Final Exam
3/25/94
Physics 5A

1 Do all five problems. These problems are quite basic and should be attempted by everyone. You will only receive credit if you explain your answers and show your work.

1. A boat is required to traverse a river that is 150 m wide. The current in the river moves with a speed of $6km/h$. The boat can be rowed on still water with a speed of $10km/h$. Set up a coordinate system in the frame of the land in which to describe the various displacements.

(a) (10 points) Using this coordinate system, write down the position vector of the boat at time t, assuming the boat moves with uniform speed and that it leaves one side with its velocity vector making an angle θ in the reference frame of the moving river.

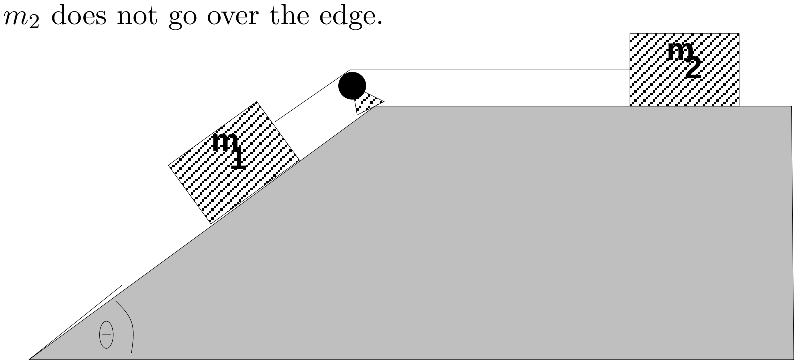
(b) (5 points) Calculate θ such that the boat lands at a point exactly opposite the starting point.

(c) (5 points) For part (b), how long will the trip take?

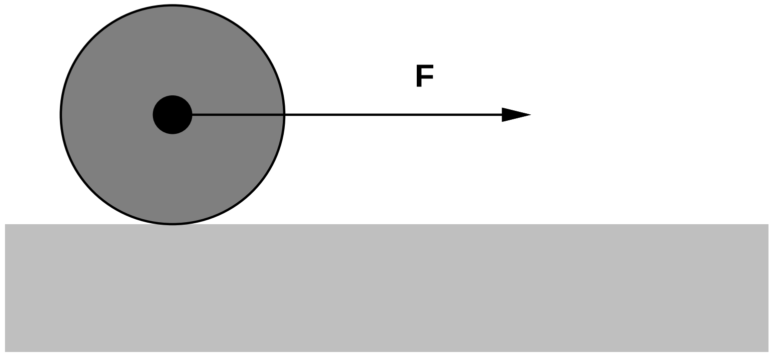
2. Masses m_1 and m_2 are connected by a taut rope. Mass m_1 is just over the edge of a ramp inclined at an angle of θ as shown below. The masses have a coefficient of kinetic friction of μ_k. At $t = 0$ the system is given an initial speed of v_0, which starts mass m_1 down the ramp.

(a) (5 points) Draw the force diagram for each mass.

(b) (15 points) Solve the equations of motion to predict the motion of the system as a function of time. Assume the rope is long enough so that mass m_2 does not go over the edge.



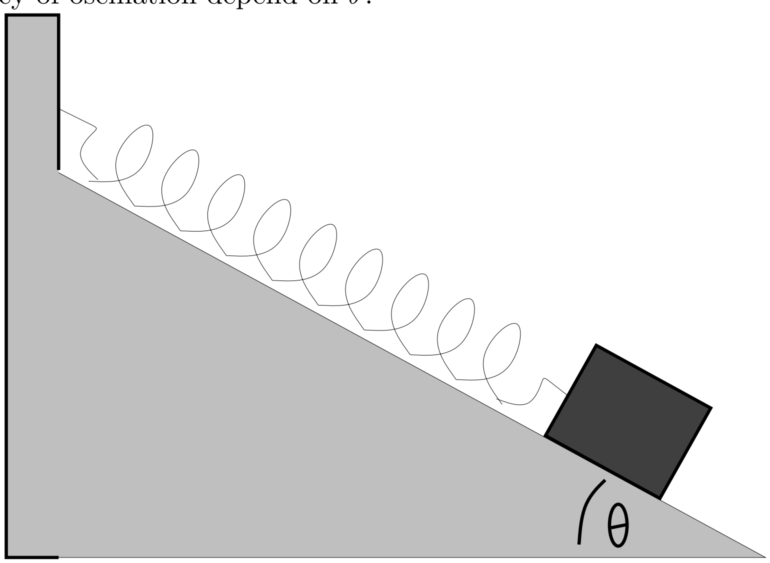
3. (20 points) A solid cylinder of mass M and radius R is pulled by a yoke connected to its axle as shown below. If the force applied to the axle is F, calculate the acceleration of the center of mass.



4. A light spring of spring constant k is fixed at one end to the top of a frictionless inclined plane as shown below. The other end is attached to a mass m.

(a) (10 points) By what length does the spring stretch when the mass is in equilibrium?

(b) (10 points) Find an expression for the frequency that the mass oscillates, after it has been displaced from its equilibrium. How does the frequency of oscillation depend on θ?



5. (20) Two identical planets are separated by a distance d much larger than their radii R. Initially they are at rest and due to their gravitational

attraction accelerate towards each other. Calculate their relative velocity right before they touch. Assume that both planets remain spherical all during this time.

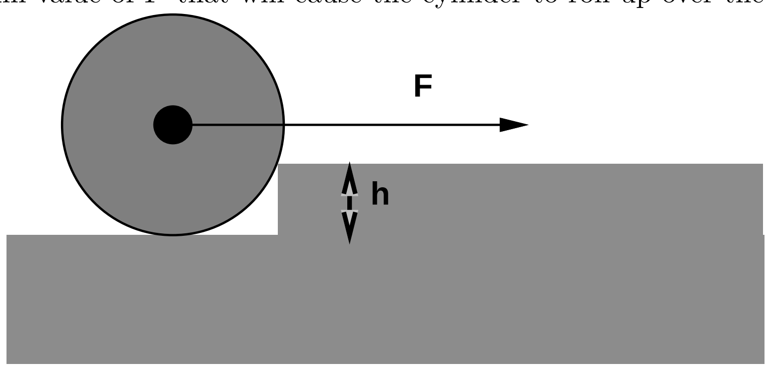
Useful information:

Moment of Inertia of a Solid cylinder rotated about center = $\frac{1}{2}MR^2$

2 Do two out of the following four problems. Only hand in work for two of these problems. You will only receive credit if you explain your answers and show your work.

6. (40 points) You throw a rock up a slope that is inclined at an angle θ with respect to the horizontal. What angle ϕ (with respect to the horizontal), should you throw the rock to get the maximum range? Remember to check your final expression in limits where you already know the answer.

7. (40 points) The cylinder and yoke of problem 2 comes up against a vertical curb of height h where $h < R$, as shown below. What is the minimum value of F that will cause the cylinder to roll up over the curb?

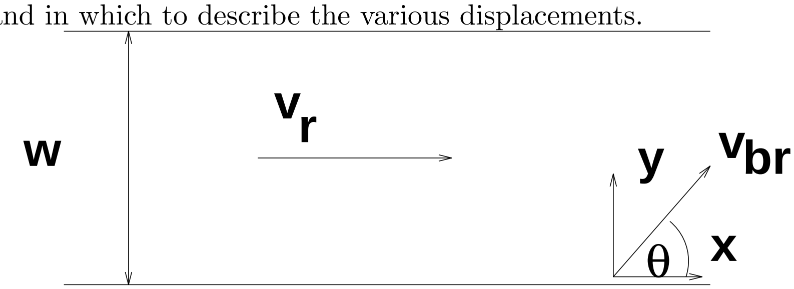


8. (40 points) A satellite is in a circular orbit of radius R around the earth. Small rockets aboard the satellite change its direction so that it has an elliptical orbit. The change causes the satellite to lose half of its orbital angular momentum but the total energy remains constant. In terms of R, what are the distances of closest and furthest approach from the center of the earth?

9. (40 points) A force acting in the xy-plane is conservative and has a potential energy function $U(x) = A(x^2 + y^2 + 2xy)$. Describe the motion of a particle of mass m that is at the point $x = y = 0$ at time $t = 0$ with initial velocity $\mathbf{v}_0 = v_0\hat{\mathbf{i}}$. *(Hint: choose a new set of perpendicular axes.)*

3 solutions for 1-5

1. A boat is required to traverse a river that is 150 m wide. The current in the river moves with a speed of $6km/h$. The boat can be rowed on still water with a speed of $10km/h$. Set up a coordinate system in the frame of the land in which to describe the various displacements.



(a) (10 points) Using this coordinate system, write down the position vector of the boat at time t, assuming the boat moves with uniform speed and that it leaves one side with its velocity vector making an angle θ in the reference frame of the moving river.

In reference frame of the river

$$\mathbf{v}_{br} = v_{br}(\cos\theta\hat{i} + \sin\theta\hat{j}) \tag{1}$$

The vector describing the velocity of the river with respect to land is

$$\mathbf{v}_r = v_r\hat{i} \tag{2}$$

So the velocity of the boat with respect to the ground is

$$\mathbf{v} = \mathbf{v}_r + \mathbf{v}_{br} = (v_{br}\cos\theta + v_r)\hat{i} + v_{br}\sin\theta\hat{j} \tag{3}$$

So the displacement vector is

$$\mathbf{r} = \mathbf{v}t = (v_{br}\cos\theta + v_r)t\hat{i} + v_{br}t\sin\theta\hat{j} \tag{4}$$

(b) (5 points) Calculate θ such that the boat lands at a point exactly opposite the starting point.

We want the x component of the displacement vector to be zero:

$$(v_{br}\cos\theta + v_r) = 0 \tag{5}$$

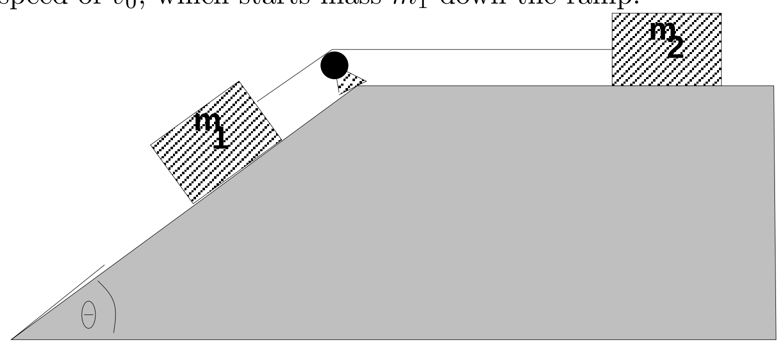
which implies

$$\theta = \cos^{-1}(-\frac{v_r}{v_{br}}) \tag{6}$$

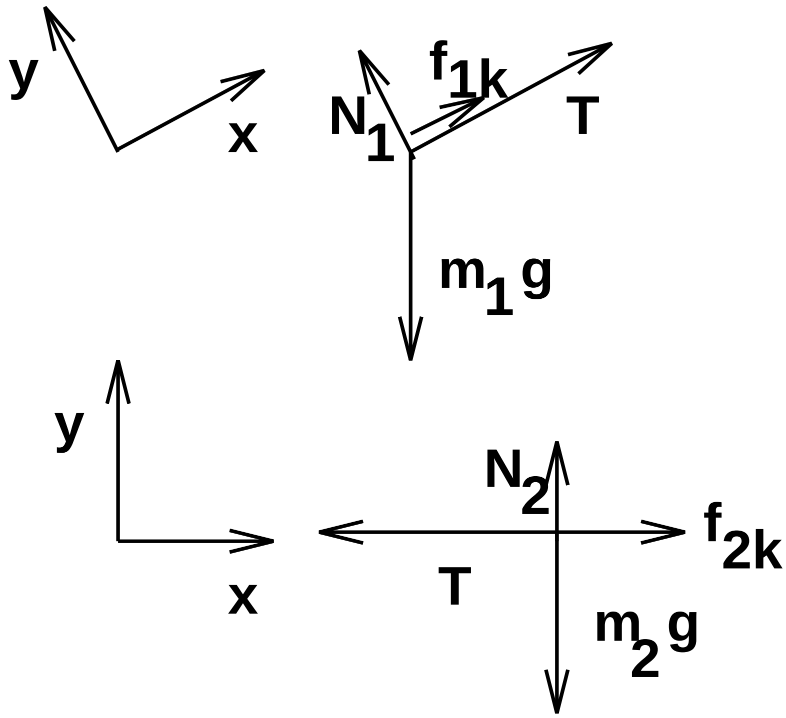
(c) (5 points) For part (b), how long will the trip take?
$w = v_{br} t \sin\theta$ so that

$$t = \frac{w}{v_{br}\sin\theta} \tag{7}$$

2. Masses m_1 and m_2 are connected by a taut rope. Mass m_1 is just over the edge of a ramp inclined at an angle of θ as shown below. The masses have a coefficient of kinetic friction of μ_k. At $t = 0$ the system is given an initial speed of v_0, which starts mass m_1 down the ramp.



(a) (5 points) Draw the force diagram for each mass.



(b) (15 points) Solve the equations of motion to predict the motion of the system as a function of time. Assume the rope is long enough so that mass m_2 does not go over the edge.

The magnitudes of the accelerations of the two masses are the same, as are the tensions.

For the first mass:

In the y direction :

$a_y = 0$. $F_{net,y} = ma_y = 0 = N_1 - m_1 g \cos\theta$

In the x direction:

$$T + f_{1k} - m_1 g \sin\theta = m_1 a \tag{8}$$

and

$$f_{1k} = \mu_k N_1 = \mu_k m_1 g \cos\theta \tag{9}$$

For the second mass:

In the y direction:

$m_2 a_y = 0 = f_{net,2} = N_2 - m_2 g$

In the x direction:

$$f_{2k} - T = m_2 a \tag{10}$$

and

$$f_{2k} \;=\; \mu_k N_2 \;=\; \mu_k m_2 g \tag{11}$$

Adding eqs. 8 and 10:

$$f_{1k} + f_{2k} - m_1 g \sin\theta \;=\; (m_1 + m_2)a \tag{12}$$

Solving for a:

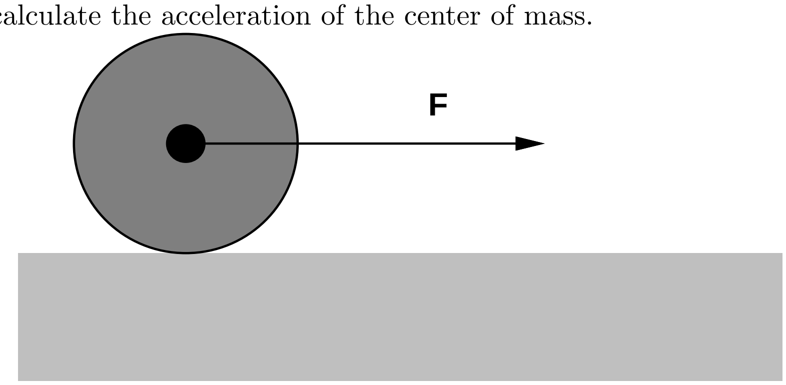
$$a \;=\; \frac{\mu_k(m_1\cos\theta + m_2) - m_1\sin\theta}{m_1 + m_2}g \tag{13}$$

The distance traveled, Δx, down the incline is

$$-v_0 t + \frac{1}{2}at^2 \tag{14}$$

with a as given above.

3. (20 points) A solid cylinder of mass M and radius R is pulled by a yoke connected to its axle as shown below. If the force applied to the axle is F, calculate the acceleration of the center of mass.



$F_{net} \;=\; ma$ where $F_{net} \;=\; F + f_s$. Therfore

$$F + f_s \;=\; ma \tag{15}$$

$\tau \;=\; I\alpha$ where $\tau \;=\; f_s R$, reckoned about the center of mass. Also $a = -\alpha R$. Combining this we obtain

$$f_s = -\frac{Ia}{R^2} \tag{16}$$

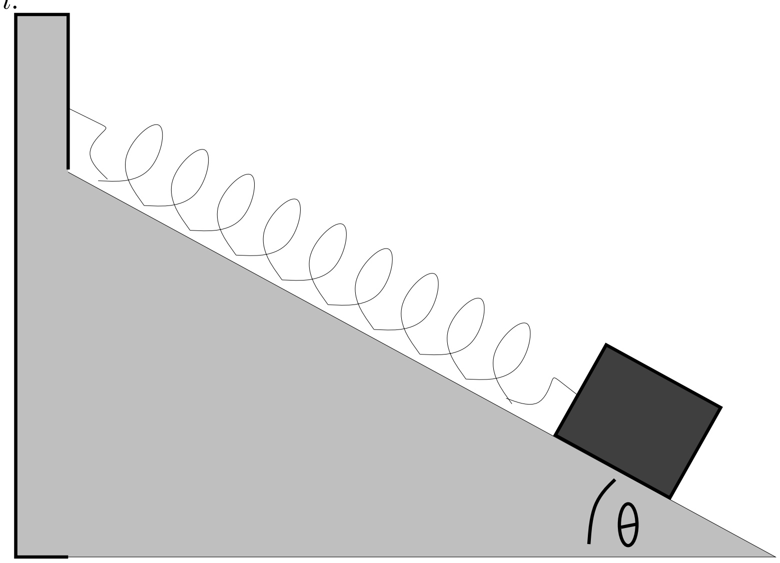
Now using eq. 15, we have $F - Ia/R^2 = ma$ which implies

$$a = \frac{F}{m + I/R^2} \tag{17}$$

But for a uniform solid disk, $I = mR^2/2$. Therefore

$$a = \frac{2}{3}\frac{F}{m} \tag{18}$$

4. A light spring of spring constant k is fixed at one end to the top of a frictionless inclined plane as shown below. The other end is attached to a mass m.



(a) (10 points) By what length does the spring stretch when the mass is in equilibrium?

In equilibrium the two forces acting on the mass along the direction of the incline must balance. The two forces are gravity which has a magnitude $mg\sin\theta$ and the force of the spring kx^*, where x^* is the equilibrium position. Equating these forces we obtain

$$x^* = \frac{mg}{k}\sin\theta \tag{19}$$

(b) (10 points) Find an expression for the frequency that the mass oscillates, after it has been displaced from its equilibrium. How does the frequency of oscillation depend on θ?

In general along the incline, $F_{net} = ma$ which means

$$mg\sin\theta - kx = ma \tag{20}$$

In terms of x^* this becomes

$$-k(x - x^*) = -k\Delta x = ma \tag{21}$$

This is the same equation as for a flat surface. Therefore

$$\omega = \sqrt{\frac{k}{m}} \tag{22}$$

or frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{23}$$

This is independent of θ.

5. (20) Two identical planets are separated by a distance d much larger than their radii R. Initially they are at rest and due to their gravitational attraction accelerate towards each other. Calculate their relative velocity right before they touch. Assume that both planets remain spherical all during this time.

The potential energy is

$$U = -\frac{Gm^2}{r} \tag{24}$$

The kinetic energy of the two masses is $K = 2\frac{1}{2}mv^2 = mv^2$. Energy is conserved, that is $K + U = const. = E$.

Initially $v = 0$, $r = d$ so

$$E = -\frac{Gm^2}{d} \tag{25}$$

Finally, $r = 2R$ so

$$mv^2 - \frac{Gm^2}{2R} = -\frac{Gm^2}{d} \tag{26}$$

Solving for v:

$$v = \sqrt{Gm(\frac{1}{2R} - \frac{1}{d})} \tag{27}$$

The relative velocity, v_{rel} between the two objects is $2v$

$$v_{rel} = 2\sqrt{Gm(\frac{1}{2R} - \frac{1}{d})} \tag{28}$$

4 Answers to 6-9

Good luck with these! Don't try and do them unless you understand more basic problems. These are really meant for those students that can do all of the homework problems with ease and want to be challenged!

6. $\phi = \theta/2 + \pi/4$

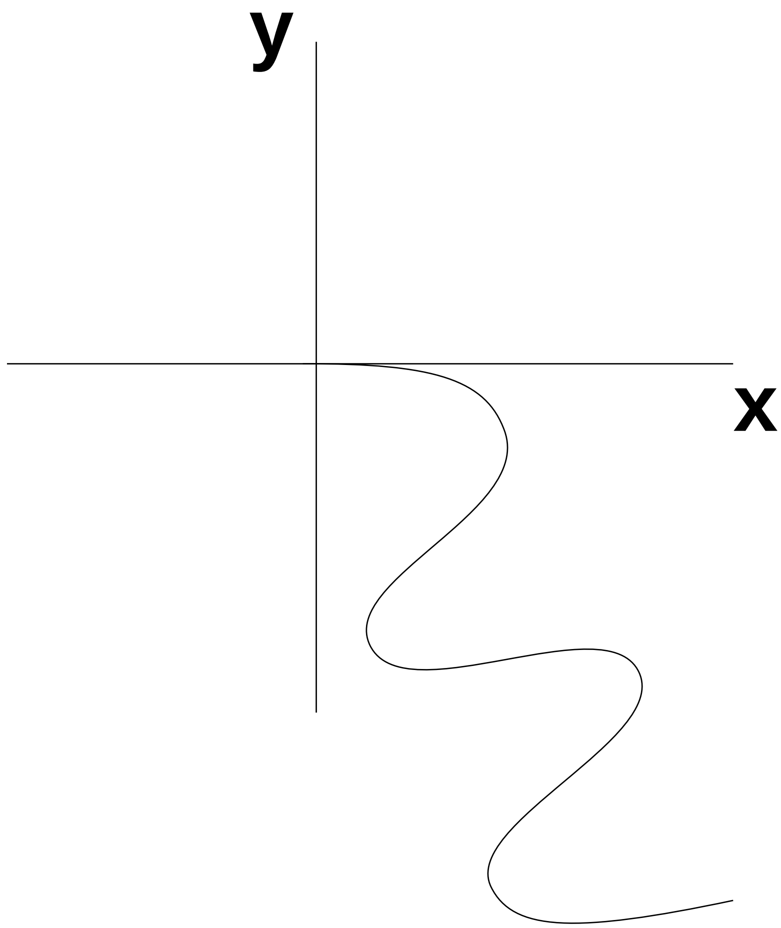
7.

$$F = mg\frac{\sqrt{R^2 - (R-h)^2}}{R-h} \tag{29}$$

8.

$$d_{\pm} = \frac{R}{4}\frac{1}{1 \pm \sqrt{3/4}} \tag{30}$$

9.


y
x