

**Homework Set 2***DUE: Thursday May 8*

1. **Power Spectrum of Perturbations.** Suppose that the power spectrum of density perturbations  $\delta \equiv (\delta\rho/\rho)$ , with index  $n$ , is  $P_k \equiv |\delta_k|^2 \propto k^n$ .

a) Derive this expression for the rms density fluctuations on a comoving scale  $\lambda$ :

$$\left(\frac{\delta\rho}{\rho}\right)_\lambda \sim k^{3/2} |\delta_k| \propto \lambda^p$$

and find the index  $p$  in terms of  $n$ .

b) Find the index  $m$  for the rms mass fluctuations on a comoving scale  $\lambda$ :

$$(\delta M/M)_\lambda \propto M^m.$$

What is the criterion on  $n$  such that structures grow in a top-down manner (large masses collapse first)? In a bottom-up manner? If the primordial spectrum has  $n = 1$  and the CDM transfer function goes as  $T(k) \sim k^{-2}$  due to the horizon entry effect, what  $m$  does this correspond to and what is the physical interpretation of the mass fluctuation spectrum?

c) The dynamics of the perturbations are driven by the fluctuations in the gravitational potential  $\phi$ . Find the index  $f$  for the rms potential fluctuations

$$\delta\phi_\lambda \propto \lambda^f.$$

What is the physical significance of the scale invariant power spectrum index  $n = 1$ ? Also explain the difference between a process being scale free and being scale invariant. Which, if either, is Newtonian gravity?

2. **Growth of Density Perturbations.** For matter density perturbations with wavelengths much greater than the Jeans length, the time evolution is given by

$$\ddot{\delta} + 2H\dot{\delta} - (3/2)\Omega_m(t) H^2 \delta = 0.$$

a) Rewrite this equation with the dependent variable being the scale factor  $a$ . Write any derivatives of  $a$  or  $H$  in terms of  $H$  and the deceleration parameter  $q$ . b) Consider the case of a flat matter universe where, on the scales considered, only a constant fraction  $\Omega_{cl}$  clumps to form structure. (Possible realizations are a cold + hot dark matter universe or a dark + baryonic matter universe.) In this case the  $\Omega_m$  in the source term of the evolution equation is replaced by  $\Omega_{cl}$ . Solve the equation to find the behavior of the growing mode:  $\delta \propto a^m$ . Interpret. Check the limits  $\Omega_{cl} = 0$  and 1. c) Consider the case of an open

universe at a time dominated by the curvature. Write the evolution equation, substituting in for  $q$  and  $\Omega_m(t)$ , keeping only the leading order for the coefficient of each term as  $a$  gets large. What happens to the source term for the growth as the universe expands? Try a solution  $\delta \propto a^m$ . Find the dominant mode and give the physical interpretation (include explanation of the roles of both the drag and source terms).

**3. Spherical Collapse.** The evolution of a spherically symmetric, overdense perturbation in an  $\Omega = 1$  universe can be solved analytically up to the point of singular collapse. As a consequence of Birkhoff's theorem (in a spherically symmetric universe, only the interior mass matters), the perturbation follows the equations of a  $k = +1$  Friedmann universe, for which we have a parametric solution. **a) Perturbation overdensity** The solutions – unperturbed for the background universe (barred quantities) and perturbed for the overdense region – for the evolution of the size of a sphere and the time are given by

$$\begin{aligned}\bar{r} &= r_0 (a/a_0) = r_0 (\eta/\eta_0)^2 & \bar{t} &= t_0 (a/a_0)^{3/2} = t_0 (\eta/\eta_0)^3 \\ r &= A(1 - \cos\theta), & t &= B(\theta - \sin\theta),\end{aligned}$$

where  $\eta$  is the conformal time and  $\theta$  is the development angle.

At early times the density perturbation must be small ( $\rho \rightarrow \bar{\rho}$ ) so the Friedmann equations for the universe and the perturbation region look the same. Enforce this by matching  $x, \dot{x}, \ddot{x}$  for  $x = r, \bar{r}$  with the respective time variables in order to find  $r(\theta)$  and  $t(\theta)$ , i.e.  $A$  and  $B$ . *Hint:* Remember the definition of the conformal time parameters  $\eta$  and  $\theta$ .

The age of the universe is unique so  $t$  and  $\bar{t}$  must be equal. Use this to derive  $\eta(\theta)$ .

Use mass conservation to express the overdensity  $\rho/\bar{\rho}$  first in terms of  $r/\bar{r}$  and then as a function of  $\theta$ .

Verify that turnaround occurs at  $\theta = \pi$  and  $\rho/\bar{\rho} = 5.55$  and that virialization occurs at  $\rho/\bar{\rho} = 178$ . Use that the radius is half the turnaround radius (implying  $V = -2K$ ), but use the time corresponding to  $\theta = 2\pi$ . (Although  $V = -2K$  at  $\theta = 3\pi/2$ , virialization requires  $\langle V \rangle = -2\langle K \rangle$ , which obtains roughly at  $\theta = 2\pi$ ).

**b) Linear regime** Show that the density contrast

$$\frac{\delta\rho}{\rho} \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}} \propto a$$

when  $\theta \ll 1$ . Show that the dimensionless velocity perturbation for  $\theta \ll 1$  is

$$\delta_v \equiv \frac{v - Hr}{Hr} = -\frac{1}{3} \left( \frac{\delta\rho}{\rho} \right),$$

where  $v = dr/dt$  is the perturbation's expansion velocity and  $H$  is the Hubble parameter of the background universe.

**c) Astrophysical application** Suppose that we observe a galaxy with rotation speed  $\sigma$  at radius  $R$ . If we attribute this rotation speed to the mass of a (spherical) dark halo and assume the spherical collapse model in an  $\Omega = 1$  universe gives an accurate description of the formation of this halo, what is the expression for the redshift of virialization  $z_v$ ? What is the value of  $z_v$  if  $\sigma = 180 \text{ km s}^{-1}$ ,  $R = 30 \text{ kpc}$  and  $H_0 = 60 \text{ km s}^{-1} \text{ Mpc}^{-1}$ ?