y_p, the primordial mass fraction of nucleons in ^{4}He, assuming that all available neutrons are incorporated into ^{4}He.

Solution

In the usual treatment, $Y_p = 2(0.15)(0.74) = 0.22$, which becomes 0.24 including the dependence on the baryon-photon ratio n_b/n_γ and agrees with observations.

With 10x shorter neutron lifetime, a much smaller fraction of neutrons exp(-269s/89s) = 0.0487 survives decay. The rest of Big Bang Nucleosynthesis is by assumption unchanged, so the primordial ^{4}He mass fraction $Y_p = 2X_n = 2(0.15)(0.0487) = 0.015 = 1.5\%$. This would badly disagree with observations!

5. Suppose that there were no baryon asymmetry so that the number density of baryons exactly equaled that of anti-baryons. Determine the final relic density of (baryons + anti-baryons). At what temperature is this relic density reached?

<u>Solution</u>

To estimate the final relic density, follow Dodelson's treatment on pp. 73-77. Dodelson defines $Y = n_X/T^3 = n_X/s$ where n_X = number density of X-particles (the protons and antiprotons in this case) after annihilation freezes out, and $s = T^3$ is the entropy.

Dodelson's eq. (3.56) is $Y_\infty \simeq 10H(m_p)/[m_p^3 \langle \sigma v \rangle] =$ asymptotic value of Y, where $H(m_p) =$ Hubble parameter when $T = m_p$. Since $H^2 = 8\pi G\rho/3 = (8\pi/3) g^* T^4/m_{Pl}^2$, it follows that $H(m_p) \simeq g^{*1/2} m_p^2/m_{Pl}$.

Since the length scale of the strong interactions is roughly $\hbar/m_\pi c = 1/m_\pi \simeq$ 1 fm, a reasonable rough estimate for the annihilation cross section is

$$\langle \sigma v \rangle \simeq m_\pi^{-2} \simeq 10^{-2} m_p^{-2}$$

Then $Y_\infty \simeq 10(g^{*1/2} m_p^2/m_{Pl}) /(m_p^3 m_p^{-2} 10^2) \sim m_p/m_{Pl} \sim 10^{-19}$.

In the real Universe with a matter-antimatter asymmetry, $Y_p = n_p/s \simeq 10^{-9}$ and $\Omega_b \simeq 0.044$. The baryon-antibaryon abundance in a baryon-antibaryon symmetric universe, where freezeout of baryon-antibaryon annihilation determines the abundance, is about 10^{-10} of that, i.e. $\Omega_b \sim 4\times10^{-12}$. Tiny!

If the abundance is given by $10^{-19} \sim \exp(-m_p/T_f)$, this implies that $T_f \sim 40/m_p \sim 25$ MeV.

There is a more detailed treatment of this in Kolb and Turner, p 127, with a very similar answer. Dodelson says that for WIMP dark matter freezeout $x_f \simeq m/T_f \simeq 10$, where T_f is the freezeout temperature, but Kolb and Turner find $T_f \simeq m_p/40 \sim 25$ MeV for the case of the freezeout of baryon-antibaryon annihilation.

The tiny abundance of baryons in a matter-antimatter symmetric universe is why cosmologists are convinced that there must be an asymmetry between matter and antimatter.