Contents

Chapter 1

Introduction

1.1 Overview

Physics tries to understand the world around us. It differs from other sciences such as biology and chemistry, in the way it approaches problems. It attempts to understand things from at a more fundamental level. If you want to ask a question in physics ask "why" enough times and you'll get a physics question. For example, the question

- "Why don't you want to try my fish sandwich?" *Because it smells bad*
- "Why does it smell bad?" *Because the fish is a bit old*
- "Why does old fish smell bad?" *Because fish decomposes, creating a chemical that smells bad*
- "Why do some chemicals smell bad?" *Certain chemicals bind to receptors inside the noise which send a message to the brain, which then interprets and decides among other things if it likes the smell or not*
- "Why do certain chemicals bind specifically to certain receptors?"

At this point, the answer will have to involve concepts in physics. A vague answer would be: *Because the forces between the electrons and nuclei that make up the molecules and the receptor sites favor them binding together.* To understand this in more detail, it is necessary to understand how electrons and nuclei interact with each other. Physicists have studied such questions quite extensively and give quite good answers to these questions.

If you get down to fundamental questions, such as how electrons interact, physics can give extremely accurate answers. However if you ask a more general question, like why you don't happen to think that *those* argyle socks go with *these* yoga pants, then physics is pretty hopeless. In principle such a question could be understood from a fundamental, physics point of view, but it would be far far too complicated to answer it. Your brain is make up of electrons, protons and nuclei so in principle it should be possible to understand it's workings if you knew exactly how your brain was constructed (which we don't).

In some sense, we're really quite lucky that things at a fundamental level are so easily understood. When I say easily understood, I don't mean that I can tell you the laws of physics, and then you'll understand all of physics. To do that would require a lot more math then you probably know at present. It would also require an intuitive understanding of physical concepts that take most people many years to acquire. All that I mean is that you can write down a set of equations, which in principle tell you how particles evolve in time. These equation are really not that complicated. You can write them down in a page or so. To solve these equations, in the case of the brain, or even in the case of three electrons and one proton, can be extremely difficult. *There is a big difference in knowing how in principle something works and figuring out what it's going to do.*

To start with we're going to study mechanics. It's a good subject to start with. It deals with the motion of big objects, such as baseballs, blocks, cars, or even people flying through the air. Big objects like this obey mechanics to a ridiculously good approximation. It's nicer to start with than electricity and magnetism because it's less mysterious. If you drop something you can actually see it fall, where as electricity is slightly more difficult to conceptualize. Hopefully you'll study electricity and magnetism later on, as it's pretty amazing.

Mechanics illustrates fundamental notions about physics. How starting from simple equations, you can deduce, with no other assumptions, how a system will behave. To do this, we'll first have to get a grip on "units".

Chapter 2

Units

2.1 Fundamental units

There are three fundamental units we'll deal with in this course: length, time, and mass. We'll see shortly how other quantities of physical interest can be made out of these three units.

You already have a concept of units. When someone asks the rather rude question "how tall are you?", you probably have an idea of what the answer is, in some units. If you live in the U.S. you'll know the answer in *feet*. If you live in most other parts of the world, you'll know the answer in meters. Scientists have tended to adopt the units of *meters*. There's no particularly good reason for this, the only thing that's good is that it's quite standard. When a scientist in Zimbabwe says something is 1.06 meters, scientists in Moscow will know exactly what is meant. If everyone had different measures of length, then it would make life far too complicated.

Same with time. There is nothing fundamental about the notion of a second, it is roughly one sixtieth of one sixtieth of one twenty fourth of the time it take the earth to spin around its axis once. It's completely historical. Nowadays, scientists have adopted some standard for what a second is to avoid any misunderstandings. You have a pretty good intuitive understanding of a second.

Mass is something that can be a little trickier to understand. When you run around on the moon, you'll notice that you seem to be able to jump higher. You actually weigh less. Does that mean to say you have less mass? Well actually no. You have the same mass (for all you afficianados of relativity out there, keep a lid on it!). You have the same mass in either case. We'll talk more about mass when we discuss Newton's laws. Mass is measured in kilograms. Mass is normally measured in kilograms. If you like working with feet instead of meters, you should know that the pound measures *force* instead of mass; you can use the associated unit of mass called a "slug", but they're slimy and I wouldn't advise it. (Like most aspects of British units, things are actually a bit more murky, but this is good enough.I'm also tired of all these UCSC slug jokes so I won't make any more stupid puns for the moment.

Here we'll mostly deal with "**SI**" units, kilograms (kg), meters (m), and seconds (s).

2.2 Other quantities

OK, now we're in a position to understand how to make other units. A good place to start is with the idea of density. Density measures, well, how dense something is. If you take a bunch of cotton candy, you'd say it has a low density because it takes up a large volume, but doesn't have much mass. So density should involve the ratio of mass to volume. We physicists like Greek symbols to make all this stuff seem more sophisticated. Here it goes: the density ρ is defined as the ratio of mass m to volume V.

$$\rho = \frac{m}{V} \tag{2.1}$$

The units of density are what? The units of mass over the units of volume. What's the units of volume? Volume always involves a length cubed. For example, the volume of a cube of side L is $V = L^3$. The units of volume are always the same, independent of the shape of the object and are $length^3$. So this means that now we can express the units of density in terms of two fundamental units as $mass/length^3$. To save writing and typing, just write this as M/L^3, ... much easier. If you are working with SI units, as we mostly will be, you can say that the unit of density is kg/m^3.

OK. That was simple enough. Now how about things with units of length + mass? Is this a reasonable physical quantity? The answer is no. You never get units like this occuring. Length × mass occurs a lot however. The reason is that physics is ultimately about predicting what's going on in the *real world*, not about fiddling around with a lot of equations. If someone says, I'm five feet and twenty seconds tall, you'll think they're mad. To make a long story short, *adding different units together doesn't make sense.*

Let's talk about what the units of force are. We'll see later on that the force equals mass times acceleration. We'll also see that acceleration has the units $length/time^2$ So we have

$$units\ of\ force\ =\ units\ of\ mass \times length/time^2\ =\ ML/T^2. \tag{2.2}$$

We can write this in SI units as kg m/s^2. Because this combination occurs so often, we give it a new name, "one Newton" or 1 N. A force of 5 kg m/s^2 is the same as a force of 5 N.

2.3 Conversion of units

Here we'll go through a number of examples of conversion of units.

2.3.1 Length conversion

Suppose you're six feet tall and want to know your height in inches. Well, 1 ft = 12 inches. I've used the standard shorthand "ft" for foot (or feet). So 6 ft = 6 x (12 inches) = 72 inches. This is really all there is to converting units.

2.3.2 Speed conversion

Let's do a slightly more complicated example. Suppose you want to convert the speed 60 miles/hour into the units ft/s (feet per second). How do you do that? Well 1 mile = 5280 ft, and 1 hour = 60 minutes = 60 x 60 s. Putting this together, we have

$$\text{60 miles/hour} \;=\; 60\,\frac{5280\text{ ft}}{60\times 60\text{s}} = \frac{5280}{60}\frac{\text{ft}}{\text{s}} = 88\text{ft/s} \tag{2.3}$$

2.3.3 Density conversion

Another example is the density of water. The density of water at "standard temperature and pressure", is 1g/cm^3 (i.e. 1 gram/centimeter^3). What is the density of water in SI units, i.e. measured in kg/m^3? To figure this out, you need to realize that one *kilo* gram = *1000* grams, and 1 meter = 100 centimeters.[1] So putting this together means that

$$1\text{g/cm}^3 \;=\; 1\frac{(1\text{kg}/1000)}{(\text{m}/100)^3} \;=\; 1000\text{kg/m}^3\ . \tag{2.4}$$

When you work with lots of equations, one easy way to get the wrong answer is to mess up with the units. Lets do an example. What is the mass of a sphere of water of radius *1 cm*? Well eq. 2.1 tells us that the mass $m \;=\; \rho V$. What is V? You might recall that the volume of a sphere is $V \;=\; 4\pi r^3/3$, where here the radius $r = 1$cm. So we have

$$m \;=\; \rho\frac{4}{3}\pi r^3 \tag{2.5}$$

So what do we do now? Plug and chug! Take the value of the density $\rho = 1000kg/m^3$ and the radius $r = 1cm$ and plug it in. So what'll we get?

$$m \;=\; 1000\frac{4}{3}\pi(1)^3 = 4189 \tag{2.6}$$

4189 what??? Is the answer kilograms, grams, or what? It better not be kilograms because our intuition should tell us that a sphere of water that size could not be anything like that much mass! What we did was very sloppy and

[1] One nice thing about SI units is that one can always use a "kilo" (k) prefix to multiply a unit by 10^3 or a "mili" (m) prefix to multiply a unit by 10^{-3} (even though no-one uses some combinations such as a kilosecond). Compare this to inch – foot – yard – furlong – mile in the British system!

gave us the **wrong answer !!**. To do this the right way, we need to put in the units. Mass and length aren't just numbers, they have units also. So what is really the correct way to do this is to say

$$m = (1000\text{kg/m}^3) \times \frac{4}{3}\pi(1\text{ cm})^3 = 4189\frac{\text{kg}}{\text{m}^3}\text{cm}^3 \quad (2.7)$$

Then we need conversion between cm and m which is 1 m = 100 cm. This gives

$$m = 4189\frac{\text{kg}}{(100\text{cm})^3}\text{cm}^3 = 4.189 \times 10^{-3}\text{kg} \quad (2.8)$$

Alternatively we could have been consistent and have done the whole thing using SI units. So we would write $r = .01m$, which is in SI units. So now

$$m = 1000\text{ kg/m}^3 \times \frac{4}{3}\pi(.01\text{m})^3 = 4.189 \times 10^{-3}\text{kg} \quad (2.9)$$

Notice that the units of m^3 cancel out leaving the unit of kg. This is the correct unit for mass. If instead we had obtained the units kg m^2, then we would have had to have made a mistake some place. This shows the importance of always **checking units**.

2.4 Dimensional analysis

Dimensional analysis is a very powerful way of reasoning about problems that can give *order of magnitude* estimates for quantities of interest. For example, if we wanted to get a rough estimate of the mass of the spherical drop of water in example 2.3.3 then we can get it to within a factor of ten as follows. That's good enough for us at present. We've decided to wave our hands around a lot. Why is such an answer useful? Often you're not even that sure of the size of the droplet and you just want to know if something is about 10^3 kg or 10^{-3} kg.

2.4.1 Density

This is an example that might be so easy that you don't see the point. Read this and then the next one. Then if you go back to this example again, and it might help.

First we want to know what are the important variables in the problem. There's the density ρ, which has units M/L^3 (mass per length cubed), and also r radius, which has units of L (length). We want to figure out how the mass is related to these two quantities. We guess that the mass should equal some constant times $\rho^p L^q$. p and q are unknown numbers that we'll try to determine. If we are successful in determining them, then we'll almost have the answer. You might be wondering why we made this guess. It'll become clearer as we work through this example. Unfortunately there's that pesky constant that we won't be able to get, but most of the time such constant don't differ from unity by more than a factor of 10.

OK so how do we determine p and q? We can do so by keeping track of units. We write

$$m \ = \ constant \times \rho^p L^q \propto \rho^p L^q \tag{2.10}$$

Here we've introduced the proportionality symbol "$\propto$". This is useful for situations of this kind when we don't want to be bothered by constants. The right hand side has units of $(M/L^3)^p L^q$ and the left hand side has units of M. We better have the same units on the left and right hand side, which then says that

$$M \ = \ (M/L^3)^p L^q \ = \ \frac{M^p}{L^{3p}} L^q \ = \ M^p L^{q-3p} \tag{2.11}$$

The left hand side $M \ = \ M^1 L^0$. The only way of getting the left and right hand sides to match is to have $p = 1$ and $0 \ = \ q - 3p$. Solving this gives $q = 3$.

So now we have p and q and so plugging this into Eq. 2.10 gives $m \ = \ constant\ \rho r^3$. Note that Eq. 2.5 is of the same form, but there we calculated the constant to be $4\pi/3$. So if we just ignore the constant, that is set it equal to one, then we end up off by roughly a factor 4, which gives the right answer to within an order of magnitude.

2.4.2 Centripetal acceleration

Later on, we'll discuss something that you're all aware of, that when you go around in a circle, you feel an acceleration. To make things concrete, let's say we're going around a speed v in a circle of radius r. You could be on a merry-go-round going at a speed of $v = 10$ ft/s relative to the ground, and at a radius of 4ft from the center. To get an idea of the acceleration, we need to know its units. The units of acceleration are L/T^2, that is length over time squared, or in SI units m/s^2. But we want to figure out a formula that relates the magnitude of the acceleration a to v and r. Let me stress that we don't expect to get the *exact* formula. Our answer will be off, but this is still very useful.

The important thing here is that we expect the acceleration to depend on only v and r, so then we can write

$$a \ = \ constant \times v^p r^q \tag{2.12}$$

Why? Well, what else could it be? It couldn't be something like $v^2 + r^3$ because this wouldn't have sensible units!. It really can only be as we guessed, where p and q are unknown. But if we thought that a depended on something else, like the distance between the Mars and Jupiter, then we couldn't write down this simple form; dimensional analysis wouldn't be anything like as useful. But fortunately, unless you really believe in astrology, it would seem quite unlikely that this is the case.

So following the same procedure as we did for density, we can write Eq. 2.12 in terms of the units of all these terms:

$$L/T^2 = (L/T)^p L^q \tag{2.13}$$

and simplifying this we have

$$L^1T^{-2} = L^{p+q}T^{-p} \tag{2.14}$$

The exponents for L and T on the two sides of the equation must match, so for T:

$$1 = p + q \tag{2.15}$$

and for L:

$$-2 = -p \tag{2.16}$$

So we have that $p = 2$ and $q = -1$. So plugging this back into Eq. 2.12 we finally get

$$a = constant \times v^2/r \tag{2.17}$$

Later on we'll see that this is the correct answer with *constant* $= 1$. Of course we shouldn't expect this to constant to be 1. For example, we could have substited diameter instead of radius in Eq. 2.12, and this wouldn't have made any difference to our argument, but then the constant would end up being 2.

You could say "what's the point of this, if you don't know that constant, if it could be anything?". Again, usually the constant is going to be around 1. Maybe it'll be 12.3, or 0.2. So this then gives you a rough estimate.

But secondly, and more importantly, this tells you about the relative acceleration. Suppose you asked what would happen if you doubled the speed? As an exercise you should be use Eq. 2.17 to see that with any constant you choose for that formula, the acceleration will go up by a factor of 4. A lot of times, that's very useful information to have.

In this example you should be able to see that with almost no physics used, just the units of quantities and sensible physical assumptions, that you can get out a lot of valuable stuff!

2.4.3 Turbulence

When liquid or gas flows through a pipe at a high velocity, it is turbulent: the flow is choppy and irregular, and changes from instant to instant. At lower velocities, the fluid moves smoothly down the pipe. How fast does the flow have to be for turbulence to occur? The only quantities that the answer can depend on are the fluid velocity v, the fluid density ρ, the diameter of the pipe D, and the *viscosity* μ of the fluid, which can be shown to have units of kg/m s. When some function of these four quantities is sufficiently large, i.e. bigger than some number, turbulence is seen. *No-one has been able to calculate what this number is!* But since numbers have no units, we have to find a combination of ρ, v, D and μ that also has no units. The units of $\rho v^p D^q \mu^r$ are

$$(M/L^3) \times (L/T)^p \times L^q \times (M/LT)^r = M^{1+r} L^{p+q-r-3} T^{-p-r}. \tag{2.18}$$

Since the combination has no units, $r = -1$, $p = 1$ and $q = 1$. Thus the condition for turbulence is that $\rho v D/\mu$ should be greater than some number, which is experimentally found to be approximately 2000.

While there is no way to calculate this number, the discussion about units already tells us, for example, that if the same fluid is flowing down a pipe of half the diameter, its speed has to be doubled in order to see turbulence. Such analyses are very important in aeronautical engineering, where one can test a small model of an airplane in a wind tunnel at high air velocity, to mimic the behavior of a full-size airplane flying under normal conditions.

Chapter 3

Motion in one dimension

In this chapter, we'll introduce the concepts of velocity and acceleration and apply them to simple situations. By one dimension, one means motion along a line, or in one particular direction. Think of a car going down a straight road, or a person running on a straight track. You could also think of an object being thrown up vertically in the air and watching it fall. It might seem a bit boring at first to work strictly in one dimension. Why not study what happens when a ball is thrown at some other angle besides straight up? Well it's true that its motion is more interesting, but it's also more complicated. One thing that you should try to learn in this course is that it's best to start with something simple and work your way up. We will discuss the more general case of motion in three dimensions, but in order to understand that, it's best to understand motion in one dimension first.

3.1 Definition of velocity

First we'll define average velocity and use that to understand the notion of instantaneous velocity

3.1.1 Average velocity

Suppose you're cruising down the highway and you go *60* miles in *1* hour. Then your *average velocity* is 60 mi/hr. Now we are going to go through this more formally as follows. Say we measure everything along a line from point. That is we were driving along a straight road and we had set our odometer to zero in San Jose. Now it reads 15 miles, and we look at our clock and it says that it's 9 A.M.. Introducing variable names to describe this, our initial position x_i equals 15 miles, and our initial time t_i equals 9 hours Later on we look at the odometer and it reads 75 miles, and our clock reads 10 AM. So we can introduce two other sets of variables to describe this. Our final position x_f equals 75 miles, and our final time t_f equals 10 hours.

Why bother to go to all the trouble of inventing four variable names? It seems like a pretentious way of saying something quite simple. Well the reason is that physics is much easier dealt with in terms of mathematical equations. If we can translate everyday happenings into a precise mathematical formulation, then we'll see that it's possible to do pretty amazing things! So just put up with this for the moment, and later on you'll see that it is indeed quite useful.

So now we are in a position to define the average velocity in one dimension $\bar{v}$. It is the ratio of the change in position $x_f - x_i$, to the change in time $t_f - t_i$.

$$\bar{v} \equiv \frac{x_f - x_i}{t_f - t_i} \tag{3.1}$$

Often as a shorthand, we'll write $\Delta x \equiv x_f - x_i$, and $\Delta t \equiv t_f - t_i$. So the Greek letter Δ can be thought of as meaning "the change in". In this way, our definition of average velocity can be written more succinctly as $\bar{v} = \Delta x/\Delta t$.

3.1.2 Instantaneous velocity

Now the idea of average velocity is something that is fairly straightforward, but the idea of *instantaneous velocity* is a little trickier. It really requires calculus to fully appreciate, but hopefully you already know what a derivative is, so this shouldn't be too hard.

Suppose the velocity of the car is varying, because for example, you're in a traffic jam. You look at the speedometer and it's varying a lot, all the way from zero to 60 mph. What is the instantaneous velocity? It is, more or less, what you read on the speedometer. I'm assuming you've got a good speedometer that isn't too sluggish and can change its reading quite quickly. Your speedometer is measuring the the average velocity but one measured over quite a short time, to ensure that you're getting an up to date reading of your velocity.

So if you measure the displacement of the car Δx over a time Δt, you can use that to determine the average velocity of the car. What we want is to take the limit as Δt goes to zero. More formally, the instantaneous velocity v is defined as

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \tag{3.2}$$

Most of the time we'll be working with instantaneous velocity, so we'll just drop the instantaneous, and call the above v the velocity.

To justify that such a limit exists is something that you've hopefully had to grapple with already. For physics problems, this limit does indeed exist and gives the derivative:

$$v = \frac{dx}{dt} \tag{3.3}$$

We can go through how this limit works out in the following example.

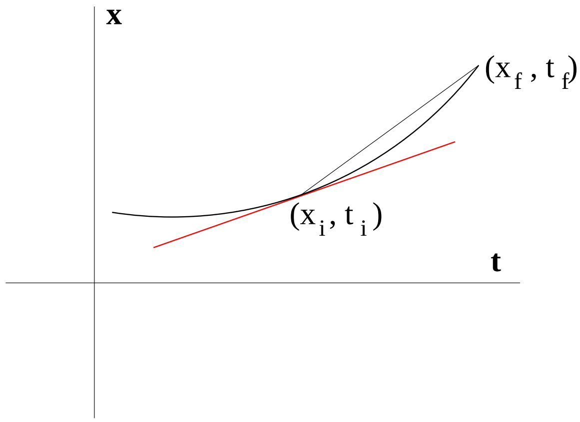


Figure 3.1: The curve is a plot of the position x as a function of time t. The average velocity between t_i and t_f is the slope of the black straight line. The instantaneous velocity at t_i is the slope of the red line, which is tangent to the curve at time t_i.

Example of a limit

Suppose you know that the position of an object depends on time as $x = t^2$ (here I'm being naughty and forgetting about units for the moment). Lets calculate the instantaneous velocity at $t = 1$.

So in this case $t_i = 1$, and $x_i = 1^2 = 1$. We'll want to try different values of t_f and verify that we do appear to converge to a sensible final answer.

Let's start with $t_f = t_i + 1 = 2$, then $x_f = t_f^2 = 4$ so from eq. 3.1 we have, $\bar{v}$ over this time interval is

$$\bar{v} = \frac{x_f - x_i}{t_f - t_i} = \frac{4-1}{2-1} = 3 \tag{3.4}$$

OK that's fine, but this is clearly not an infinitesimal interval. Let's shrink the interval by 1/2 so that $t_f = 1\ 1/2$. Then

$$\bar{v} = \frac{x_f - x_i}{t_f - t_i} = \frac{9/4-1}{3/2-1} = 5/2 \tag{3.5}$$

If we shrink the interval Δt even further, so that $t_f = 1.1$ then going through the same steps gives $\bar{v} = 2.1$. If we now try $t_f = 1.001$, then $\bar{v} = 2.001$.

It looks pretty clear that as $\Delta t \to zero$ we're coming up with an instantaneous velocity of 2.

This is what you'd expect since the derivative of t^2 is $2t$. Evaluating this at $t = 1$, we get 2.

3.1.3 Speed

Speed is defined as the absolute value of the velocity. In other words, If you're driving the wrong way down the freeway, then besides it being a really stupid and dangerous thing to do, your velocity has reversed sign. Your speed could quite possibly remain unchanged.

When it says the speed limit is 65 mph, you can't get around it by driving in the reverse direction. If they had posted a velocity limit instead then this trick would work.

3.2 Acceleration

Acceleration is another concept that you should have an intuitive feel for. If you're accelerating you actually *can* feel it. The acceleration is the rate of change of velocity with respect to time. So when you're at a stop sign and you step on the gas, you kind of feel pinned to your car seat. If you suddenly step on the brakes, then you lurch forward. At least that's the way it appears in the movies. We're real safe drivers here.

So now we'll go through similar steps to our above exposition of velocity, and as a consequence, I'll cut down a bit on the verbiage.

3.2.1 Average acceleration

The average acceleration is defines in analogous manner to the average velocity. It at some time your velocity is 30 ft/s, and a second later its 40 ft/s, then your average acceleration is 10 ft/s^2.

Again, in terms of equations

$$\bar{a} \equiv \frac{v_f - v_i}{t_f - t_i} \tag{3.6}$$

Here v_f is the final velocity measured at time t_f and v_i is the initial velocity measured at time t_i.

3.2.2 Instantaneous acceleration

OK, I promised to cut out the crap, so here it is. The *instantaneous acceleration* is defined as

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \tag{3.7}$$

As with instantaneous velocity, we'll spare people's aching ears by referring to the instantaneous acceleration as just the acceleration.

Now using eq. 3.3 we then see that

$$a = \frac{dv}{dt} = \frac{d\frac{dx}{dt}}{dt} = \frac{d^2x}{dt^2} \tag{3.8}$$

So the acceleration is the second derivative of the position with respect to time.

3.3 Motion with constant acceleration

There are many important examples where an object experiences constant acceleration, most notably an object in “free fall”. That is, an object that has been thrown into the air and is not traveling so fast that air resistance becomes important.

In the case of free fall on the surface of the earth, the acceleration is approximately $-9.8 m/s^2$, or $-32 ft/s^2$. This is for objects moving vertically up or down, so our coordinate x is now in the vertical direction. (Actually we'll tend to use y to denote motion in this direction. Note that one can use any symbol that one pleases, such as $\heartsuit$, but that would be silly).

We'll study the case of motion with constant acceleration in some detail, first in one dimension, and then in two and three dimensions.

3.3.1 Velocity as a function of time and initial conditions

So now we get to see the use of all this math that we've been doing! Take the definition of acceleration eq. 3.7 and turn it around. Notice that now a is a *constant* (in time).

So we want to solve this equation for velocity v. v will be some function of time (because the system is accelerating), and we want to know what it is. So how do we solve

$$\frac{dv}{dt} = a \quad ? \tag{3.9}$$

Such an equation is a simple example of a *differential equation*, because it involves a derivative and also an equals sign. That doesn't help us much in solving it, but at least we get to impress our friends with all this fancy math terminology. But to actually solve it isn't beyond the reaches of the human intellect.

- The above equation says that the derivative of a function of time is a constant. So what function has a constant derivative? Have you got it yet?
- Another way of saying it is: what function when differentiated yields a constant, in other words, what is the anti-derivative of a constant?

From either way of looking at it you get, a straight line

$$v(t) \;=\; at + C \tag{3.10}$$

Now I wrote $v(t)$ because I wanted you to notice the the velocity depends on time, and not say, the color of the socks you're wearing. You don't have to write it this way, you're perfectly all right if you write it as just v, but then you have to remember what you're trying to solve for: how the velocity varies with time. Now C is an arbitrary constant. It could be anything with the information provided. So this is a bit strange. We're saying we don't know exactly what the solution to the equation is! The answer has this unknown constant floating around.

How can we determine that constant? Well the answer is, with the information given, we can't. In order to determine the constant, we need some additional piece of information.

A common type of problem is when you're given the initial value of a quantity at some time, say $t = 0$. This is called being given an "*initial condition*" . For example, you may say that initially the velocity of the object was $1.2m/s$. We'll see now that with an initial condition, the constant can be determined and we have a unique solution to this equation, in other words we have completely determined its velocity as a function of time.

The strategy here is to apply the initial condition to what we've just figured out, that is eq. 3.10. At $t = 0$ we're saying we know that $v = v_0$, some given initial velocity. So applying this we've got $v_0 = a0 + C \ = \ C$. So now we know C. It's just v_0! Plugging this into eq. 3.10, we have

$$v(t) \ = \ at + v_0 \tag{3.11}$$

You should be able to see that this answer makes sense intuitively. At $t = 0$, the velocity starts out at v_0, as we wanted it to. Then as time goes on, it increases linearly at a rate a.

Applying an initial condition, like we just did, is an important part of mechanics, and more generally the study of differential equations. It might seem weird at first, but you soon get used to it. The point is that depending on the initial conditions of a system, you're going to get different subsequent motion. If you toss a ball up into the air, then you're giving it one initial condition. If you throw it towards the ground, you're giving it another. What you see the ball do is different in these two cases, because they have two different initial conditions.

3.3.2 Position as a function of time.

How do we find out how the position of a object varies with time, when it is moving under the influence of constant acceleration?

Well we can start by using how the velocity depends on time. How does it? We just figured it out, it's Eq. 3.11. So how can we get from v to x? We use the definition of instantaneous velocity, eq. 3.3. As in the above, we just turn around this equation, in symbols:

$$\frac{dx}{dt} \ = \ v(t) \ = \ at + v_0 \ . \tag{3.12}$$

So now we have a slightly more complicated differential equation to solve. Following the same line of reasoning as above, we ask: what function has as its derivative $at + v_0$? In other words, what is the anti-derivative of $at + v_0$? You can answer this question for both the at and the v_0 terms separately, and then just add the answers together. You easily see the correct anti-derivative is

$$x(t) \ = \ \frac{1}{2}at^2 + v_0 t + \ D \ . \tag{3.13}$$

Again we have some arbitrary constant D that we can't determine with the information given. We need yet *another* initial condition to determine what that constant is.

So we follow the same steps as above. We say that the position of the object at $t = 0$ is given, call it x_0. Then at $t = 0$ eq. 3.13 becomes $x_0 = 0 + 0 + D = D$. So now we know D. It's just x_0. Plugging this back into eq. 3.13 gives

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0 \quad . \tag{3.14}$$

So if you know the initial position, the initial velocity, and the acceleration, then you can determine the position of the object as a function of time.

3.3.3 Straightforward example

Suppose a camel on a bicycle has a constant acceleration of 1 m/s^2. You have a stop watch and when it passes some point you like to call "go", the camel has a velocity of $4m/s$. Where is it relative to "go" after $2s$, and how fast is it going at that time?

Don't look at the solution until you've thought about this for a bit.

Solution

Well we know that the answer to this has to do with the formulas we just derived, so that's a pretty big hint. Don't go on until you've thought about how to apply these formulas.

How do you apply these formulas? We first have to *translate* this word problem into mathematical problem.

- *Identify the relevant pieces of information in the problem.* Well most of what was stated was relevant, the only thing that was really irrelevant was the fact that the object that was moving was a camel riding a bike. The answer would of course be the same if it was say a tortoise riding a moped. We can see that the positional and velocity information in the problem affects the final result and is therefore relevant, Same with the time shown on the stop watch.

- *Give names to all the relevant variables.* Call the position and velocity of the camel at time t, x and v respectively. x measures the position of the camel relative to "go", so that when $x = 0$, the camel passes go. Also when the camel passes "go" the stop watch was started. So we identify this point with $t = 0$. Note that the position and velocity at $t = 0$ have been given: $x_0 = 0$ and $v_0 = 2m/s$.

- *Identify what equations describe the physical situation we are analyzing.* We have an object moving with constant acceleration and want to know its position and velocity at a time $t = 2s$. We noted above that we also know its initial position and velocity. Well the correct equations to use (well errr.... the only equations to use at this point) are eq. 3.11 and 3.14.

Now we have everything we need to solve this problem. Try to solve it now if you haven't managed so far.

With the values of v_0, a, and t as given we have

$$v = (1m/s^2)2s + 4m/s = 6m/s \tag{3.15}$$

and

$$x = \frac{1}{2}(1m/s^2)(2s)^2 + (4m/s)2s + 0 = 2m + 8m = 10m \tag{3.16}$$

Note the strategy we used in solving this problem. Even if you found this example to be really easy, you can learn something from the technique that was used to solve it. We divided the problem up into several simpler questions that we could answer immediately. When dealing with more complicated problems, you should keep your cool and try and do the same thing.

3.3.4 A little less straightforward example

OK, now suppose you have the same bicycling camel but you don't have a stop watch. Instead you can see the speedometer on the camel's bike. The camel passes "go" at the same initial velocity of $4m/s$ and its acceleration is still $1\ m/s^2$.

So you notice that at some point the camel is going at $6m/s$. At this point, how far is the camel from "go"?

Hint: look at the final answers in the solution to the last example.

Solution

We saw in the last example that at $t = 2s$, the velocity of the camel was $6m/s$. That's exactly the point in time we're interested in! How lucky! We already calculated the position at that time. It is $x = 10m$. That's all there's to it.

3.3.5 An even less straightforward example

Well now we won't be so lucky. The same set-up as the previous example, but now at some point we notice the camel going at $5m/s$. How far is the camel from "go" now?

Think about this for a while before looking at the solution.

Solution

Hmm, well it looked pretty easy before because we serendipitously knew the time t when $v = 6m/s$. But we don't know it now, so how can we solve the problem? Maybe we can figure out t when $v = 5m/s$?

So now we have a sub-problem that'll help us solve the main one:

- Given, the initial velocity of a camel, $v_0 = 4m/s$, a constant acceleration of $a = 1m/s^2$, at what time is the final velocity $v = 5m/s$?

  To solve this we want an equation relating the quantities v, v_0, a and t. Can you find such an equation? Sure, you already used it, its eq. 3.11 . We just need to solve this for t.

$$t = \frac{v - v_0}{a} = \frac{5m/s - 4m/s}{1m/s^2} = 1s \tag{3.17}$$

Fine, we solved this sub-problem, now we know t. So we have a problem just like the first camel example, section 3.3.3, except now we want to know the position of the camel when the stop watch reads $1s$, instead of $2s$.

Following the same logic as in the solution to that example (3.3.5),

$$x = \frac{1}{2}(1m/s^2)(1s)^2 + (4m/s)1s + 0 = 4.5m \tag{3.18}$$

3.3.6 Relations between other quantities

The last two examples, in sections 3.3.5 and 3.3.4 posed a different kind of problem then we considered before. We started off in section 3.3.2 by asking

- given a,x_0, v_0, and t, what is x?

The last two examples asked the question

- given a,x_0, v_0, and v, what is x?

Which is an equally legitimate question. Do we want to go through the same rigmarole as in the last example? It'd be better to find the general formula relating the above quantities a,x_0, v_0, v, and x. We almost had that formula but we blew it, I'm sorry to say. Yes, I'm going to get preachy again and mention yet another rule that you should always follow, (well maybe not always, but most of the time).

Do not substitute numerical values into formulas until you've got to the last stage of the problem. Do everything in terms of symbols, not numbers.

I'll illustrate that for you with the problem at hand. Eq. 3.17 gave the general expression relating t to v, v_0 and a. We screwed up getting a nice general formula by foolishly substituting in for the numerical values of quantities, giving us $t = 1s$. That wasn't necessary at all. We could have waited to do the substitution. Let's see what happens if we do that now.

Following the same logic we used in that example, the next thing to do was to substitute t into eq. 3.14. Instead of substituting in $1s$ let's substitute what's after the second equals sign in eq. 3.17, and keep all other quantities as symbols.

$$x = \frac{1}{2}a(\frac{v - v_0}{a})^2 + (v_0)(\frac{v - v_0}{a}) + x_0 \; . \tag{3.19}$$

This can easily be simplified. It's just a little algebra. You finally get

$$v^2 - v_0^2 = 2a(x - x_0) \tag{3.20}$$

There we are. A general formula relating a,x_0, v_0, v, and x! That's what we wanted. With this formula, you can solve a lot of problems similar to examples 3.3.4 and 3.3.5 quite simply.

Another reason why doing things in symbols is so important is that it allows you to check you answers to see if they make sense. Look at eq. 3.20. Check what happens if $a = 0$. Well then you see that the speed of the particle doesn't change. That sounds correct. How about if $x = x_0$? Yes the initial and final speeds are the same in that case also. You know in these limits, the answer is right. Also you can easily check that the units work out. This gives you some confidence that the formula is indeed correct. If you had instead plugged in the time as we did in the previous example (3.3.5), you couldn't check all these limits and would be far more likely to make some mistake.

It takes a while to get used to solving problems with symbols instead of numbers. It seems far too abstract for most people at first. You can get over this problem by using the same approach we did here. First solve the problem the more comfortable way by plugging in numbers as you go along. But after that *go back*, and go through the same steps but this time keep all your symbols. It's the same logic in both cases. You're just replacing a lot of multiplication and addition with algebraic manipulations.

Chapter 4

Vectors

Vectors aren't really necessary to understand physics but they're really cool and simplify understanding of a lot of problems enormously. Without vectors mathematical descriptions are much more cumbersome. A vector is something with both a magnitude and a direction. It is often thought of as an arrow like so: $\nearrow$. The length of the arrow is its magnitude, and obviously the direction that its pointing is the direction (duh). You can define operations on vectors analogous to the addition of regular (real) numbers. You can add and subtract vectors, and there are two common ways of multiplying them together.

Vectors are most commonly notated in books by using bold face. A vector named "A" would be notated **A** to distinguish it from a regular real number. Another common way to write vectors is to represent them by placing a little arrow right over the top of the letter (e.g. $\vec{A}$).

As I just said, it's nice to think about a vector as being an arrow, having a magnitude (the length of the arrow) and a direction. Two vectors (e.g. $\swarrow$ and $\swarrow$) that have the same length and go in the same direction are equal, even if they don't start off at the same point in space, as with the two arrows above. However they are different if either their magnitudes or directions differ (e.g. $\swarrow$ and $\nearrow$.

From a mathematical point of view, it is good to represent vectors by real numbers and that's what we'll talk about now.

4.1 Representation of vectors in Cartesian coordinates.

The most commonly used method to represent a vector is with Cartesian coordinates. The units can be anything, but to start with, we could just consider the "displacement vector" which describes the difference in position of two points in space. You just plonk your vector on top of an x-y grid and read off the numbers on the x and y axes. Like shown in Fig. 4.1

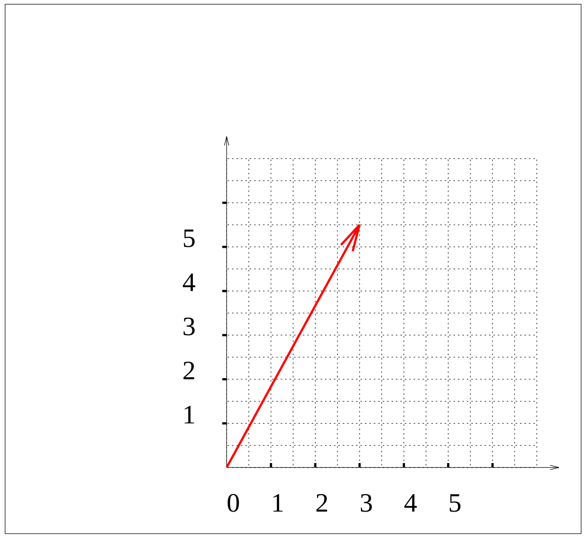


Figure 4.1:

So this (two dimensional) vector, call it **A** is represented by a pair of numbers. The first one is the x component of the vector, A_x, which you get by reading it off this figure. You can see that it's 3. The y component, A_y, can also be read of and is 5.5. So you could write this vector **A** as (3,5.5). If this vector was three dimensional (which is more difficult to draw), then it would be represented by three numbers.

But why did I place the grid the way that I did? Wouldn't I have been equally justified in plonking the vector on top of a grid going at some other angle, like so, as shown in Fig. 4.2?

Sure, that that seems fine too. After all this nothing better about one orientation rather than another. They're both equally valid coordinate systems. But when now I figure out the components of the vector, I see they're different: (5.4, 2.3).

Which one is right? They're both right. You just have to be clear what you're doing. Your specifying the components of the vector *with respect to* a particular coordinate system. So when you say $A_x = 3$, $A_y = 5.5$, it's with respect to a particular coordinate system that we plonked down. Those numbers are *meaningless* unless you specify what coordinate system you're using!

4.1.1 Scalars

When talking about vectors, sometimes it can get confusing because one also talks about real numbers like 2 , π or e at the same time. When mentioning good old regular numbers at the same time as vectors, it's useful to give these regular numbers a name to distinguish them from vectors. We call them scalars.

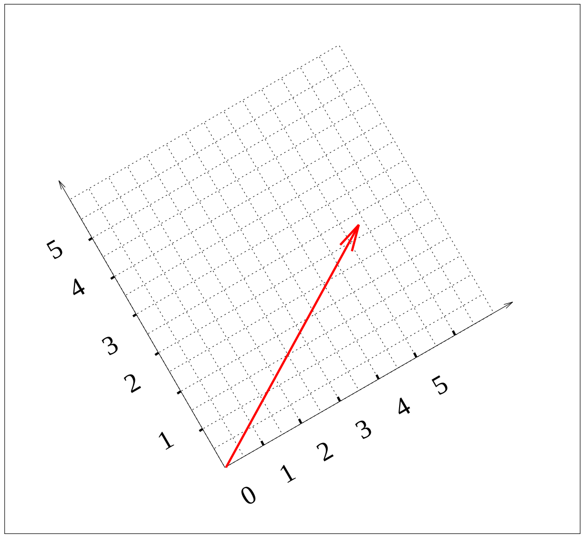


Figure 4.2:

Well, actually there's a bit more to it than that. I'm being awfully cavalier. To explain this subtle point, take as an example the length of the vector **A** above, so called $|\mathbf{A}|$. Note that when we rotated the coordinate system, $|\mathbf{A}|$ didn't change. So that's a scalar. Now if instead you just look at the x component of a vector, then we saw that this did change. So you wouldn't want to call A_x a scalar, though it is just a regular number. You might want to keep this subtlety in mind, though it probably won't come up very much here: a scalar is a number, or a quantity whose value is a number, provided that the number does not change if we rotate our coordinate system.

4.2 Representation of vectors in polar coordinates.

You can also use polar coordinates to represent a two dimensional vector. You just specify the angle the vector makes with respect to the x axis. In the case of the first coordinate system we used, the angle is 56 degrees as shown:

The the x and y components of the vector are also shown. From trigonometry we can relate the angle of 56 degrees, (call it θ for more generality) to the x and y components, $A_x = 3$, and $A_y = 5.5$:

$$\tan\theta = \frac{A_y}{A_x} \tag{4.1}$$

Then you have to talk about the length or "magnitude" of the vector. The magnitude of the vector can be denoted by the absolute value symbol $|\mathbf{A}|$.

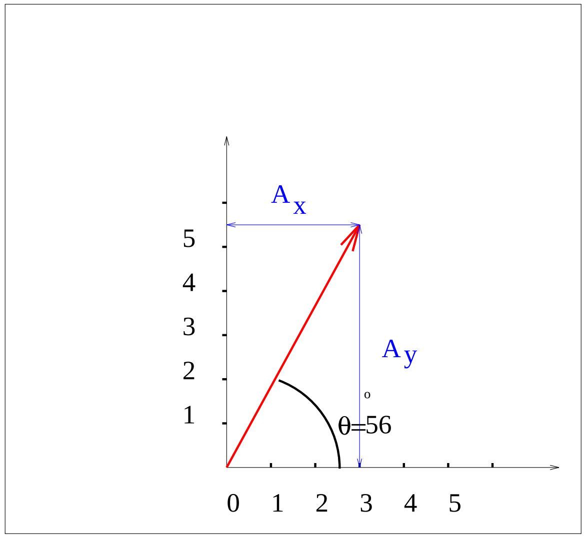


Figure 4.3:

Sometimes one doesn't want to write it as $|\mathbf{A}|$ because it looks too complicated, instead one can write it as A (not bold face any more). If you are writing it by hand, instead of writing $|\vec{A}|$ you'd also write A. So from good old Pythagoras we have

$$A = \sqrt{A_x^2 + A_y^2} \tag{4.2}$$

In three dimensions this is just $A = A_x^2 + A_y^2 + A_z^2$.

Sometimes this polar representation is a more useful way of understanding a final answer then by looking at A_x, A_y. You know immediately the magnitude of the vector, and the direction it's pointing in. But using x and y components tends to be the most useful way of wading through intermediate steps of a problem. We'll see that operations performed on vectors are easiest done in terms of these components.

We can go the other way and relate the x and y components of $\mathbf{A}$ to A and θ. Again, it's just trigonometry:

$$A_x = A\cos\theta \ , \ A_y = A\sin\theta \ . \tag{4.3}$$

4.3 Examples of vectors and scalars

We already briefly mentioned an example of one type of vector, the displacement vector, which describes the difference the relative position of two points in space, but the magnitude and direction.

Another example, which will talk about extensively in the next chapter is the velocity vector. The direction of the vector tells you the direction an object is moving, and the magnitude, its speed.

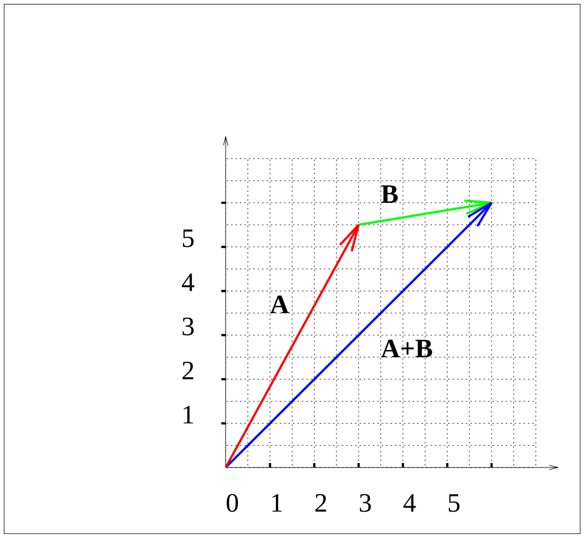


Figure 4.4:

A chapter after that, we'll start discussing forces. These are also vectors. The direction of a force acting on an object determines in what direction the force is pushing the object. Its magnitude tells you hard its being pushed.

An example of a scalar, is the speed of an object, like a bird or an electric cabbage. The energy of an object, something we'll talk about even later, is also a scalar. It doesn't matter what coordinate system you use to measure the energy. You always get the same answer.

I'm giving you lots of examples of things which are vectors. Well perhaps it'd be useful to give an example something that is *not*. OK well here is a completely random example. The x component is the number of socks that I am currently wearing (1), and the y component is the time showing on my watch (11:11PM). This is not a vector because if I use a different coordinate system, I don't find that these two numbers change. They should if this was a vector as we saw in section 4.1.

4.4 Addition of vectors

Pictorially vector addition is pretty straightforward. If you want to add the vectors **A** and **B**, you just take the second vector, represented by an arrow, and translate it so that it's tail is at the head of the first vector. $\mathbf{A} + \mathbf{B}$ is just a vector that starts at the tail of the first arrow and goes to the head of the second, so the whole thing, vectors **A**, **B**, and $\mathbf{A} + \mathbf{B}$, form a triangle, as shown in Fig. 4.4:

Now the nice thing about this definition is that it's easily seen that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, and also the rule for adding in terms of components couldn't be simpler. Look at the x components for example. What is the x component of the vector $\underline{A} + B$?. Its just the addition of the x components of the two vectors, $A_x + B_x$. This is easily seen by the following picture. We just show "project" the components of A, B, and A+B down close to the x - axis. You see there's a of length A_x, one of length B_x, and one which is the x component of $\mathbf{A} + \mathbf{B}$ which I labeled $(\mathbf{A} + \mathbf{B})_x$. You see from that picture that the total length of the red and green lines, is the length of the blue line. That's what we wanted to show. The same holds true in the y direction, so that the y component of $\mathbf{A} + \mathbf{B}$ is just $A_y + B_y$.

So vectors add nicely in terms of their x and y components. It's much messier in terms of the vectors' polar representation.

4.5 Multiplying vectors by real numbers

One would also like to have a good definition of multiplication. Well, there are a couple of common ways of multiplying a vector by a vector, but let's first ask how to multiply a vector such as $\mathbf{A}$ by a real number. For example what would be a good definition of $2\mathbf{A}$? Well you'd think it'd be good if $\mathbf{A} + \mathbf{A} = 2\mathbf{A}$. I think it's good too, so let's say that. Then by the procedure for addition discussed above, we see that we're taking one vector and adding an identical copy of it to itself, so the head of the first is at the tail of the second (I hope these vectors don't smell). This gives a vector going in the same direction but with twice the magnitude.

So in general that's what multiplication of a real number r by a vector $\mathbf{A}$ is going to do. It'll give us something going in the same direction with r times its original magnitude. In terms of components $r\mathbf{A}$ has x component rA_x and y component rA_y.

If r equals -1, then the resultant vector points in the opposite direction. For example if $\mathbf{A}$ looks like $\swarrow$, then $-\mathbf{A} = \nearrow$.

4.6 Subtracting vectors

From this it can be seen how to subtract to vectors. You just write $\mathbf{A} - \mathbf{B} = \mathbf{A} + -\mathbf{B}$. In terms of components, the x component of $\mathbf{A} - \mathbf{B}$ is $A_x - B_x$ and similarly for the y component.

4.7 Unit vectors

Sometimes it'll be convenient to deal with vectors that always have a magnitude of 1 but can have any direction. Such vectors are called "unit vectors". So take again our often used vector $\mathbf{A}$. If we want to make it into a unit vector, we have to construct something with the same direction but a magnitude of 1. We typically but silly little party hats on the top of unit vectors to make it clear to everyone that these are unit vectors. How the party hat notation got started beats me, but it does work out pretty nicely. Anyway getting back to the problem at hand, we'd call this unit vector $\hat{A}$. To actually write down a formula for $\hat{A}$ in terms of A isn't too tough. Just write

$$\hat{A} = \mathbf{A}/|\mathbf{A}| \ . \tag{4.4}$$

The right hand side means the vector **A** multiplied by the real number $1/|\mathbf{A}|$. Let's check that this is correct. First of all, does it have the right magnitude? From the definition of multiplication above, when we multiply the vector **A** by a real number number, in this case $1/|\mathbf{A}|$, the resulting vector has magnitude of $1/|\mathbf{A}| \times |\mathbf{A}| = 1$. That's correct. How about the direction? That's right too, because when you multiply a vector by any nonzero number, it also points in the same direction as the original vector.

So this indeed is the right expression for $\hat{A}$.

4.8 i, j and k

There are three particularly useful unit vectors often called either $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$ or $\hat{x}, \hat{y}$, and $\hat{z}$ depending on what book you read. When you plonk down your coordinate system, the vector $\hat{\imath}$ points along the direction of the x axis, and $\hat{\jmath}$ points along the direction of the y axis. Similarly for $\hat{k}$ and the z axis. And you already know because of the little hats that these are unit vectors. That's all to these cute little vectors. Why are they useful? Well it's mostly a notational thing. Suppose I want to write the vector **A** in terms of its components. Then using $\hat{\imath}$ and $\hat{\jmath}$ you can write (in two dimensions)

$$\mathbf{A} = A_x\hat{\imath} + A_y\hat{\jmath} \tag{4.5}$$

Why is this right? Well because $A_x\hat{\imath}$ is a vector kind of like $\rightarrow$ and $A_y\hat{\jmath}$ like $\uparrow$, so when you add them together you get the original vector **A**. Of course in three dimensions the equivalent formula is $\mathbf{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$.

So this is a very convenient way of representing a vector in terms of its components.

4.9 Scalar product

As I said before, there are two ways of multiplying vectors together that make sense, the *scalar product* and the *cross product*. We'll postpone the discussion of cross products until a time closer to when we'll need them which will be in a moon or two from now.

The scalar product is also referred to as the *dot product*. It takes two vectors, say **A** and **B** and multiplies them together forming a scalar. Scalars were defined above in section 4.1.1. This is often denoted by $\mathbf{A} \cdot \mathbf{B}$.

It is defined as follows:

$$\mathbf{A} \cdot \mathbf{B} \equiv A_xB_x + A_yB_y + A_zB_z. \tag{4.6}$$

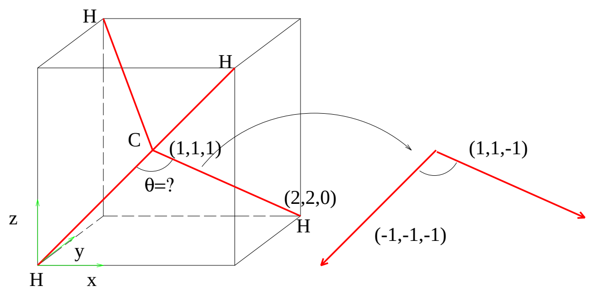


Figure 4.5:

How do we know that this is a scalar? Well, the square of the magnitude of any vector is a scalar. Therefore

$$|\mathbf{A}+\mathbf{B}|^2-|\mathbf{A}|^2-|\mathbf{B}|^2 = [(A_x+B_x)^2+(A_y+B_y)^2+(A_z+B_z)^2]-[A_x^2+A_y^2+A_z^2]-[B_x^2+B_y^2+B_z^2] \tag{4.7}$$

is a scalar too. Expanding out the right hand side, we obtain twice $\mathbf{A}\cdot\mathbf{B}$ as defined in Eq.(4.6).

Since $\mathbf{A}\cdot\mathbf{B}$ is a scalar, we can calculate its value in any coordinate system. In particular, if the x- axis is lined up with $\mathbf{A}$, we have $\mathbf{A}\cdot\mathbf{B} = A_xB_x = |\mathbf{A}|B_x$, i.e. the product of $|\mathbf{A}|$ and the projection of $\mathbf{B}$ on $\mathbf{A}$. (Similarly, $\mathbf{A}\cdot\mathbf{B}$ is also the product of $|\mathbf{B}|$ and the projection of $\mathbf{A}$ on $\mathbf{B}$.) From simple trigonometry, the projection of $\mathbf{B}$ on $\mathbf{A}$ is the magnitude $|\mathbf{B}|$ multiplied by the cosine of the angle between $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A}\cdot\mathbf{B} \equiv |\mathbf{A}||\mathbf{B}|\cos\theta \tag{4.8}$$

where θ is the angle between the two vectors. In particular, this means that $\hat{\imath}\cdot\hat{\jmath} = 0$ and so on, because the angle between the two unit vectors is 90 degrees, and that $\hat{\imath}\cdot\hat{\imath} = |\hat{\imath}|^2 = 1$, and likewise for $\hat{\jmath}\cdot\hat{\jmath}$ and $\hat{k}\cdot\hat{k}$.

This definition of the scalar product looks much nicer, involving simple geometrical (ok, trigonometrical) concepts. But the definition is Eq.(4.6) is often easier to work with, as we will now see.

4.9.1 Example: methane

In methane you have four hydrogen atoms bonded to a carbon atom. The four hydrogen atoms form the corner of a tetrahedron. What is the angle between different hydrogen atoms?

In Fig. 4.5 you see that the atoms of hydrogen can be put at corners of a cube of width 2. The carbon atom goes in the middle at coordinate (1,1,1). So we want to determine the angle θ.

We consider just two of these red bonds in the figure on the right, and calculate what displacement vectors these correspond to. They are $\mathbf{A} \equiv (-1,-1,-1)$ and $\mathbf{B} \equiv (1,1,-1)$.

To calculate the angle between these two vectors we just use the definition of dot product:

$$\mathbf{A} \cdot \mathbf{B} \;=\; |\mathbf{A}||\mathbf{B}| \cos\theta \;=\; \sqrt{3}\sqrt{3}\cos\theta \tag{4.9}$$

But we also know how to calculate the dot product by multiplying out the components. So $\mathbf{A} \cdot \mathbf{B} \;=\; (-1)(1) + (-1)(1) + (-1)(-1) \;=\; -1$.

Putting these two things together we have $\cos\theta = -1/3$, or $\theta \;=\; 109.47$ degrees.

Chapter 5

Motion in two and three dimensions

Now that we've learned about motion in one dimensions and about vectors, we can combine these ideas to understand motion in more than one dimension. We'll apply this to motion of projectiles. The name projectiles does not necessarily simply that you're dealing with some ghastly military device such as a cannon ball. It's just a general term, to describe anything moving freely through the air, such as a piano.

So we'll follow much the same path we did for motion in one dimension, defining velocity and acceleration in higher dimension.

5.1 Definition of velocity

First we'll define average velocity and use that to understand the notion of instantaneous velocity

5.1.1 The average velocity

Suppose you're a camel on a large flat plane, and start out 1 mile east of Timbuktu and 2 miles north. You trot for an hour and then find that you're 4 miles east and 6 miles north. You're average velocity in the west-east direction is (4-1) miles per hour which equals 3 miles per hour. Your average velocity in the south-north direction is (6-2) miles per hour which equals 4 miles per hour. This is your average velocity. In terms of vectors, you could define an x-y coordinate system with its origin in Timbuktu oriented so that the x axis was west-east. Then you could write the average velocity in vector form as $\bar{\mathbf{v}} = (3\hat{\imath} + 4\hat{\jmath})$ mi/hr. The speed of the camel is then just $|\bar{\mathbf{v}}| = \sqrt{3^2 + 4^2}$ mi/hr $=$ 5 mi/hr.

More formally we can denote the final position of the camel by the vector

$\mathbf{r}_f$ and the initial position by the vector $\mathbf{r}_i$. Then in analogy to eq. 3.1 we have

$$\bar{\mathbf{v}} \equiv \frac{\mathbf{r}_f - \mathbf{r}_i}{t_f - t_i} \;=\; \frac{\Delta \mathbf{r}}{\Delta t} \tag{5.1}$$

We can write this in terms of components as

$$\bar{\mathbf{v}} \;=\; \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} \tag{5.2}$$

5.1.2 The instantaneous velocity

Now we can define instantaneous velocity in analogy to the derivative definition in one dimension eq. 3.3. So we have

$$\mathbf{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} \;=\; \frac{d\mathbf{r}}{dt} \tag{5.3}$$

Writing this in components gives

$$\mathbf{v} \;=\; \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} \;=\; v_x\hat{i} \;+\; v_y\hat{j} \tag{5.4}$$

Here the subscripts x and y on the velocity denote the components in those respective directions.

So in words, the definition of the velocity vector is fairly easy to see. It just says to get the x-component of the velocity, you measure the rate the x-component of position changes in time. That's what we called v_x. The same idea holds for all the other components.

Again, these are just definitions. They don't contain any physics. We'll see that physical laws are nicely expressed in terms of these definitions and that's why we're defining them the way we are.

5.2 Definition of acceleration

This is very similar to what we just did with the velocity. So not to get too boring, I'll simply write write down the appropriate definitions:

5.2.1 Average acceleration

The average acceleration from initial time t_i to final time t_f depends on the initial and final velocities $\mathbf{v}_i$ and $\mathbf{v}_f$. as

$$\bar{\mathbf{a}} \;\equiv\; \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i}\;. \tag{5.5}$$

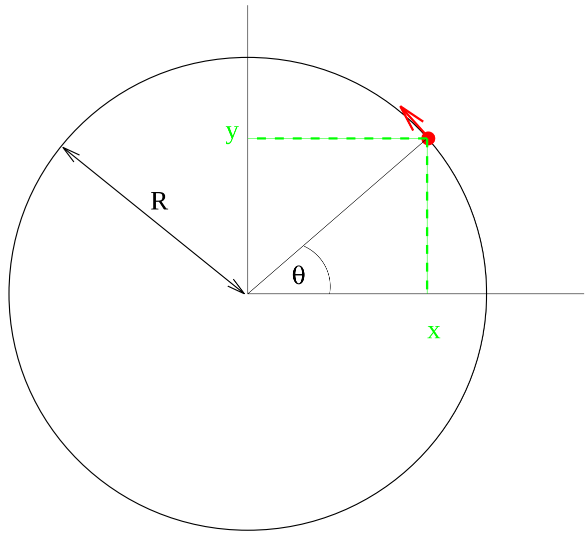


Figure 5.1:

5.2.2 Instantaneous acceleration

The instantaneous acceleration is also easily generalized from the one dimensional definition, eq. 3.7:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d\frac{d\mathbf{r}}{dt}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \ . \tag{5.6}$$

You're probably getting definitioned out, so we'll now go on to an application of how to use these definitions.

5.3 Circular motion

Suppose a centipede is bicycling around a circular track at a constant speed. We want to be able to describe the position, velocity and acceleration of this noble creature as a function of time. We'll call the radius of the track R and the the speed of the centipede v.

The angle θ varies linearly with time so we can say that $\theta = \omega t$ where ω is a constant. It's often referred to as the *angular velocity* since its the rate an angle changes with time. In the present example it is just some constant with dimensions of inverse time.

So now we can describe the position of the centipede in terms of its x and y components as seen in green in the figure. A little trigonometry gives the $x = R\cos\theta$ and $y = R\sin\theta$. So substituting $\theta = \omega t$ and expressing the result in terms of vectors:

$$\mathbf{r} = R(\cos(\omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath}) \tag{5.7}$$

It is important to note that we are measuring θ in *radians* here; it is only in radians that we can use standard results from differential calculus such as $d(sin\theta)/d\theta = \cos\theta$. Therefore ω is measured in radians per second. (Actually, since angles are related to the ratios of lengths, the radian has no units, and the unit for ω is s^{-1}.)

Good, now we have a well defined mathematical expression for the position of the centipede as a function of time. We can now differentiate it once to obtain the velocity

$$\mathbf{v} \;=\; R\omega(-\sin(\omega t)\hat{\imath} + \cos(\omega t)\hat{\jmath}) \tag{5.8}$$

So what does this say about the speed? We know how to take the magnitude of a vector right? Remember Pythagaros? (Did you know he worshiped beans?) So we have

$$|\mathbf{v}| \;=\; R\omega\sqrt{\sin^2(\omega t) \;+\; \cos^2(\omega t)} \;=\; R\omega \tag{5.9}$$

So $v \;=\; \omega R$. So that tells us how to relate this mysterious ω to v. Now we can differentiate the velocity again getting

$$\mathbf{a} \;=\; -R\omega^2(\cos(\omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath}) \tag{5.10}$$

But notice this is the same as $\mathbf{a} \;=\; -\omega^2\mathbf{r}$ (see eq. 5.7). That says that the acceleration points in the opposite direction to the radius vector $\mathbf{r}$. The *acceleration is pointing towards the center of the circle*. Its magnitude is just $a \;=\; R\omega^2$. But from eq. 5.9 this just says:

$$a \;=\; v^2/R \tag{5.11}$$

This is called the *centripetal acceleration*.

5.3.1 Question

Beethoven asks:

- I thought if the velocity of an object was constant, it had no acceleration. Doesn't that mean that the centripetal acceleration of the centipede, or the centipedal acceleration, is zero?

Think about this before you read the answer!

Answer:

Well the flaw in this line of reasoning is that Beethoven has stupidly misunderstood vectors. Look, if the velocity of an object is constant, it has no acceleration. But the velocity of this centipede is not constant! The speed is constant but its direction keeps changing. Again its important to distinguish between vectors and scalars. Velocity is a vector, it has a magnitude *and* a direction. You can't blame Beethoven for getting this wrong, he's quite busy decomposing.

5.4 Motion with constant acceleration

Now we come to the problem of how something falls through the air. Not very happily if it we're talking about a camel. So let's talk about something else instead. What is the position of a baseball as a function of time? What is its velocity?

Free fall is an example of motion with constant acceleration. We already dealt with this in one dimension but things look a lot more exciting in two dimensions. Baseball is not an exciting sport in one dimension, but in three dimensions it seems to have a mesmerizing effect on *certain* people.

In the case of free fall, the acceleration vector points vertically downward. Again, its value has a magnitude of about $9.81m/s^2$.

We can follow the same line of reasoning we did in one dimension. Everything is the same except that now we have vectors. So we can march through all the same steps, and come up with a very similar set of equations. Eq. 3.11 becomes

$$\mathbf{v}(t) \;=\; \mathbf{a}t + \mathbf{v}_0 \tag{5.12}$$

Here $\mathbf{v}_0$ is the initial velocity of an object at $t = 0$. To get the position vector $\mathbf{r}$ as a function of time is equally easy to generalize. Instead of eq. 3.14 we have

$$\mathbf{r}(t) \;=\; \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_0 t + \;\mathbf{r}_0 \quad . \tag{5.13}$$

Here $\mathbf{r}_0$ is the initial position vector of the object.

5.4.1 Application to free fall

In the case of free fall, we can take our x axis to be in the horizontal direction and the y axis to be in the vertically up. The acceleration of gravity is then $\mathbf{a} = -9.81\hat{j}m/s^2$ Instead of writing 9.81, let's just call it g, so $\mathbf{a} = -g\hat{j}$. We can write down Eqs. 5.12 and 5.13 in component form. Again we'll follow the standard notation of using subscripts x and y to denote the components of a vector quantity, so

$$v_x \;=\; v_{0x}\;,\; v_y \;=\; -\,gt + v_{0y}\;, \tag{5.14}$$

Note that the x component of the velocity is constant, which it should be since there's no acceleration in that direction.

Now for position:

$$x(t) = v_{0x}t \;+\; x_0 \tag{5.15}$$

and

$$y(t) \;=\; -\,\frac{1}{2}gt^2 + v_{0y}t \;+\; y_0 \tag{5.16}$$

This says that motion of an object in free fall has its x and y components completely decoupled. That is they're independent of each other. If you change v_{0x} it doesn't affect the y motion at all. The motion in the x direction is that of an object going at a constant velocity. The motion in the y direction is exactly the same equation as in one dimension for a particle in free fall. By combining these two components, you get the overall motion.

The shape of the trajectory

So now we know how the position of an object in free fall depends on time. But what if we want to know the shape of arc drawn out by the object? We want to know how y depends on x. Let's take the initial position of the object to be at the origin $\mathbf{r}_0 = 0$. Then we just eliminate time in eqs. 5.15 and 5.16. So first solve for time in eq. 5.15, that gives $t = x/v_{0x}$. Substitute this into 5.16 and we get:

$$y = -\frac{1}{2}\frac{g}{v_{0x}^2}x^2 + \frac{v_{0y}}{v_{0x}}x \qquad (5.17)$$

This is the equation of a parabola. A stream of water, such as produced by a garden hose, or a fountain, gives us nice conformation of these parabolic shapes.

The range of a projectile

Now suppose that we kick a ball on the ground up into the air with some initial velocity. How far away does it land? Again take the initial position to be at the origin of our coordinate system. To find where it lands we ask when does it's y component equal zero. Setting the left hand side of eq. 5.17 to zero, we can solve for x, giving where the ball will land. We'll call this point the "range" R. It is then $R = 2v_{0y}v_{0x}/g$.

Often we want to express the initial velocity in terms of its polar represent ion (see 4.2). So $v_{0x} = v_0 \cos\theta$ and $v_{0y} = v_0 \sin\theta$. Using the trigonometric identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ we can rewrite the range as

$$R = \frac{v_0^2}{g}\sin(2\theta) \qquad (5.18)$$

Now you might suspect that some mistake could have crept into this derivation. How can you tell if it seems reasonable? Well first check the units. Yep they're right. OK, now lets look at various limiting cases. Take θ to be zero. The range is zero. That makes sense because if you kick a ball from the ground level in the horizontal direction, it doesn't fly through the air at all. It hits the ground immediately. OK, how about when θ is 90 degrees? Well you get zero also. That seems right too because in that case it's going straight up, and it's liable smack you on the way down if you don't get out of the way.

If you maximize the range over all angles you get that it's largest at 45 degrees.

The maximum height of a projectile

How do you find the maximum height of a projectile? Well there are a few ways you could go about it. You could figure it out from eq. 5.17, but let's instead do it from eq. 5.14. At the maximum height, the y component of the velocity is zero. So we can solve for t_m, the time when this happens. It's just $t_m = v_{0y}/g$. Now plug this into eq. 5.16 to get the maximum height, again with $\mathbf{r}_0 = 0$:

$$y_m = -\frac{1}{2}g\frac{v_{0y}}{g}^2 + v_{0y}\frac{v_{0y}}{g} = \frac{1}{2}v_{0y}^2/g \qquad (5.19)$$

5.4.2 Example: The Monkey and the Evil Emperor

Well I don't really know what to call this example. In my day, it was called the monkey and the hunter. It had to do with a hunter shooting a monkey. Nowadays people have come up with some pretty PC alternatives, like the monkey and the game warden. Holy mother of mercy, this is just some stupid totally unrealistic physics example for goodness sake, but it doesn't stop the publishers from getting nervous about getting sued! The monkey is an endangered species and is being protected from impending doom by being shot with a tranquilizer so as to be relocated in a more salubrious environment. Sure, sure, sounds pretty lame. Perhaps we could resuscitate the idea of the monkey getting shot by saying it was carrying ebola and was about to cause the deaths of many many other monkeys. But then we might have to add a trigger warning and tell the reader that they need to go to their safe space afterwards and do meditation exercises etc. I know, how about if this was actually a toy monkey being shot by another toy, the evil emperor Zurg? Yah, that sounds good, we'll call this shooter "Z".

OK, so Z is in the jungle and spies the monkey dangling from the branch of a (plastic) tree laughing at Z. The monkey is a clever little creature, and Z has quite a slow gun (it's child safe). Or it could even be a blow gun if Z was a native of the jungle and was simply interested in eating the monkey as it was a traditional cultural practice. So the instant the trigger is pulled, the monkey drops from the branch, in free fall towards the ground.

So the question is, where should Z aim? Above, below, or at that laughing monkey?

Solution

- *Identify the relevant pieces of information in the problem.* The initial positions and velocities of the monkey and Z are relevant. We're interested in whether the projectile and monkey collide. That is, is there a time when the coordinates of the monkey equal the coordinates of the projectile? So the coordinates as a function of time are also relevant.

- *Give names to all the relevant variables.* Place Z at the origin. It's always a good idea to make the simplest possible choice of coordinate system. Call the initial position of the monkey r_{0M}. Call The initial velocity of the projectile $\mathbf{v}_{0p}$. The initial velocity of the monkey is zero, since it's dropping from rest. Call the position vectors of the monkey and Z at a time t, $\mathbf{r}_M$ and $\mathbf{r}_p$ respectively.

- *Identify what equations describe the physical situation we are analyzing.* Both the projectile and the monkey are in free fall so we can use eq. 5.13 to describe both of them.

The strategy we'll take to solve this problem is to write down the two equations that describe the coordinates of the monkey and projectile and then set then equal. We set the coordinates equal because that means the the projectile and the monkey have collided. This will tell us what condition $\mathbf{v}_{0p}$ must satisfy in order to hit the monkey.

OK, so let's write down the two equations:

$$\mathbf{r}_M = \frac{1}{2}\mathbf{a}t^2 + \mathbf{r}_{0M} \tag{5.20}$$

and

$$\mathbf{r}_p = \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_{0p}t \tag{5.21}$$

Now let's set these equations equal. Notice that the $\frac{1}{2}\mathbf{a}t^2$ terms cancel. This means that the acceleration (i.e. gravity) cancels out completely. So at this point there are two ways to proceed:

1. Well if gravity doesn't matter, just take it to be zero from the beginning. The monkey could be in deep space where there is no gravity (it's actually plastic). If it tries to drop now, it will discover to its dismay, that it's not dropping. So if Zurg aims directly at the monkey, he'll be bound to hit it!

2. Let's look at the equation we get: $\mathbf{r}_{0M} = \mathbf{v}_{0p}t$. That says that the vectors $\mathbf{r}_{0M}$ and $\mathbf{v}_{0p}t$ have the same direction. So if Z wants to hit the monkey, the gun should be pointed directly at it.

Either way you see that Z will hit the monkey by aiming directly at it, despite the fact that it is falling. The point is that the decrease in height due to gravity is exactly the same for both objects, $\frac{1}{2}\mathbf{a}t^2$, despite the fact the bullet is going much faster than the monkey. So gravity is irrelevant to the relative position of two objects in free fall.

5.5 Motion in different reference frames

Suppose you have a bear that is running about inside a moving train. You're trapped on board the train watching the bear quite intently. You monitor the bear in from a coordinate system on the train. Call the position vector as seen from this coordinate system $\mathbf{r}_{bt}$, that is the coordinates of the bear relative to the train. Call the coordinates of the train seen from a coordinate system on the ground $\mathbf{r}_{tg}$. A person on the ground wants to know what are the bear's coordinates seen in the reference frame on the ground, $\mathbf{r}_{bg}$. How can we determine what this is? We can see all this pictorially in Fig. 5.2 below.

From the figure we see the following relation between the above vectors:

$$\mathbf{r}_{bg} = \mathbf{r}_{bt} + \mathbf{r}_{tg} \ . \tag{5.22}$$

Differentiating the above formula with respect to time, we see the velocities are related the same way:

$$\mathbf{v}_{bg} = \mathbf{v}_{bt} + \mathbf{v}_{tg} \ . \tag{5.23}$$

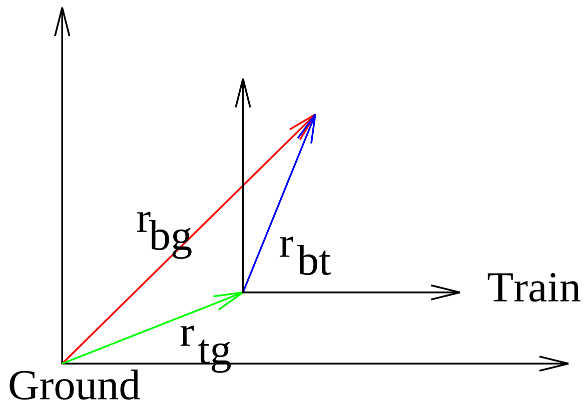


Figure 5.2:

So to get the velocity of the bear as seen from the ground, you add up the velocity of the train to the velocity of the bear relative to the train.

Differentiating again, one sees that the relation between the accelerations is

$$\mathbf{a}_{bg} \;=\; \mathbf{a}_{bt} + \mathbf{a}_{tg}\ . \tag{5.24}$$

So if the train is traveling at constant velocity, the acceleration of train is zero. That means the acceleration of the bear as seen from the ground and as seen from the train are identical.

5.5.1 Cycloidal motion

Suppose you have a reflector mounted right at the rim of a bicycle wheel. If you hold the wheel up in the air and spin it, the reflector traces out a circular path. The equation describing the position is as a function of time is give by eq. 5.7. But when a camel is riding the bicycle, the wheel is on the ground so that the center of the wheel is moving. We'll see in a month or so that the velocity of the center of the wheel is $-R\omega\hat{i}$. That implies that the the position of the center of the wheel as a function of time is $\mathbf{r}_c \;=\; -R\omega t\hat{i}$. So from equation 5.22 we have that the position of this reflector relative to the ground is the sum of two terms. The position of the reflector relative to the center of the wheel, plus the position of the center of the wheel relative to the ground:

$$\mathbf{r} \;=\; R((\cos(\omega t) - \omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath}) \tag{5.25}$$

This traces out a "cycloid" as shown in the Fig. 5.3. Notice the series of cusps that occur when the reflector is close to the ground. When you look at a bicyclist at night, if they're being good and have reflectors on their wheels, you should be able to see these cycloidal trajectories.

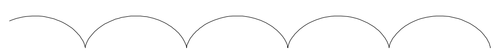


Figure 5.3:

Chapter 6

Forces

6.1 Basic ideas

Now we actually come to the first bit of real physics. You see, what we've done up to now has been entirely math: vectors, and problems involving derivatives. We didn't posit any laws of nature, just what happens if say a particle has a constant acceleration. But constant acceleration isn't a general law of physics. It's an approximation to what happens when quite large heavy objects are thrown close to the surface of the earth. So now we'll see how physics *really* works and discover the amazing predictive power of what is known as "Newtonian mechanics". By the way, we shouldn't be too quick to attribute all of this to Sir Isaac. He was a brilliant man but history has a way of burying the names of many worthy people, for example Robert Hooke, who probably has a lot more to do with the invention of mechanics than most physics textbooks will admit.

So what we'll be working with are "forces". These have the units of mass times acceleration. Forces are *vectors*. That's crucial to remember. Failure to do so will likely result in wrong answers. Many forces act on a single object. One obvious example is the force of gravity. It's what keeps pushing you down so you don't accidentally fly off into outer-space. But what keeps you from being pushed down to the center of the earth? (Which has a habit of being rather unpleasant, even at this time of year.) Well if you're sitting down, then the chair is pushing up against your tushy. If you're standing up, the ground pushes up against your feet. If you have a dog, then that dog might be pulling at your pants. This also results in a force. So there are all these forces acting on a single object. Some pushing left, others right, some up, some down, some sideways. You get the idea. What's a poor object to do?

Well assuming your object is a point, then we know what to do. Unfortunately this is not very likely in my case seeing all the twinkies I've been eating. But no matter, even if I wasn't a point, but extremely rigid, then we can also say what to do about all these forces. There's a special point in an object called the *center of mass*, that we'll discuss a bit later. And this point acts in a very simple way, sort of like all the mass has been concentrated to that location. So when we talk about an extended object, like a block, we'll be thinking of it mainly as a single point and the particular location of that point turns out the be this so-called center of mass.

The upshot of all this is that to get the motion of the object (or more correctly its center of mass) what's important is the *net force* acting on the object. By net force I mean you add up all the force vectors acting on the object. That's your net force.

6.2 Newton's Laws

So now what we want to do is ask how do the forces acting on an object relate to its motion? Well that was a tricky business for a few millennia. The presence of air resistance and other frictional forces hid the fundamental physics of the situation. So instead of going through the history, we'll just say what the answer is without too much justification, the justification being that it works really well. But bear in mind that it took the human race a long time to figure this out. So one's first guess is that the bigger the force, the bigger the velocity. That's not quite right. You see in outer space, you can have a spaceship going at very high velocity relative to the earth, but hardly any forces acting on it. The force doesn't relate directly to the velocity, but to its derivative: the acceleration.

Scientists, by which I mean just logical and critical people, will break down this task that we're embarking on, and try to understand what we're trying to accomplish in minute detail. So when we say we'd like to see if there's a relationship between force and acceleration, they'd naturally ask what happens if the force is zero:

- "But wait a second! You think you can find a law relating the force acting on an object to its acceleration? If you believe that, do I have a bridge for you! Oh come on, I can show you two cases where you have the same forces acting on the objects, but their motion is different. Let me explain why this is an ill defined quest:

  - Assume you have a ball in outer space inside a spaceship, no gravity. [This sort of reasoning is called doing a *gedanken* experiment.] There's no force acting on the ball so it will just be floating in the ship, cruising along at a constant velocity, no acceleration.
  - *Or will it?*. Suppose you turn on the thrusters on your ship. Now if you stand inside the ship and look at the ball, you'll see the ball accelerating towards the back of the ship where the thrusters are. You haven't applied any force and now it's accelerating.

  So with zero force applied, depending on the state of the spaceship, you'll either see acceleration, or you won't. Therefore it appears that there is no definite relationship between force and acceleration".

Hmm..., that sounds like a good point! So to salvage our line of reasoning, and to find a relationship between force and acceleration, we'll have to decree that we're not allowed to have the thrusters on when we do an experiment. That would be considered a *non-inertial* reference frame. So would a reference frame inside a car that's turning a curve. We want to only consider this force–acceleration question in an *inertial* reference frame. (Actually Albert Einstein would have used that terminology). And operationally, how do you know you're in an inertial frame? Well you know you're in one when you see that the ball isn't accelerating when no forces act on it. It'll just move at a constant velocity (or won't move at all).

To make this precise, we'll make this a law, Newton's first law. But note this isn't so much of a law, as it is a choice of *reference frame*. So Newton's first law says, more or less:

- **The next two laws are only true in what'll be called an inertial reference frame. In such a reference frame, if no forces are acting on an object, it'll just move at constant velocity.**

So how about Newton's two other laws? The second law says that the acceleration of an object is proportional to the net force acting on an object. Proportional? That's not very precise. Well the coefficient of proportionality depends on the *mass* of the object as follows.

The more massive, the less the object responds to a force. The idea of mass is really related to this second law. If you have two objects that are acted on by the same force, say that one accelerates at $1m/s^2$ and the second one at $2m/s^2$, then the first one is twice the mass of the second. In more generality, one could say that the ratio of their accelerations in inversely proportional to the ratio of their masses. Using this idea of *mass*, we can write down Newton's second law:

$$\mathbf{a} = \frac{\mathbf{F}}{m} \tag{6.1}$$

Where **a** is the acceleration of the object, **F** the net force acting *on* the object, and m is its mass.

So far we've seen that Newton's first law isn't so much a law, as it is a choice of reference frame. The second law, would seem pretty intuitive if you lived in outer space with no air resistance, that the acceleration is proportional to the net force on an object.

Now how about the third law? It says that forces always come in *pairs*. They are equal in magnitude but opposite in direction. This is a bit counterintuitive. Let's illustrate the strangeness of this with the example of a humble banana and the Earth. While the banana is accelerating towards the Earth, it has a force acting on it due to the Earth's gravitational field. Call this $\mathbf{F}_{EB}$. There is another force however that isn't so obvious. Its the force the banana exerts on the Earth. Call this $\mathbf{F}_{BE}$. Well the Earth is much bigger so it's natural to think that the first force is much bigger than the second, but no! The two forces have same magnitude and opposite direction, in equations, $\mathbf{F}_{EB} = -\mathbf{F}_{BE}$.

It seems confusing to think that such a small object like a banana could be as strong in this respect, as the Earth. But hey, that's the way it is. It turns out that Newton's third law is related to conservation of momentum, which in turn is a consequence of the fact that the laws of physics are independent of location. And zillions of experiments bear out that this is indeed the case. So despite the counterintuitive feeling you might get, us scientists are quite confident that it's true (at least for mechanics that we're studying here).

You might have heard the third law stated something like this: **For every action there is an equal and opposite reaction**. So one of the forces is called an *action* and the other is called a *reaction*. Thus these two forces form an *action-reaction pair*. All this is just terminology you might come across googling for physics problems, but these are just fancy words to describe what we just discussed.

Another example of force pairs are those between your feet and the ground. As I said earlier, there's a force the ground exerts on your feet. And there is another force due to your feet that act on the ground. You might think that in some circumstances, these to forces wouldn't be equal. For example, in an earthquake. The ground is shaking around in this case, so for the same reasons that we discussed with the spaceship, maybe this makes this law invalid because our feet are in a non-inertial reference frame. But don't worry, it still works just fine, otherwise it wouldn't be all that useful.

But there is a well known paradox that is quite instructive and you should try to think about: A horse starts to pull a cart at rest and exerts a force $\mathbf{F}$ on it. By the third law, the cart exerts a force equal and opposite, of $-\mathbf{F}$ on the horse. So the total force on the horse-cart system is $\mathbf{F} - \mathbf{F} = 0$. Since there is no net force, the cart *cannot move*! But we know from experience that horses do pull carts with some considerable degree of success. So doesn't this mean that Newton's third law is for the birds?

Most students that hear this paradox often come up with reasons why the horse does successfully move the cart, and most of them put the cart before the horse by violating Newton's third law. But really it isn't violated. The resolution is like this: The acceleration of an object, say the cart, is determined by the net force acting on it. The pair of forces: [horse-on-cart, cart-on-horse] *do not both act on the same object*. The only one acting on the cart is the first one: the force that the horse exerts on the cart. For example, to find out if you're going to be accelerated, you don't include the force of the planet Neptune on Jupiter (hmm... how about Uranus?). In any case, getting back to the cart, the other force in the pair: cart-on-horse, is not a force acting on the cart: we don't include it in figuring out the net force on the cart. So we don't sum both forces in the pair together, since they're acting on *different objects*. There will be a net nonzero force acting on the cart, and therefore *it will accelerate*. So physics really isn't quite as silly as it might first seem.

6.3 The force of gravity

Unlike the two forces that were talking above, the force of gravity provides a clear example of "action at a distance". Two objects don't need to be in contact to attract or repel each other. Try having a garlic burger if you want to see how this works.

If we take a banana of mass m and drop it, it'll accelerate with an acceleration of g pointing down. It's the only force acting on the banana so Newton's second law says that the force of gravity is $f = mg$ pointing down.

6.3.1 Example: Your weight

So how big is the force of gravity acting on you? Well how much do you weigh? The answers are the same! To see this, step up onto a scale. If you happen to look like a robot, you'd have the diagram of Fig 6.1:

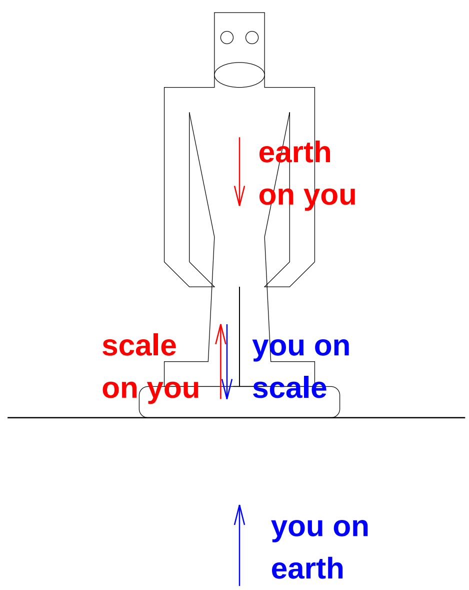


Figure 6.1:

The red arrows indicate the forces acting on you. They are the force of gravity, and the force the scale exerts on you. The force the scale exerts on you is an example of a "normal force". The blue arrows indicate forces you exert on everything else. So at this point it is always a good idea to consider only the forces that act on one object at a time. So lets consider the red arrows first. We want the sum of these two forces to be zero because you are not accelerating. Therefore the force of gravity $\mathbf{F}_G$ plus the force of the scale on you $\mathbf{F}_{SY}$ has to be zero, i.e. $\mathbf{F}_{net} = m\mathbf{a} = m0 = 0$. This means they must have the same magnitude. So $|\mathbf{F}_{SY}| = mg$. Now what about the force you exert on the scale,

$\mathbf{F}_{YS}$? By the third law, that has the same magnitude as $\mathbf{F}_{SY}$. So the scale registers a weight of mg.

6.3.2 Example: Weights in elevators

Now what happens if you weight yourself in an elevator? Suppose the elevator is accelerating with an acceleration of $a\hat{j}$ vertically upward?

This will modify the last example. Now the net force acting on you, which is the force of gravity $\mathbf{F}_G$ plus the force of the scale on you $\mathbf{F}_{SY}$ should no longer be zero. From the second law we have

$$\mathbf{F}_{net} = \mathbf{F}_G + \mathbf{F}_{SY} = m\mathbf{a} \tag{6.2}$$

The force of gravity is $-mg\hat{j}$, so solving for $\mathbf{F}_{SY}$, we have

$$\mathbf{F}_{SY} = m(g+a)\hat{j} \tag{6.3}$$

And now the rest of the argument is the same as the previous example. What's the force you exert on the scale, $\mathbf{F}_{YS}$? By the third law, that has the same magnitude as $\mathbf{F}_{SY}$. So the scale registers a weight of $m(g+a)$.

So if the elevator is accelerating upward, the scale weighs more. If the elevator has a downwards acceleration, then you weigh less.

Your mass doesn't change. Just the force that you exert on the scale.

6.4 Tension

Now a lot of problems we'll discuss will involve ropes and strings. Most of the time, we'll assume that they are massless, or more precisely, that the mass of a string is much smaller than the other masses in the problem. As you know from everyday life, if you grab a weight by a string, you feel the full force of weight transmitted through the string. This is illustrated in Fig. 6.2.

Here we have a person holding a string that's attached to a weight. The different colored arrows correspond to the forces acting on different objects. The red arrows correspond to the forces acting on the weight. The blue correspond to the forces acting on the string, and the green arrow, the force acting on the hand.

We want to figure out what is the force acting on the hand. So we consider each object separately. Call the force exerted by gravity on the weight $\mathbf{W}$. Call the force of the weight on the string $\mathbf{F}_{WS}$. These names are listed in the figure. Then

$$\mathbf{F}_{WS} + \mathbf{W} = ma = 0 \tag{6.4}$$

It equals zero because the weight isn't moving so the acceleration is zero.

Now lets look at the equation for the string, which involves the blue forces:

$$F_{HS} + F_{WS} = m_{string}a = 0 \tag{6.5}$$

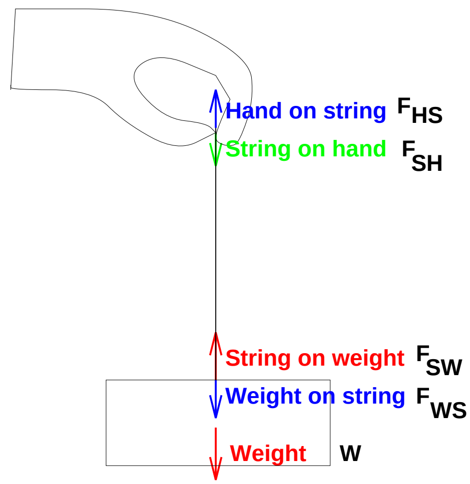


Figure 6.2:

This equals zero even if the acceleration is not zero. This is because we're assuming the mass of the string is zero. The force exerted by the string on the weight, is called the "tension".

Now let's apply Newton's third law. We know that forces come in pairs so that $F_{WS} = -F_{SW}$, and $F_{HS} = -F_{SH}$.

Putting this all together, we quickly see that $F_{SH} \;=\; W$.

As you can see from the above equation, the tension transmits the force of gravity that acts on the weight, to a force acting on the hand. This is a general feature of massless strings, and is true even if there is acceleration: **the magnitude of the force (i.e. tension) exerted at one end of a massless rope is the same as on the other side**, which can be seen from Eq. 6.5.

6.5 Problem solving strategies

There is no short cut to learning this stuff. It takes a lot of patience and effort to master how to logically apply Newton's laws to obtain the correct answer. Remember this is physics, not philosophy. The answers you get can be tested experimentally. *There are never two right answers to well posed physics problems.* So if you get stuck **do not:**

1. Write down something that you know is wrong and continue.
2. Guess an equation.
3. Guess an answer.
4. Fudge your solution because you think you know the right answer and it's just not working out the way you think it should.

Instead, realize that the answers are derivable from Newton's laws. You just need to be logical enough to see how to do it. In order to aid you, you should follow the following advice. If you don't, you have nobody but yourself to blame when you screw up.

1. Draw a diagram. If you want, draw a rough sketch to get yourself off the ground. Then identify all the forces in the problem. You saw an example of this above with the picture of the hand and the weight in fig. 6.2.

2. Draw a "free body diagram" for each object. That is, draw a picture of an object, and draw *only those forces that act on the object*. Make sure to include all the forces that act on the object.

3. Figure out a good coordinate system to use for each object. That is how the coordinate system is to be oriented. Them apply $F_{net} = ma$ in component form. Do this for the x and the y components separately. **Remember to use symbols such as m_1, m_2 etc. instead of numbers like** *3 kg*.

4. Solve the equations. If you've done everything right, you should be able to solve for all your unknowns.

5. Check limiting cases. What happens if this mass goes to zero? This one goes to infinity? What if this angle becomes 90 degrees? You get the idea. If all the limiting cases seem to make sense, and the units check out, then you've probably solved the problem correctly. If they don't check out, at least now you'll have a clear logical description of your work and will be able to go over it to check for mistakes.

Item 5 is particularly hard for most students to master. It's easier for most people to think concretely rather than abstractly. But if you don't use symbols, you won't be able to take limits to check your answers. You'll find that often you're doing way more work than you need to do. For example, if you use symbols, you might get an expression like $yx/y - x^2/x$, which equals zero. You won't need to plug in any numbers at all, and in fact if you do, you'll be likely to have rounded and get an nonzero answer like $0.001kg$, which would be wrong. So my advice to you is to work your way up to doing problems symbolically. Work out problems with the numerical values, and then *go back* and do it substituting in symbols. You'll soon get used to using them. Although this also takes discipline and practice, it's well worth the effort. To help you, it's good to have a convention for symbols. Don't call distances t, or masses v. Stick with the convention that's common and that I'm using. A mass could be called m, or m_1, if there's more than one in the problem. Velocity $\mathbf{v}$, and the y component would be v_y, and speed v. Distances in two or three dimensions, r, in one dimension, or for components, use x, y, or z. Forces $\mathbf{F}$, but tension (which is force) is T. Acceleration is $\mathbf{a}$.

One of the surprising things that happens at this point in the course is that a lot of the hot shot students that actually like physics, find that they mess up, because they don't solve problems systematically, but intuitively. When a problem gets sufficiently complex even with great intuition, you can't figure out the right solution to a problem. If you're talented at physics you'll still need to follow the above advice or you'll quickly go from getting all the problems right to messing them all up. Having intuition is great, but there's more to physics than that.

Now let's looks at some examples of how you can systematically solve mechanics problems.

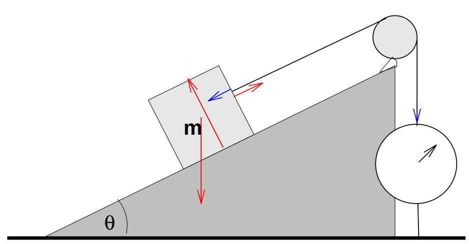
6.5.1 Tension on an inclined plane

Consider a block on a "frictionless" inclined plane. We'll talk about friction in a little while so don't get too bothered by this term. Basically it means a very very well greased surface, so that there is no force acting along any direction on the surface of the plane. We have this block of mass m that is attached by some string to a force meter. We want to know how much force is registered by the meter. If the angle θ that the inclined plane makes with the horizontal is zero then this force will be zero. But for other angles what is the answer?

Let's follow the handy-dandy rules of above to quite effortlessly solve this problem. Once you've gone through this once, you should try to solve it again without looking.

Draw a diagram

Here's the diagram. All forces acting on the block are in red. The ones acting on the string are in blue.



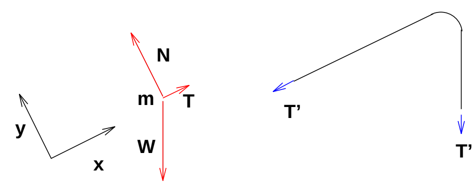
What are the forces acting on the block?

- There is the "normal force". This force acts normal to the surface and is exerted by the plane on the block. The reason it's normal is that the plane is frictionless and can't exert forces along the direction of the plane. So it has to be normal.
- There is the weight of the object due to the force of gravity. This acts down.
- There is the force the string exerts on the mass. That force is parallel to the plane

Now how about the forces acting on the string? There are two drawn in blue. Because it's a string and its going over a frictionless wheel, the two blue forces have the same magnitude. By the way, the force the string exerts on the mass is the same as the force the mass exerts on the string by the third law.

Draw free body diagrams

So now we get rid of the actual shape of the block which is nothing to get sad about seeing the block was real boring looking. We just draw all the red vectors together starting from the same central point. It makes the whole thing look simpler.

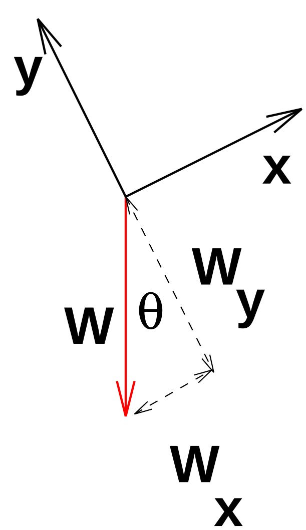


We've also drawn all the blue arrows together. This part is quite boring. I've just included to let you know you often have to draw more than one free body diagram to solve problems involving multiple objects. All this diagram says is that the tension is transmitted all the way along the string.

F=ma in component form

Now we need to figure out a good coordinate system to use and apply $F_{net} = ma$ for both the x and y components.

Funnily enough, the best coordinate system to use is the one tilted along the inclined plane as shown in the figure below.



This will take some practice to figure out. You can use any coordinate system you want, but its good to use a simple choice as the answers tend to come out a lot more easily that way. With this choice you can see that the only vector that we have any trouble decomposing is the weight. The normal force and the tension lie in the direction of the y and x axes respectively. This way we have less trigonometry to do.

So decomposing the weight we have

$$W_x = -W \sin\theta \ , \ W_y \ = \ -W \cos\theta \tag{6.6}$$

Because the mass is not moving, the acceleration of it is zero, so from the second law, that means that the net force acting on the block is zero.

$$\mathbf{N} + \mathbf{T} + \mathbf{W} = 0 \tag{6.7}$$

This is a vector equation. We want to write it now at component form. (Remember my advice!).

In the y direction, the tension is zero, but the normal force is entirely in this direction. so

$$N + W_y = 0 \tag{6.8}$$

This says that

$$N \ = \ W \cos\theta \quad . \tag{6.9}$$

In the x direction, the normal force is zero, but the tension entirely lies along this direction. Therefore

$$T + W_x \ = \ 0 \tag{6.10}$$

Therefore

$$T = W \sin\theta \quad . \tag{6.11}$$

Now as we said above, from the third law, this means that the scale will read the same thing.

Remembering $W \ = \ mg$, we have deduced that the scale will read $mg \sin\theta$.

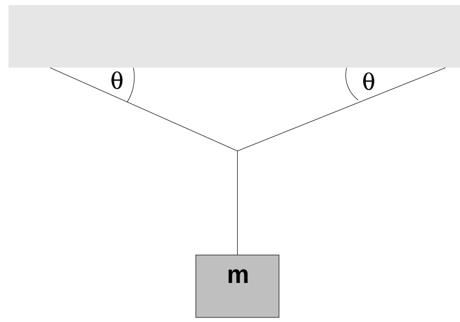
Limits

Do the limits look right? When the angle θ is zero, the scale reads zero. When it is 90 degrees, the scale reads mg. So yes the answer makes sense. Also the units look right.

You see that all we did was to follow the rules, take out time, and everything worked out. Pretty soon we'll be able to solve much more complicated problems by using the same strategy.

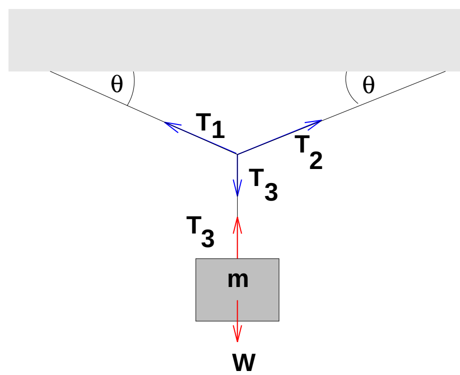
6.5.2 Mass held by two ropes

Consider a mass being held up by two ropes as shown. What are the tensions in the ropes?

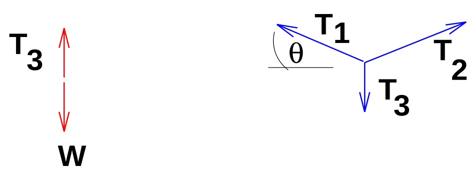


Draw a diagram

Now let's add in labels for all quantities we'll need.



Draw free body diagrams



The first free body diagram labels all forces acting on the mass m. There is the force of gravity, that is the weight **W** and the tension of the string, that I've labeled $\mathbf{T}_3$.

Now something a little weird is necessary. We've drawn a free body diagram for the junction between the three strings. The point is that because the acceleration of the junction is zero (and its mass is also zero), the net force acting on the junction must be zero, by the second law. We had to realize that this is an important piece of information that we must use to solve the problem.

F=ma in component form

This time it's pretty clear that a simple horizontal-vertical coordinate system is the best one to use.

The two vectors that we'll have to decompose are $\mathbf{T}_1$ and $\mathbf{T}_2$. So

$$T_{1x} = -T_1 \cos\theta, T_{1y} = T_1 \sin\theta \tag{6.12}$$

and

$$T_{2x} = T_2 \cos\theta, T_{2y} = T_2 \sin\theta \tag{6.13}$$

Again because the accelerations are zero, the net force on all objects is zero. This tells you that $\mathbf{T}_3 = W$ as expected.

The next thing to do is worry about the junction. The vector equation for that is

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 = 0 \tag{6.14}$$

Solve the equations

Now let's deal with the x direction. T_3 points in the y direction so we don't need to worry about it. So we have $T_{1x} = -T_{2x}$. Expressing this using Eqs. 6.12 and 6.13 we have $-T_1 \cos\theta = -T_2 \cos\theta$ or $T_1 = T_2$. That makes sense. The tensions in the two strings are the same because the angles are the same. It would seem bizarre if this was not true.

Now the y direction. We have to include all the tensions, $T_{1y}+T_{2y} = -T_{3y}$. But we know $T_{3y} = W$ and $T_{1y} = T_{2y} = T_2 \sin\theta$. So that means that $2T_2 \sin\theta = W = mg$, or

$$T_2 = \frac{mg}{2\sin\theta} \tag{6.15}$$

Limits

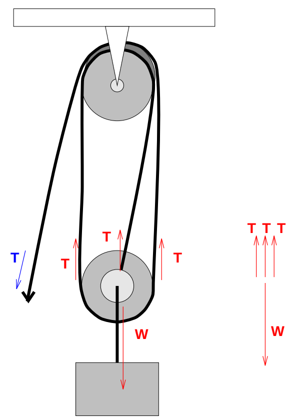
Well, the units are right. How about the limits? When θ is 90 degrees, then each rope supports half of the weight which seems right. How about as $\theta \to 0$? The tensions go to infinity! That means its impossible to support any weight at all when the two strings are totally horizontal. This is actually true. It's a good thing to see if this agrees with your intuition.

6.5.3 Pulleys

Another rather important example is that of pulleys that have been used for thousands of years to lift up large weights. The figure below illustrates a pulley.

How hard do you have to pull to keep the weight hanging, or moving up at a constant velocity?

Assuming that the ropes can slide freely over the wheels, the tension along the rope is the same everywhere. So we have the free body diagram illustrated above on the right. We've also assumed that the weight of the pulley and rope is negligible compared to the weight that's hanging.



If the system is not accelerating, then the sum of the forces in this free body diagram must add up to zero. That says that $3T = W$, or the tension $T = W/3$. So you have to supply a tension of a third the weight of the object to keep it hanging.

Of course if you wrapped the rope around the wheels more times, it'd be even easier to lift the weight. The problem is that you'd have to pull a lot more rope that way than if you wrapped it just a few times. We'll see later why this has to be in order to conserve energy.

6.6 Friction

If you push a heavy box across the floor, you have to apply a force in order to get the box to start moving, and then keep applying a force to maintain its motion. This force is needed in order to balance the friction force between the box and the floor. There are some *approximate* facts about the friction force:

- The friction force points in the opposite direction to the force you apply. (More generally, it is opposite to the net force that tries to move the box across the floor.) If the applied force is small, the friction force is just big enough to cancel it exactly.

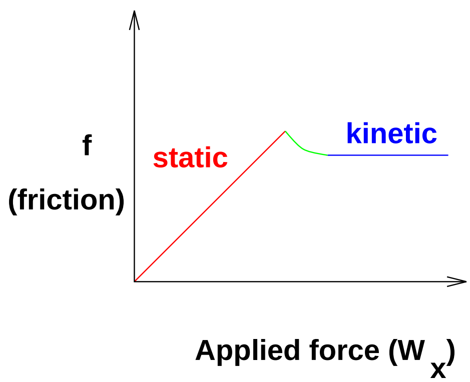
- There is a maximum possible force of friction; if the applied force is greater than that, the box starts to move.

- When the box is moving, there is a friction force that points in the direction opposite to the direction in which it is slipping across the floor. This force of "kinetic friction" is usually slightly less than the maximum possible force of "static friction."

- The maximum force of static friction, and the force of kinetic friction, are approximately independent of the area of the bottom of the box, and are related to the weight of the box by

$$\begin{aligned} f_{s,max} &= \mu_s N \\ f_k &= \mu_k N. \end{aligned} \tag{6.16}$$

- More generally, whenever two objects are sliding past each other, or trying to slide past each other, there is a friction force between them that is approximately related to the normal force pushing the two objects towards each other according to Eqs.(6.16).



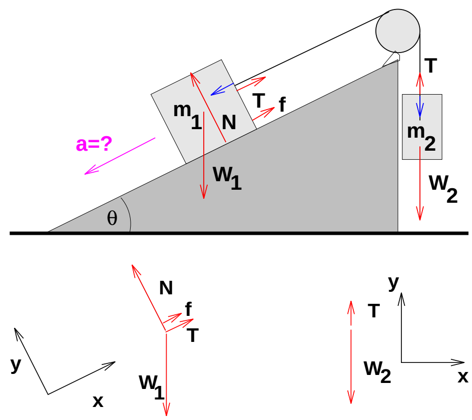
Normally μ_s and μ_k are numbers around one or less, and depend on the nature of the two surfaces. A shoe on a banana peel will be expected to have a small coefficient of static friction, where as Good running shoes are designed to have good traction and hence a large coefficient of static friction. Sometimes, the difference between μ_k and μ_s is ignored, and one speaks of "*the* coefficient of friction."

We emphasize that the curve drawn in Figure 6.6 is only approximate; the actual friction force that is experienced depends can vary if one tries the experiment many times. The green region in particular is bogus. It is not at all reproducible. It depends on whether you start with a static object and increase the applied force, or you start with a moving object and decreases the applied force. The physics of friction is a very interesting but complex subject and is currently under investigation by a lot of researchers.

6.6.1 Example

Here we have pictured a block going down an inclined plane with coefficient of kinetic friction μ_k. Assume it is sliding to the left as shown. Calculate the acceleration, given the masses m_1 and m_2.

The tension T is the same throughout the string because we're assuming that the wheel at the top of the diagram is massless and that it can turn freely (no friction).



Free body diagrams

F=ma in component form

We've chosen the coordinate systems as shown. Note that it is perfectly consistent to use two separate coordinate systems for the different masses. Keep this point in the back of your mind when you go through this problem. Do you notice anything illegitimate?

Let's look at the first mass. We'll denote its acceleration by $\mathbf{a}_1$.

We have $\mathbf{F}_{net} \;=\; m\mathbf{a}_1$ or

$$\mathbf{N} + \mathbf{W}_1 + \mathbf{T} + \mathbf{f} \;=\; m\mathbf{a} \tag{6.17}$$

The decomposition into components for the weight W_1 is the same as in section 6.5.1. See Eq. 6.6. So there are two forces acting in the y direction. The weight has a component W_{1y} in this direction and the normal force entirely in this direction. The acceleration in the y direction is zero, since the block is sliding along the x direction. So the y component of the above 6.17 is the same as Eq. 6.9 which says

$$N \;=\; W_1 \cos\theta \tag{6.18}$$

The y component is

$$T + f - W_1 \sin\theta = m_1 a_1 \tag{6.19}$$

But we also know that $f \;=\; \mu_k N$. Putting this all together, the y equation becomes

$$T + \mu_k W_1 \cos\theta - W_1 \sin\theta = m_1 a_1 \tag{6.20}$$

Now on to block 2. Here all the action occurs in the y direction, We have

$$T - W_2 \;=\; m_2 a_2 \tag{6.21}$$

Now let's size up the situation here. We have two equations 6.19 and 6.21. They contain three unknowns, the tension, and the two accelerations. So we can't solve for these uniquely without some extra info. Aren't we missing something? As things stand now, we haven't said there's any relation between the two acceleration. But clearly they closely related. If the first mass goes down the plane one inch, the second mass must go up the same amount. So the accelerations are equal and opposite (think about the coordinate systems we're using)

$$a_1 = -a_2 \tag{6.22}$$

Solve the equations

Now we have a third equation so can go ahead and solve for the acceleration. Subtracting 6.21 from 6.19 we can eliminate the tension giving

$$W_2 + \mu_k W_1 \cos\theta - W_1 \sin\theta = m_1 a_1 - m_2 a_2 = (m_1 + m_2) a_1 \tag{6.23}$$

Now remembering that $W_1 = m_1 g$ and $W_2 = m_2 g$, we can solve for the acceleration

$$a_1 = \frac{g}{(m_1 + m_2)} \left[m_2 + (\mu_k \cos\theta - \sin\theta)\, m_1 \right] \tag{6.24}$$

Check answer

Well the units look right. How about what happens if the acceleration is zero and $m_2 = 0$? Solving that case, and replacing μ_k with μ_s, we have $\mu_s = \tan\theta$. This is the slope at which the block first starts slipping.

Chapter 7

Work and Energy

7.1 Basic ideas

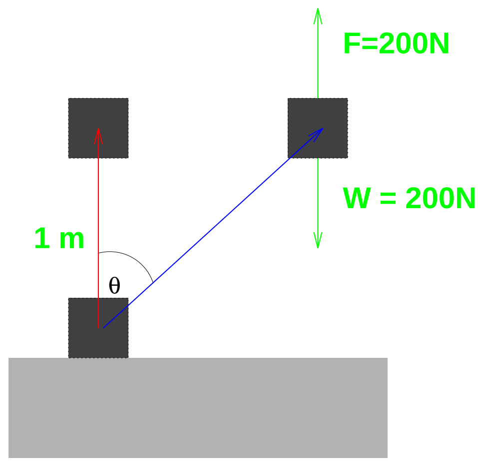
Last chapter we covered Newton's three laws, which can be used to predict how things move in time. You have to know what the forces are between objects, but assuming that, Newton's laws tell you everything. So what is there left for us physicists to do? Actually a hell of a lot, like surf, play bridge, sit around a talk about how smart Newton was. But seriously, buried in Newton's laws there are some extremely important concepts that aren't at all obvious. Most importantly is the idea of *conserved quantities*. These have very important consequences to physics. The first one we'll talk about is *conservation of energy*. We'll define this thing called "energy" in a while. And the neat thing about energy is that it is conserved, that is it doesn't change in time. You should understand that we don't postulate that it doesn't depend on time. We can actually derive it as a consequence of Newton's laws. That's one nice thing about physics. That you don't have to remember very much. A few basic equations is all you need and the rest can be derived. Of course you need quite an big CPU to be able to this in practice, so it's good to try to remember other things as well, like conservation of energy.

We'll start now by considering a thing related to energy called *work*.

7.2 Work with constant forces

Imagine you have to vertically lift a crate off the ground by 1 meter. The crate weighs 200 N. You have to apply a force of 200N to the box in order to barely lift it. The *work* you do is just this force, 200 N times the distance you move it, 1 meter. So the work you do is 200 Nm. We'll define a new unit called a "Joule" (J) which is a Newton times a meter. So you do 200 J of work lifting this crate vertically up.

This is illustrated in the figure below by the red arrow:



Now suppose you don't move it vertically up but at some angle θ with respect to the vertical, as illustrated by the blue arrow. So you still end up 1 meter higher than where you started but some distance away horizontally. If we want a sensible definition of work, we'll want the work to be the same. So the work you do is the force times the distance gone in the vertical direction, or more generally, the distance gone in the direction of the force. The force **F** is vertically upward and the vector representation of the distance traveled is the blue arrow. So how do we come up with a nice concise definition of work involving vectors? It's easy with the dot product. We just take the dot product of the blue displacement vector $\Delta\mathbf{r}$ with the force **F**. So the work is

$$W = \mathbf{F} \cdot \Delta\mathbf{r} \tag{7.1}$$

This is the definition of work when you have a force that doesn't change as you move, and also the path you take is straight. We'll now consider in one dimension how the work is related to velocity and introduce the concept of *kinetic energy*.

7.3 Kinetic energy with constant forces

The kinetic energy K of a particle with mass m and speed v is somewhat mysteriously defined as

$$K = \frac{1}{2}mv^2 \tag{7.2}$$

This is true even in three dimensions but we'll start by considering it in just one dimension. We see now why this is a sensible definition.

Consider applying a constant net force F to an otherwise free particle. It will accelerate with constant acceleration because $F = ma$. Remember that with constant acceleration we showed earlier that over some time interval you could relate initial velocity v_0, final velocity v, and distance traveled Δx as

$$v^2 - v_0^2 \;=\; 2a\Delta x. \tag{7.3}$$

Multiply both sides by $\frac{1}{2}m$, and this becomes $K_f - K_i \;=\; ma\Delta x \;=\; F\Delta x$. As usual, we'll write this change in kinetic energy as $\Delta K \equiv K_f - K_i$. But in one dimension, you don't have to worry about the dot product so $F\Delta x \;=\; W$, which is the work done by the force F, so we end up with the simple looking formula

$$\Delta K \;=\; W \tag{7.4}$$

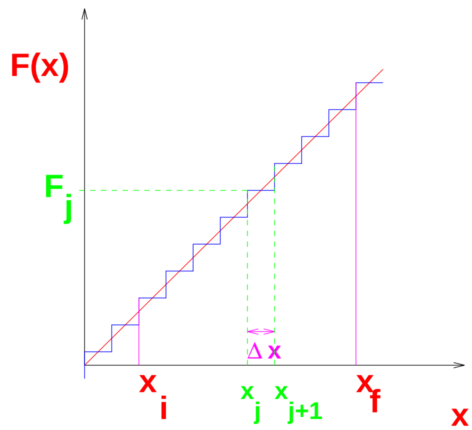
This is called the "work-energy theorem". In words this says that the change in kinetic energy is equal to the work done by the force F on mass m. We had only considered one force here, but it's the same argument when we have many forces and this F becomes the net force. So this kinetic energy is a pretty neat thing. If you know its change, you can figure out how much work was done by all the forces acting on the the particle.

7.4 Work with variable forces

Now what if the force is variable, that is it depends on position? The simplest example of this is a spring. If a mass is attached to the end of a spring, and the spring is stretched, the force exerted on the mass is $F(x) = -kx$, where x is the amount of stretching. The negative sign tells us that the mass is being pulled back by the spring if the stretching is positive; if the spring is compressed, x is negative and the mass is pushed out. The parameter k is the "spring constant", and is bigger for a stiffer spring. Of course, this is an approximation: if you stretch or compress the spring too much, the force is no longer proportional to x (and eventually the spring breaks or is scrunched up to its smallest size), but it works well if x is not too big. The fact that a spring exerts a force $F(x) = -kx$ is called Hooke's law; it is an approximate result like the friction 'law' we saw in the last chapter, rather than something like Newton's laws which are exact.

Can we figure out what the work is in this case? Of course, I wouldn't have mentioned otherwise. Look at the diagram below. We're applying a force to the spring in order to move it from position x_i to x_f.

The red line is the force as a function of position x. (If this was a spring it'd be a negative slope but I'm not assuming anything about the force function here). Let's calculate the total work in going from x_i to x_f. Well we know how to get the answer if the force is constant, but here it's clearly varying. So we'll try to get an approximate answer by replacing the smooth curve in red by the staircase shown in blue. We're replacing it with this because we know how to calculate the work in going along one of the steps, since the force is constant on a step. In the diagram we see the work done in going along between x_j and x_{j+1} can be calculated because the force, F_j, is constant in that interval. It's



just $F_j\Delta x$. So to get the total work we have to sum up over all these intervals:

$$W \approx \sum_j \Delta x F_j \tag{7.5}$$

Now we'll do the old calculus trick of taking the limit as $\Delta x \to 0$. In this limit we get an integral so the work becomes

$$W = \int_{x_i}^{x_f} F(x)dx. \tag{7.6}$$

This can be interpreted, as usual, as the area under the curve $F(x)$.

The integral is nothing more than inverse differentiation: if you can guess a function $V(x)$ such that $V(x+\Delta x) - V(x) = (\Delta x)F(x)$, then $W = V(x_f) - V(x_i)$. In the limit $\Delta x \to 0$, this means guessing a function $V(x)$ such that $dV/dx = F(x)$. (Instead of guessing, people memorize the inverse differentials of a large number of functions and learn tricks to change unfamiliar integrals into one of the standard forms.)

In the case of the spring, the force $F = -kx$ so the work done by the spring in moving an object from x_i to x_f is

$$W = -\frac{1}{2}k(x_f^2 - x_i^2) \tag{7.7}$$

as can be verified by checking that $dW/dx_f = -kx_f$.

7.5 Kinetic energy with variable forces

Now what happens to Eq. 7.4 when we have variable forces? Basically nothing. This can be seen as follows. Still using this staircase figure, we see that along a step we can apply Eq. 7.4. The incremental work moving a distance Δx is still the change in kinetic energy.

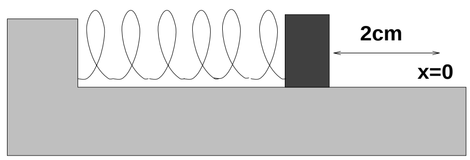
Now if we move two steps the total work is just the sum of the work from the individual steps $W_1 + W_2$, say. This is just the sum of the two changes in kinetic energy. But the total change in kinetic energy is just the sum of the two changes. So that means that for two steps Eq. 7.4 is still true. So generalizing this, you can see that it will be true for the addition any number of steps. So Eq. 7.4 is true for any variable force in one dimension!

Big yawn? OK, let's do an example and you can see how powerful this result actually is!

7.5.1 Example

This is a great exam problem so listen up!

A spring is anchored to some point on a table. The free end is placed at the origin of a coordinate system $x = 0$. The spring constant is $k = 1000$ N/m. The other end is compressed 2 cm and a mass of 1.6 kg is attached to it as shown.



(a) When you let go of the mass it accelerates towards $x = 0$. What is the velocity of the mass at $x = 0$? Assume that the table is frictionless.

Hint: use the work energy theorem

(b) Now assume that there is a frictional force of 4N between the block and the table. What's the velocity in this case?

Solution

(a) So how do we use the work energy theorem? Well first we calculate the work. The initial position $x_i = -.02$ m and the final position $x_f = 0$. From Eq. 7.7 we see that

$$W = -\frac{1}{2}k(0 - x_i^2) = \frac{1}{2}kx_i^2 \tag{7.8}$$

What about the change in kinetic energy? The initial kinetic energy is zero so $\Delta K = \frac{1}{2}mv_f^2$.

Equating ΔK to W we have

$$\frac{1}{2}mv_f^2 = \frac{1}{2}kx_i^2 \tag{7.9}$$

which simplifies to

$$v_f = \sqrt{\frac{k}{m}}|x_i| = \sqrt{1000 \text{ Nm}/1.6 \text{ kg}}.02 \text{ m} = 0.5 \text{ m/s} . \tag{7.10}$$

(b) In this case we have an additional force acting on the block, It is a constant force and opposes the direction of motion. Therefore the work done on the block due to this frictional force is $W_f = f\Delta x$ where $f = 4$ N.

So get the net work on the block we have to add in this new force:

$$W_{net} = \frac{1}{2}kx_i^2 - f\Delta x \tag{7.11}$$

Again, by the work energy theorem, this must be equal to the change in kinetic energy, so we have

$$\frac{1}{2}kx_i^2 - f\Delta x = \frac{1}{2}mv_f^2 \tag{7.12}$$

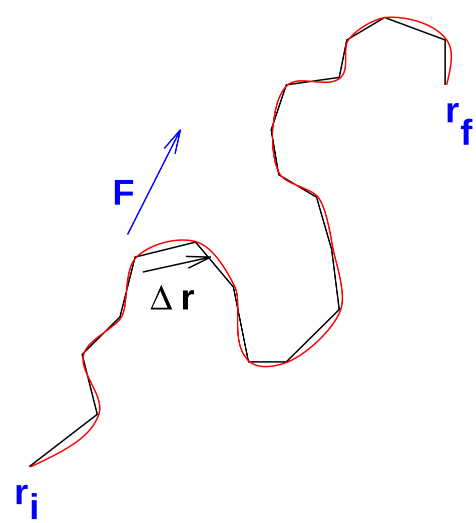
Solving for v_f, we have

$$v_f = \sqrt{\frac{k}{m}x_i^2 - 2\frac{f\Delta x}{m}} = .39 \text{ m/s} \tag{7.13}$$

7.6 Work with variable forces in 3d

Let's repeat the same logic we did in one dimension. We'll think about a force now that's a function of position in three dimensions. What's an example of this? Well the earth going around the sun. The force of the sun on the earth depends on where the earth is located. Another example is a bungee cord with one end anchored down. The force exerted by the other end depends on how far away it is from the anchor.

Following the same logic, we'd think about moving an object along some three dimensional path as shown below.



To get the total work, we'll approximate the smooth red path by the black line segments as shown. Then we'll approximate the force as being constant on a particular line segment. The reason for doing this, is that we know how to calculate the work done by a constant force when going in a straight line. For one of these line segments its just $\mathbf{F} \cdot \Delta\mathbf{r}$. Now we want to sum over all these different line segments. You could write it as

$$W \approx \sum \mathbf{F} \cdot \Delta r \tag{7.14}$$

Now we'll take the same old calculus limit as before, but this time we'll get what's known as a *line integral*

$$W = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} \tag{7.15}$$

The meaning of the line integral is just as we described. You some up all the incremental work going along this path, taking infinitesimal steps as you go along it.

7.6.1 Work Energy theorem in 3d

The derivation of the work energy theorem is really the same idea as we just did in section 7.5, but with more opaque notation. In general it is true that

- the change in kinetic energy is equal to the total work done on an object.

7.7 Potential Energy

We just introduced one form of energy, the kinetic energy, but there is another equally important form: *potential energy*.

We'll talk about it now, but first I'll let you know a good way to think about it. You can think of it as stored energy, that could be converted to kinetic energy to do work. An example of this is a waterfall. The water at the top of the fall has a fairly low kinetic energy but a lot of potential energy. By the time it gets to the bottom of the fall, it has lost a lot of its potential energy and has gained a lot of kinetic energy.

When is this a useful concept? We have seen how to calculate the work done on an object when it moves along a path. If the work done between the points $\mathbf{r}_i$ and $\mathbf{r}_f$ can be expressed as a difference

$$U(\mathbf{r}_f) - U(\mathbf{r}_i) = -W = -\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r}, \tag{7.16}$$

independent of the path taken to go from $\mathbf{r}_i$ to $\mathbf{r}_f$, then the force is called a *conservative force* and $U(\mathbf{r})$ is the potential energy at the point $\mathbf{r}$. Since we have seen that the change in kinetic energy along any path is always

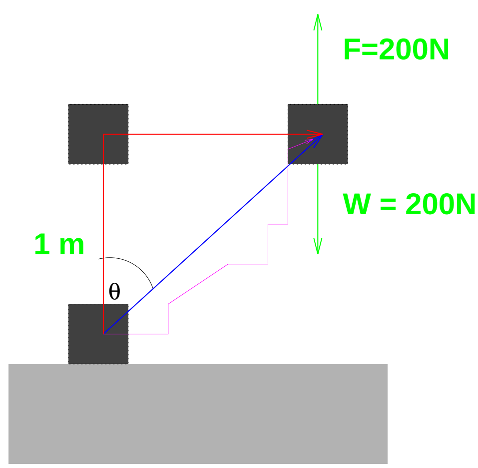
$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2, \tag{7.17}$$

the sum of the kinetic energy K and the potential energy U is constant, i.e. independent of time. We call $K + U$ the total mechanical energy E.

Notice that, if the forces are conservative and a potential energy can be defined, all we ever deal with are differences in the potential energy between two points. Therefore, we can change $U(\mathbf{r})$ to $U(\mathbf{r}) + const$ using any constant, without messing up anything. For those of you who are familiar with integral calculus, this is nothing more than the "constant of integration". This isn't a problem; we will see that we can get a lot out of the fact that, for conservative forces, the sum of the kinetic and potential energy is independent of time.

7.8 Examples of conservative and non-conservative forces

Recall the crate example we began this chapter with (7.2). The work done lifting the object straight up was the *same* as lifting to the same height at a different angle. Now what happens if we want the final position of the crate to be the same in both case, as illustrated below?



Well the total work is the same because the work along the horizontal part of the red line is zero, so again, the work for these two paths are the same. In fact you could choose the magenta path and you'd get exactly the same answer. You should be able to show this by reasoning similar to how we saw that the red and blue path gave the same work.

This force, the gravitational force, is an example of a conservative force. *It doesn't matter what path you take, to go between two points, the work is always the same if you have a conservative force.*

The gravitational potential energy is easy to calculate now. The work done in moving an object by a small distance $\Delta\mathbf{r}$ is $\mathbf{F} \cdot \Delta\mathbf{r} = -mg\hat{j} \cdot \Delta\mathbf{r}$. But $\Delta\mathbf{r} \cdot \hat{j} = \Delta y$, the difference in the vertical coordinates. So we have the work is just $-mg(y_f - y_i)$. We can define the potential energy uniquely by saying we want

$U = 0$ at $y = 0$. So from the definition of potential energy, Eq. **??** we have have

$$U(y) = -W = mgy \tag{7.18}$$

Another example of a conservative force is the force of a spring, from Eq. 7.7 we see that it depends on only the end points, which means that it must be conservative. If one chooses the potential energy to be zero when $x = 0$, then $U(x) = \frac{1}{2}kx^2$.

In fact, for any one dimensional problem where the force just depends on the position, we can define a potential energy as

$$U(x) = -\int_{x_i}^{x} F(x)dx + U(x_i). \tag{7.19}$$

The integral can always be obtained, even if it isn't easy. If the force is *not* just a function of x, this argument fails. For example, in the problem we considered earlier with a spring connected to a block that slides on a table that exerts a force of friction, the friction force points to the left (right) when the block is moving to the right (left), and therefore the force at a point x cannot be expressed as $F(x)$. In three dimensional problems, the condition that $\int \mathbf{F} \cdot d\mathbf{r}$ should be independent of the path between the end points is harder to achieve.

You can easily convince yourself that the work done by a conservative force going on a closed path, that is one where the beginning and end points are the same, is zero. Give it a try!

Example

You throw up a rock vertically with an initial speed v. What is the speed when it comes down to the starting point?

Solution

Well the force of gravity is the only force acting on the rock. It is conservative. It ends up where it started meaning that the total work must be zero. By the work energy theorem this must be equal to ΔK. So the final and the initial speeds are the same.

What about the force of friction? Is that conservative? Well if you go around the room in a big circle pushing a crate on a rough surface and end up where you started, do you do any work? Of course you do. The work is the circumference times the frictional force. Therefore this can't be a conservative force.

Even in the case of the rock thrown vertically, air resistance slows it down when it is going up, and *slows it down even more* when it is coming down. Therefore the system is not exactly conservative, and the final speed is slightly less than the initial one. For the kind of speed at which you can throw the rock, ignoring the air resistance is a good approximation.

In everyday situations, conservation of mechanical energy is at best a good approximation. (It can be a very *bad* approximation, as with the crate pushed around on the rough surface.) But it can make it so much easier to solve problems, that we use the approximation whenever it is reasonable. And there are certain situations, such as the path of planets in the solar system, where the approximation is extremely good: there is no friction or air resistance when a planet moves in vacuum, in space.

For non-conservative forces, the mechanical energy changes with time. But conservation of *energy*, not just mechanical energy, is more general than this. Thus if you have non-conservative forces such as friction, some mechanical energy goes into heat, that is mechanical motion at a molecular level. If you were to add in this extra energy, you'd see that energy is still conserved. If a ball bounces on the table, some mechanical energy goes into heat and into sound (you hear the ball bounce). The *total* energy is *always* conserved. That's a general principle of physics.

The reason for the conservation of energy has to do with the simple observation that the fundamental laws of physics don't change with time. If you do an experiment on an electron on Tuesday, you'd expect to get the same results on Friday. Unless of course it's Friday the 13th, when no experiments work. This simple idea that the laws of physics don't change with time, leads to conservation of energy. It is not obvious how one leads to the other, but if you take more advanced physics classes, you'll eventually see why that's so.

7.9 Using the conservation of Mechanical Energy

We now consider various examples where we can use the fact that the forces are conservative, and mechanical energy is conserved, to solve problems efficiently.

7.9.1 Example: Pendulum

Consider the pendulum below. It has a length L and starts at rest at an angle of θ_0 with respect to the vertical as shown.

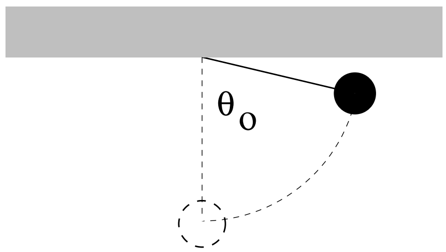
What is its speed at the lowest point?

Solution

Well what forces are acting on the pendulum? We're ignoring friction, so the only forces are the tension and the force of gravity. The tension acts perpendicular to the direction of motion and hence does no work. The force of gravity is conservative, so we can use conservation of energy to solve this problem.

Define $y = 0$ at the point the pendulum is attached to the ceiling. We'll first talk about the initial state of the pendulum, and then its final state.

- **initial time:** The initial y value of the pendulum is $y_0 = -L\cos\theta$. Therefore the initial potential energy is $-mgL\cos\theta_0$. The initial kinetic energy is zero. Therefore the total energy is $K + U = -mgL\cos\theta_0$.



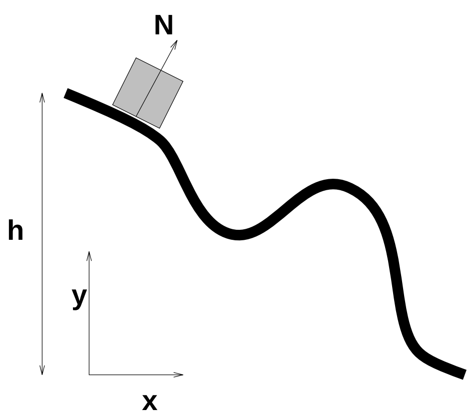
- **final time:** The final value of y is $-L$ and we'll call its final kinetic energy $\frac{1}{2}mv^2$, so the final energy is $\frac{1}{2}mv^2 + (-mgL)$

The initial and final energies are the same, by conservation of mechanical energy. Therefore we can solve for v:

$$v = \sqrt{2gL(1 - \cos\theta_0)} \tag{7.20}$$

7.9.2 Example: Slide

Consider the slide below, and assume that its quite frictionless. If you prefer, we could be more adventurous and imagine that this is a roller coaster. Don't get too carried away, we got to keep our minds on the physics here.



If the object shown starts out at rest at a height h and is let go, what is its speed at the bottom?

Solution

We'll solve this in the same way as the pendulum example, using conservation of mechanical energy. Here there are two forces acting on the sliding object, the force of gravity and the normal force **N**, but here again, the normal force is acting perpendicular to the direction of motion, and hence does no work, so we only need to consider the force of gravity to get the work. It is a conservative force, so we can use conservation of energy.

We'll first talk about the initial state of the slide, and then its final state.

- **initial time:** The initial y value of the object is h and its initial kinetic energy is zero. Therefore the initial energy is $E = K + U = mgh$.
- **final time:** The final value of y is zero and its final speed we'll call v. So the final energy is $\frac{1}{2}mv^2$.

So now we equate the initial and final energies, since energy is constant. We get

$$v = \sqrt{2gh} \tag{7.21}$$

One thing you should think about is: how would you solve this problem using Newton's laws? In principle it's possible because conservation of energy is a consequence of Newton's laws, however it is by no means obvious how to do it. Note that the final answer is independent of the shape of the path. Is that really true?

7.9.3 Example: Ball in circle

Consider a ball on a string being whirled around vertically. Assume there is no friction so that energy is conserved.

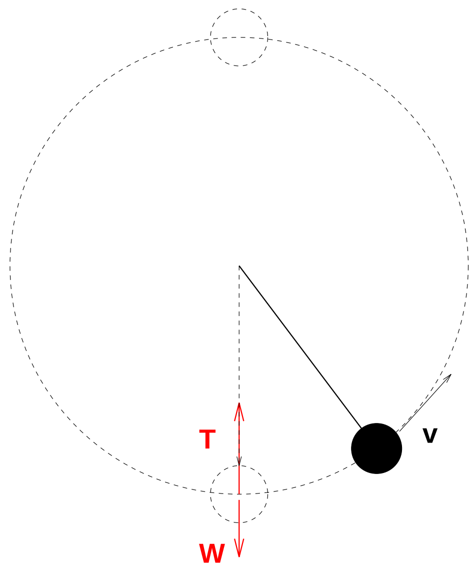
Show that the difference between the tension at the bottom and the top is six times the weight of the ball. This problem is a bit trickier than the previous examples we just did, because it involves conservation of energy *and* Newton's laws. But if you remember how we solved for tensions and centripetal acceleration, then it's not so hard.

Solution

We need to use Newton's laws because this problem explicitly involves forces; conservation of energy alone can't be sufficient to solve this problem. But we've seen from the pendulum example above, energy considerations are extremely powerful and so we'll use it in addition to Newton's laws.

So how about Newton's laws applied to the ball at the bottom? Well $\mathbf{F}_{net} = m\mathbf{a}$. What's the acceleration? It's not zero because the ball is traveling around in a circle. We know that it points up and has a magnitude of v^2/R. What's the net force? The tension points up, but the weight mg goes down so we have

$$T_{bottom} - mg = mv_b^2/R \tag{7.22}$$



where v_b is the speed at the bottom.

The tension at the top is similar but we have to realize though that the acceleration and the tension are now pointing in the opposite directions, so

$$-T_{top} - mg \ = \ mv_t^2/R \tag{7.23}$$

where v_t is the speed at the top.

We were asked for the difference between these tensions so adding these equations, we have

$$T_{bottom} - T_{top} - 2mg \ = \ m(v_b^2 - v_t^2)/R \tag{7.24}$$

So this is all we'll use of Newton's laws. Now we need to figure out $m(v_b^2 - v_t^2)$, which looks awfully closely related to kinetic energy, so now we'll use conservation of energy to figure it out.

Let's but $y = 0$ at the bottom of the circle. Then the energy at the bottom is $K + U = \frac{1}{2}mv_b^2$, and the energy at the top is $K + U = \frac{1}{2}mv_t^2 + mg2R$. Equating these two energies, we see that

$$\frac{1}{2}m(v_b^2 - v_t^2) \ = \ 2mgR \tag{7.25}$$

Using this is Eq. 7.24 we have

$$T_{bottom} - T_{top} \ = \ 2mg + 4mg \ = \ 6mg \tag{7.26}$$

which is what we wanted to show.

Again this problem would be possible, but very hard to solve using Newton's laws alone.

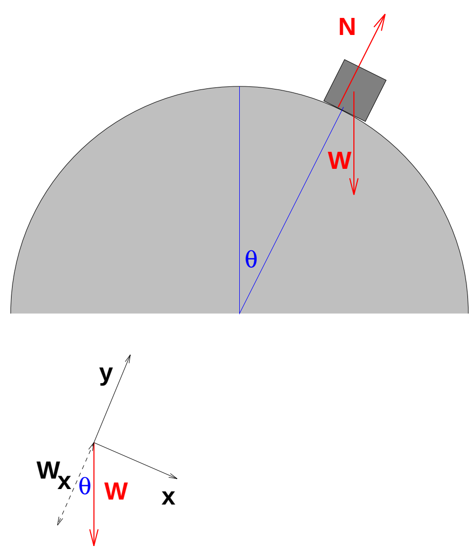
7.9.4 Example: Block on sphere

A block is placed on the top of a sphere and released from rest. At what height does it lose contact with the sphere? Assume there is no friction.

Solution

Another problem where conservation of energy is useful.

We draw a picture of the situation below



The red arrows represent a free body diagram of the block.

There are two forces, the weight and the normal force. What's the condition for when the block looses contact with the sphere? It's when the normal force is zero.

So lets apply Newton's laws to this situation. When the normal force is zero, there's only one force, that is gravity acting on the block. Using the coordinate system shown, the acceleration is in the negative y direction. It has a magnitude of v^2/R. The component of gravity in the y direction is, from the diagram, $-mg\cos\theta$. So we have

$$-mg\cos\theta = -mv^2/R \quad (7.27)$$

or

$$mv^2 = mgR\cos\theta \quad (7.28)$$

Now let's use conservation of energy, choosing that the potential energy is zero at the center of the sphere. Then we have initially $E = mgR$ and finally $E = \frac{1}{2}mv^2 + mgR\cos\theta$. Eliminating E, we have

$$mgR = \frac{1}{2}mv^2 + mgR\cos\theta \quad (7.29)$$

Now using Eq. 7.28 this gives

$$mgR = \frac{1}{2}mgR\cos\theta + mgR\cos\theta \tag{7.30}$$

Doing a little algebra this gives $\cos\theta = 2/3$. This is a height of

$$h = R\cos\theta = \frac{2}{3}R \tag{7.31}$$

7.10 Non-conservation of mechanical energy

We saw that when we have just conservative forces, that energy is conserved. But what happens if a non-conservative force such as friction is in a physical situation? Can we still get something out of energy arguments? Well the answer is yes we can. Things aren't quite as powerful as before, but valuable information can still be obtained.

Let's try to go through the derivation of conservation of mechanical energy and see what happens when non-conservative forces are included.

Let's separate out work done on an object into two parts, those derived from conservative forces, such as gravity, and those from non-conservative forces such as friction. The total work W is the some of the conservative and non-conservative contributions,

$$W = W_c + W_{nc} \tag{7.32}$$

For the conservative part W_c it is possible to express it in terms of a potential U, because just as before $W_c = -\Delta U$. So $W = -\Delta U + W_{nc}$. But also we recall that $W = \Delta K$. Together this gives

$$\Delta(K + U) = W_{nc} \tag{7.33}$$

So the *change in mechanical energy is equal to the work done by non-conservative forces.*

7.10.1 Example

A pendulum starts from rest, making an angle of 60 degrees with the vertical. After swinging one cycle and coming back to rest, it now makes an angle of only 45 degrees with the vertical. What is the work done by frictional forces?

Take the mass of the pendulum to be 1 kg and its length to be 1 m.

Solution

Call the length of the pendulum L, its mass m, and the initial and final angles θ_i and θ_f, respectively.

We want to calculate the change in the mechanical energy and that'll give us the work done by frictional forces.

The initial and final kinetic energy of the ball is zero. So the change in mechanical energy is just $\Delta U = U_f - U_i$.

$\Delta U = mg\Delta y = mg(y_f - y_i)$. Using trigonometry we can express the difference in heights Δy in terms of angles, so we have

$$W_{nc} = mgL(\cos\theta_f - \cos\theta_i) = -2.03 \text{ J} \tag{7.34}$$

7.11 Power

The power delivered to a system is the rate energy is transfered to it. That means it has the dimensions of energy/time. In SI units this is Joules per second. We call that *Watts* for short. That should be a familiar unit to you, if you and your friends have ever tried to change a light bulb. If you've ever heard a friend bragging about a car, you might be familiar with the unit of horsepower (hp). 1 hp = 550ft lb/s = 746W. So a 25 hp jalopy can keep going a lot of light bulbs!

The average power supplied to say, light bulb, over a time t is

$$\bar{P} = \frac{\Delta W}{\Delta t} \tag{7.35}$$

and the instantaneous power (or power for short) is

$$P = \frac{dW}{dt} \tag{7.36}$$

Suppose we have an object and we are pushing it with a force $\mathbf{F}$, what power do we deliver to the object? Well in a time t we do work $\mathbf{F} \cdot d\mathbf{r}$. So the power is

$$P = \frac{dW}{dt} = \frac{F \cdot dr}{dt} = F \cdot v \tag{7.37}$$

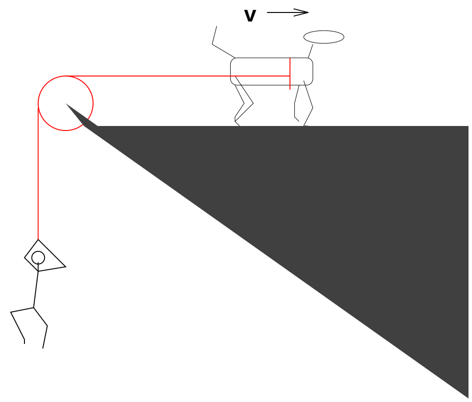
7.11.1 Example

A horse pulls up a 225 lb man at a speed of 2 ft/s as shown below. What is the power exerted by the horse in lifting this man?

The tension in the rope is the weight of the man. And from the equation above the power should be this tension man times the velocity the horse is moving. This is 225 lb $\times$ 2 ft/s = 550 ftlb/s = 1 hp. Now that's one strong horse!

7.12 Stability of equilibria

Potential energy is such a gosh-darned useful concept that many times it's easier to think about physics in terms of it, rather than forces. We'll try to illustrate this in one dimension and use potential energy to describe the "stability of equilibria". So if now we want to think in terms of potential energy $U(x)$, we



would like to know how to relate the potential energy of an object to the force acting on it. Recall that

$$U(x) \;=\; -\int_{x_i}^{x} F(x)dx + U(x_i) \tag{7.38}$$

which is equivalent to

$$F(x) \;=\; -\frac{dU}{dx}. \tag{7.39}$$

Equilibrium points can be thought of as follows. If an object is placed at rest at an equilibrium point, it'll stay there for all time. This is therefore a point where the force acting on the object is zero. This therefore corresponds to a point where the slope of the potential energy curve is zero.

Now let's discuss shapes of various potential energies in one dimension, and see how this leads to ideas of stability.

First, we'll consider this nice concave shaped potential function pictured below

Below it is sketched the force, obtained by taking the negative derivative of the potential energy. On top is pictured the force a red ball will feel when at various x positions. If the ball is to the left of the minimum, the force is positive and therefore points to the right. Conversely, if the ball is to the right of the potential minimum, it'll feel a force to the left. It feels no force at the minimum because there the derivative is zero.

So if the ball starts off at rest at the minimum, it'll stay there. If it starts at rest away from it, it's pulled towards the minimum. Suppose it starts out like the red ball on the right. It'll start moving towards the minimum, transferring potential energy to kinetic energy. Since energy is conserved, the maximum potential energy is when the kinetic energy is zero. That energy E, is shown by the horizontal dashed line. So this means that the particle can't move up to a higher starting point than this horizontal line. This means that the motion is *bounded*. We call this minimum, for obvious reasons, a *stable equilibrium point*.

Now let's look at the upside-down version of the same potential

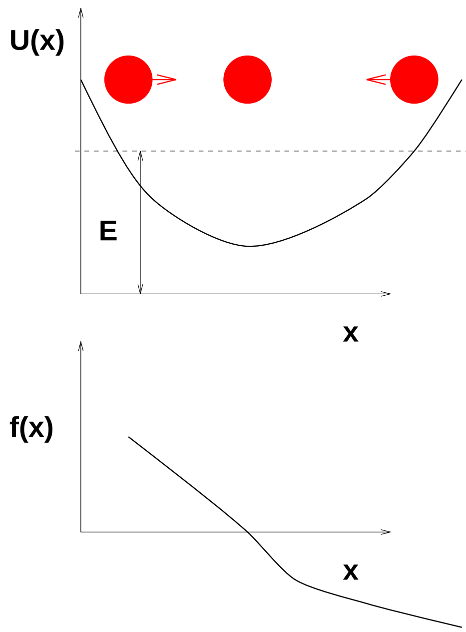


Figure 7.1: Stable equilibrium

Now everything is reversed. The force is the negative of what it was before. So if a red ball starts on the left of the potential maximum, it'll be pushed away from it. If it starts on the right it'll also be pushed away from the maximum. It feels no force at the maximum itself so if a ball starts out at rest at the maximum, it'll stay there for all time.

But what we can see is that if it starts out at rest just slightly away from the maximum, then it'll continue to move away. In this case the motion is *unbounded*. We call this maximum, for equally obvious reasons, an *unstable equilibrium point*.

An equilibrium point of a higher dimensional system can be stable in some directions and unstable in others.

Lastly, consider the potential shown below

Here the force is zero over a finite range in x. If a particle starts off at rest any point on the top of this curve, it'll remain there. It's neither pulled one way or the other. Such a situation is called a *neutral equilibrium*.

Intuitively, it seems that the potential energy is pretty easy to understand. Take the case of the first curve we discussed, that of a stable equilibrium. You could think of the potential energy as being a salad bowl, and the red balls as cherry tomatoes (organic whole ones obviously) that are placed, with a little lubricating dressing, at various starting positions inside the bowl. They'll always roll down towards the center of the bowl. See, the person that invented the salad bowl must have understood something about stable equilibria. If instead

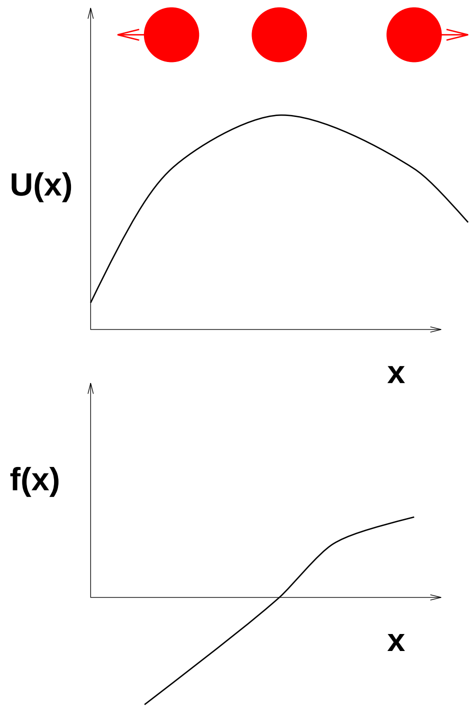


Figure 7.2: Unstable equilibrium

the inventor had got their minus signs mixed up, they would've probably mis-invented an upside-down salad bowl. That'd be bad news at a dinner party. The guests would see the tomatoes careening across the table, spreading your vinagrette all over your new Kmart tablecloth, and eventually on someone's lap. This is the case of an unstable equilibrium, also pictured above.

The case of a neutral equilibrium is what I like to use at my place, use no bowl at all, you just place things directly on the table. This saves on washing up.

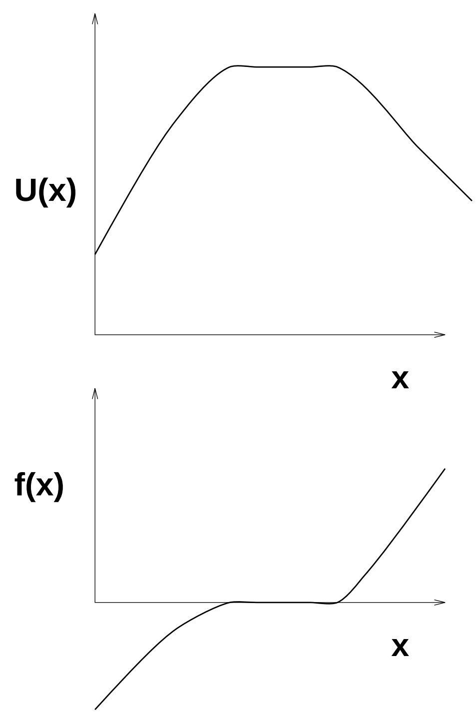


Figure 7.3: Neutral equilibrium

Chapter 8

Momentum

Momentum is a very useful concept owing largely to the fact that as we shall see shortly, it is *conserved*. Like conservation of energy, this allows you to solve a lot of problems that would be very difficult to solve any other way. One important application of this conservation law, is to the study of *collisions*. Collisions occur when two objects, or people, bump into each other. Besides exchanging pleasantries such as "Excuse me, I didn't mean to break your nose", they exchange some of their energy and momentum. We will see how to use this to find out the trajectories of the objects *after* the collsion. But first we need to define momentum.

8.0.1 Definition of Momentum

The *momemtum* of a point particle of mass m going at velocity $\mathbf{v}$ is

$$\mathbf{p} \;=\; m\mathbf{v} \tag{8.1}$$

As a simple example, consider a ball with mass of $0.5kg$ and with a velocity in the positive x direction of $2m/s$, the magnitude of the ball's momemtum is then $0.5 \times 2kg\ m/s\ = 1kg\ m/s$.

Now if you have two particles, with masses m_1 and m_2 and velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, the *total* momentum of these particles , $\mathbf{p}_{tot}$ is

$$\mathbf{p}_{tot} \;=\; \mathbf{p}_1 + \mathbf{p}_2 \;=\; m_1\mathbf{v}_1 + m_2\mathbf{v}_2 \tag{8.2}$$

That is, the momenta add normally as vectors are supposed to. As an example, consider the ball we just considered, and another ball, with mass $1kg$, travelling in the negative x direction with a speed of $-1m/s$. One ball is going to the right, the other to the left. If you sum the two momenta together, you get a total momentum of zero.

Now how about for three particles? How about four particles? How about 45820349 particles? Hmmm, looks like a lot work. We can get around this by getting a little more sophisticated and write the definition of the total momentum of N particles, of masses $m_1, \ldots, m_N$ and velocities $\mathbf{v}_1, \ldots, \mathbf{v}_N$ as

$$\mathbf{p}_{tot} \;=\; \sum_{i=1}^{N} \mathbf{p}_i \;=\; \sum_{i=1}^{N} m_i \mathbf{v}_i \tag{8.3}$$

8.0.2 Relation between momentum and kinetic energy

Sometimes it's desirable to express the kinetic energy of a particle in terms of the momentum. That's easy enough. Since $\mathbf{v} \;=\; \mathbf{p}/m$ and the kinetic energy $K \;=\; \frac{1}{2}mv^2$ so

$$K \;=\; \frac{1}{2}m(\frac{p}{m})^2 \;=\; \frac{p^2}{2m} \tag{8.4}$$

Note that if a massive particle and a light particle have the same momentum, the light one will have a lot more kinetic energy. If a light particle and a heavy one have the same velocity, the heavy one has more kinetic energy.

8.0.3 Relation between momentum and force

We'll start by considering one objects and then move on to consider two or more objects.

One object

By the magic of calculus, we can easily see the relation between force and momentum for a single object. Let's say the a point mass is being acted on by a force $\mathbf{F}$. Then we know its acceleration, right? It's just $\mathbf{F} = m\mathbf{a}$. So what about momemtum? Well if you go into hypnosis you might be able to regress to a time a few weeks ago and then remember the jolting definition of acceleration

$$\mathbf{a} \;=\; \frac{d\mathbf{v}}{dt} \tag{8.5}$$

Using these two equations, we have

$$\mathbf{F} \;=\; m\frac{d\mathbf{v}}{dt} \;=\; \frac{dm\mathbf{v}}{dt} \tag{8.6}$$

But $m\mathbf{v}$ is just the momentum we've just defined, so when I snap my fingers and wake you up, you'll immediately see that

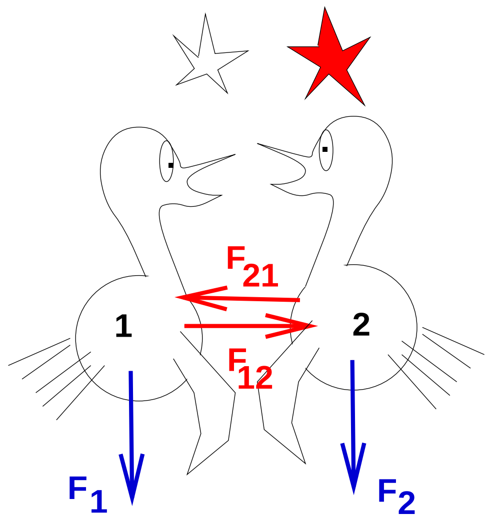
$$\mathbf{F} \;=\; \frac{d\mathbf{p}}{dt} \tag{8.7}$$

This is the *total net force* acting on this one object.

Now let's try to understand two objects.

Two objects

Now we have to distinguish between two different kinds of forces, *internal* and *external*. Think about two booby birds as being our system. Suppose they fly into each other.



There are forces between the birds when they hit, the force that 1 exerts on 2 and the force that 2 exerts on 1, $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ respectively. These are internal to the booby bird system. But both birds are also acted on by the force of gravity, so the weight of each bird, $\mathbf{F}_1$ and $\mathbf{F}_2$, are external forces due to the earth acting on each bird.

The total force acting on booby 1 is the sum of the internal, F_{21}, and the external, F_1, forces, $\mathbf{F}_{t1} = \mathbf{F}_1 + \mathbf{F}_{21}$. Similarly for the second bird, $\mathbf{F}_{t2} = \mathbf{F}_2 + \mathbf{F}_{12}$.

Now let's calculate the *net* external force on these boobies. It's $F_{net,ext} = \mathbf{F}_1 + \mathbf{F}_2$. This is the net external force acting on the system. Remember the birds have forces between each other too, $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$. But I didn't include these. However even if I did *the answer would be unchanged*. That's because of Newton's third law that the internal forces are equal and opposite, so they cancel.

(Of course the forces don't cancel for each booby, they definitely feel the force of their brethren bird. But $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ act on different objects, not on the same bird.)

So we have the net external force:

$$\begin{aligned} \mathbf{F}_{net,ext} &= \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}_1 + \mathbf{F}_2 + [\text{sum of all internal forces (i.e. 0)}] \qquad (8.8) \\ &= [\text{sum of all internal and external forces}] = \mathbf{F}_{t1} + \mathbf{F}_{t2} \end{aligned}$$

But we just learned that

$$\mathbf{F}_{t1} = \frac{d\mathbf{p}_1}{dt} \qquad (8.9)$$

and

$$\mathbf{F}_{t2} = \frac{d\mathbf{p}_2}{dt} \tag{8.10}$$

So combining these we see that

$$\mathbf{F}_{net,ext} = \mathbf{F}_{t1} + \mathbf{F}_{t2} = \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = \frac{d\mathbf{p}_1 + \mathbf{p}_2}{dt} = \frac{d\mathbf{p}_{tot}}{dt} \tag{8.11}$$

At this point I'll drop the *ext* subscript, writing $\mathbf{F}_{net,ext} = \mathbf{F}_{net}$, because we saw that the total net force and the total net external force were the same from Newton's third law. Plus too many extra subscripts make these equations harder to read.

By using summation notation you can easily see how to extend this result to N particles, so that

$$\mathbf{F}_{net} = \frac{d\mathbf{p}_{tot}}{dt} \tag{8.12}$$

Good. Now we see that the rate of change of total momentum is the net (external) force acting on the system. This momentum thing is starting to look more useful!

8.0.4 Conservation of momentum

In many cases the net force acting on a system of particles is zero, or at least very close to it. For example, if you are on a nearly frictionless surface such as ice, the net force in the horizontal direction is zero and you will quickly slip and fall down. If you're wearing skates or rollerblades you might notice that there is little friction in the direction of motion so the net force acting in that direction can be quite close to zero. Since you are a system of particles, actually quite a sizeable number of particles (sorry for getting a bit personal here), then this is an example of a system (namely yourself on skates) that has zero net force acting on it in the horizontal direction. Other systems of particles, such as isolated atoms and molecules, provide another example of cases where often zero net force acts on the system. We'll talk a little bit more later about why even with friction, conservation of momentum can often still apply pretty accurately!

So let's derive conservation of momentum when $\mathbf{F}_{net} = 0$. Eqn. 8.12 becomes

$$0 = \frac{d\mathbf{p}_{tot}}{dt} \tag{8.13}$$

Now if the derivative of something is always zero, that something is a *constant*. This means that

$$\mathbf{p}_{tot} = constant \tag{8.14}$$

That is the total momentum of a system is conserved, if the net force acting on the system is zero.

Let's talk about how poweful this conservation law is. It's kind of like energy conservation, inside the system, momentum can be transfered from one particle to another. But when this happens, the total momentum must stay unchanged.

Let's see how this works through an example.

8.0.5 Example

A spacecraft is launched from the ground, going up vertically. The engineers who designed it didn't learn their mechanics very well because they spent too much time partying. As a result, the spacecraft quickly breaks up 1000 m off the ground into two pieces. The first has a mass of 10000 kg and the second a mass of 20000 kg. The first one is seen flying off initially in the horizontal direction with a velocity of 100 m/s due east. What was the horizontal velocity of the second piece? Ignore air resistance.

8.0.6 Solution

So can we apply conservation of momentum? Well, in the horizontal direction we're ignoring air resistance, so the net force on the pieces is zero. Now there were certainly some large forces acting on the the spaceship, otherwise it wouldn't have exploded. Don't these screw things up? No. Because those forces are between atoms *that are part of our system*. The different atoms are pushing and pulling each other, in a hideously complicated way, but they're all internal to the system. The net (external) force acting on the system in the horizontal direction is still zero. So we *can* apply conservation of momentum.

The initial momentum in the horizontal direction is zero. So the final momemtum should also be zero:

$$m_1 v_1 + m_2 v_2 = 0 \tag{8.15}$$

Solving for v_2, we have

$$v_2 \; = \; -(m_1/m_2)v_1 \; = \; (10000/20000)100 \text{ m/s} \; = \; 50 \text{ m/s}. \tag{8.16}$$

8.1 Center of Mass

Now we introduce an important concept that often allows us to think of a collection of particles in terms of a single vector. Take for example a baseball. We don't really want to think of atom 1029890123890 being at position (1.112323,-2.21334123,0.123340984) cm and atom 1029890123891 being at position (1.112324,-2.21334122,0.123340985) cm *etc.*. It would take a long time to describe that baseball. We have the feeling that there is a simpler way to describe the baseball. It's got a radius and a mass, and a center coordinate. What we'll see now, is that the proper center coordinate to use for a ball, or any system of particles is the *center of mass*.

Let's start with two particles, a mass m_1 at position $\mathbf{r}_1$ and m_2 at position $\mathbf{r}_2$. The total mass $M \equiv m_1 + m_2$. The center of mass $\mathbf{r}_c$ is defined as

$$\mathbf{r}_c \; = \; \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M} \tag{8.17}$$

For N masses, using summation notation we can write

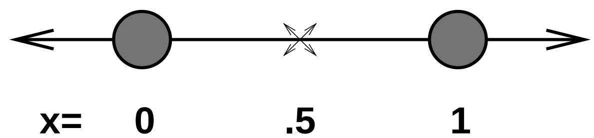
$$\mathbf{r}_c = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i} \tag{8.18}$$

8.1.1 Example

Consider two heavy lead balls, each with mass 10 kg seperated by a distance of a meter. Where is the center of mass of this system?

8.1.2 Solution

Let's draw a diagram to see what's going on:



Define coordinates for this problem. Put the first mass m_1 at the origin $\mathbf{r}_1 = 0$, and put mass m_2 at the coordinate $\mathbf{r}_2 = (1, 0, 0)$. Let's apply definition Eq. 8.17. Note that the y and z coordinates of the center of mass are zero, the only non-trivial one is the x coordinate. So in the x-direction

$$x_c = \frac{1 \times 0 + 1 \times 1}{2} = 0.5m \tag{8.19}$$

Note that if we made m_1 much heavier than m_2 than the center mass would be shifted to the origin. If the converse where true, the center of mass would be shifted towards m_2. When they're equal, the center of mass lies right in the middle between the objects.

8.1.3 Neat trick for calculating center of mass

Suppose we've arduously calculated the center of mass of a system of 6 masses $m_1, \ldots, m_6$. And now we want to include another 3 $m_7, \ldots, m_9$. Do we have to redo the entire calculation? No! A little mathematical manipulation should convince you that we can the calculations of the center of masses of masses 1-6 and 7-9. to calculate the center of mass of masses 1-9 . This is illustrated pictorially in Fig. 8.1.

Call the center of mass of 1-6 $\mathbf{r}_{1-6}$. It has a total mass of M_{1-6}. Similarly call the center of mass of 7-9 $\mathbf{r}_{7-9}$. It has a total mass of M_{7-9}. Then what happens if we calculate the center of mass of these two "pseudo" particles? We get

$$\frac{M_{1-6}\mathbf{r}_{1-6} + M_{7-9}\mathbf{r}_{7-9}}{M_{1-6} + M_{7-9}} \tag{8.20}$$

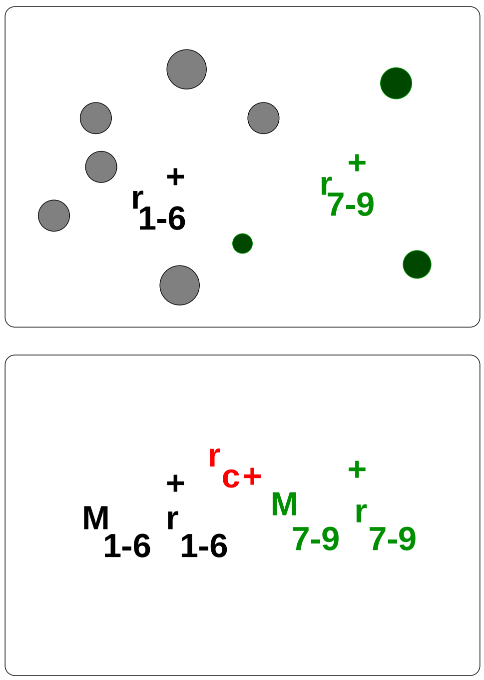


Figure 8.1: Illustration of how to get the center of mass for 9 balls, given you already know the center of mass of the first 6 and the remaining 3. Above you see 6 balls in grey, and their center of mass is at the black "+" symbol. The remaining 3 balls have their center of mass at the green "+" symbol. Below you the center of mass for all 9 balls r_c is the red "+" symbol. It's obtained by considering the total mass of the first 6 balls M_{1-6} to be concentrated at the center of mass r_{1-6}. And similarly for the remaining balls 7-9. The center of mass of these two concentrated points gives you the center of mass of the entire system of 9 balls.

By plugging in the definitions of center of mass, you should be able see that you get quite a few cancellations and finally convince yourself that this is equal to

$$\frac{m_1\mathbf{r}_1 + m_1\mathbf{r}_1 + \cdots + m_9\mathbf{r}_9}{m_1 + m_2 + \ldots m_9} \tag{8.21}$$

which is just the center of mass of the complete 9 mass system.

8.2 Center of mass for a continuous bodies

What is the center of mass of a uniform sphere? That isn't too bad. By symmetry it's at the sphere's center. But what if the sphere is not of uniform density? The top hemisphere is made out of balsa wood and the bottom hemisphere is made out of roquefort cheese? That's a bit more tricky, and to solve this sort of problem, we'll have to formulate it in terms of integration. Let's start off with the one dimensional case. Suppose that you have a rod that has a mass per unit length $\lambda(x)$ (that can vary with position x. We choose coordinates so that the left hand side of the rod is at 0 and the right hand side is at L. To obtain the center of mass, we break up the rod into millions of tiny section each of length Δx. Then the center of mass is the sum of all the individual masses m_i times their corresponding position x_i. So

$$x_c = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_N x_N}{m_1 + m_2 + \cdots + m_N} \tag{8.22}$$

Now one of these little masses $m_i = \Delta x \lambda(x_i)$ so

$$x_c = \frac{\Delta x \sum_{i=1}^{N} \lambda(x_i) x_i}{\Delta x \sum_{i=1}^{N} \lambda(x_i)} \tag{8.23}$$

You might remember from calculus expressions of this kind. If we take the limit as $\Delta x \to 0$ then these become integrals

$$x_c = \frac{\int_0^L \lambda(x) x dx}{\int_0^L \lambda(x) dx} \tag{8.24}$$

Since $\lambda(x)dx = dm$, an infinitessimal of mass, this integral can be more elegantly, but less usefully written as

$$x_c = \frac{\int x dm}{\int dm} \tag{8.25}$$

Example

Let's apply 8.24 to the case where the density of a rod varies linear with position, that is, $\lambda(x) = cx$, where c is a constant that won't make a difference to the final answer. (If you are not familiar with integral calculus, an alternative solution will be given later.)

Solution

In this case we get

$$x_c = \frac{\int_0^L (cx)x dx}{\int_0^L (cx)dx} \tag{8.26}$$

The c's cancel and we get

$$x_c = \frac{x^3/3\Big|_0^L}{x^2/2\Big|_0^L} = \frac{2}{3}L \tag{8.27}$$

As expected, the center mass is shifted for the middle towards the region of higher density.

8.2.1 Higher dimensions

In two or three dimensions, we can perform the same kind of analysis to to obtain the center of mass $\mathbf{r}_c$ of a system

$$\mathbf{r}_c = \frac{\int \mathbf{r} dm}{\int dm} \tag{8.28}$$

This is now a multidimensional integral that needs to be performed.

Example

Let's illustrate how this works for an icoscoles right triangle. (Again, a solution without integration will be given later.) One corner is at the origin. The right angled corner is at $(L, 0)$ and the top 45 degree corner is at (L, L). The triangle is taken to be of constant density σ. The triangle is shown in Figure 8.2.

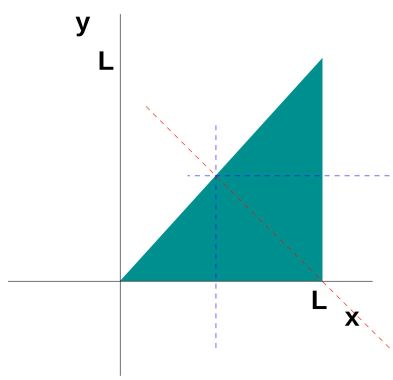


Figure 8.2:

Solution

We break up the integration into x and y directions. By symmetry if we get the answer along the x direction, we can easily figure out what it should be along the y direction. In fact it's intuitively clear that the center of mass lies along the red dashed line that goes 45 degrees through the right angled corner, as seen in Figure 8.2.

So let's get the answer for the x direction.

$$x_c = \frac{\int x dm}{\int dm} = \frac{\int_0^L \int_0^x x\sigma dy dx}{\int_0^L \int_0^x \sigma dy dx} \tag{8.29}$$

This means the following. As shown in Figure 8.3, e've divided up the integration into vertical strips. We first integrate over y to find out the answer in one strip. Then we integrate over all x to find the final answer, integrated over all strips.

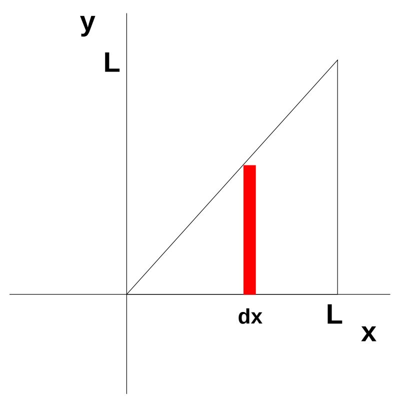


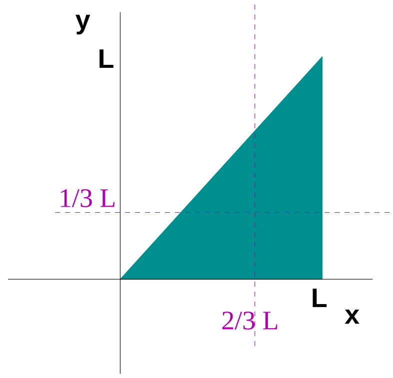
Figure 8.3:

Doing the top and bottom integrals over dy we have

$$x_c = \frac{\int_0^L x^2 dx}{\int_0^L x dx} \tag{8.30}$$

By complete chance, we just did this calculation for the one dimensional rod, see Eq. 8.27. The answer was $\frac{2}{3}L$. So we know the center of mass in the x direction. The y-direction is almost the same problem, but slightly different. In the x-direction, the triangle is becoming thicker with increasing x, in the y direction it is becoming thinner with increasing y. So it makes sense just to say the center of mass in the y direction is $\frac{2}{3}L$ from the thinner end. This means it's $\frac{1}{3}L$ from the thicker end, so it has coordinate $y_c = \frac{1}{3}L$.

Tricks like these save you from having to do tedious calculations. You can do the integral the more complicated way if you're unsure that this reasoning is correct.



As promised, the same result can be obtained without any calculus. In Figure 8.2, the triangle is cut into four identical isoceles right angle triangles by the red dashed line and the two blue dashed lines, with one of the triangles being upside down. By symmetry, the distances from the x axis of the four centers of mass are $a, a, L/2-a$ and $L/2+a$ for some (as yet) unknown a. Since all the four triangles have the same size, the center of mass of their combination will be at the average of all four centers, i.e. at $y = a/2 + L/4$. But the big triangle is simply twice as big as each of the small triangles, and therefore this distance must be equal to $2a$. Therefore $2a = a/2 + L/4$, i.e. $a = L/6$, and the center of mass of the big triangle is at a distance $2a = L/3$ from the x axis. The distance from the y axis can be obtained by symmetry to be $L - L/3 = 2L/3$. (It is easy to generalize this to an arbitrary triangle, since any triangle with its base oriented horizontally can similarly be cut into four identical pieces.)

As a byproduct, we see that if we squeeze the triangle into a rod along the x axis, with all the mass in a vertical slice dx (see Figure 8.3) moved down to the x axis, we obtain a rod whose mass per unit length is $\lambda(x) \propto x$, and its center of mass is at $x = 2/3L$.

We got lucky with the triangle, being able to find its center of mass using symmetry. For an irregular shape, doing integrals is the only way to find the center of mass.

8.2.2 Gravitational potential energy of a body

An interesting application of the above formulas is in calculating the gravitational potential energy of a body. Suppose you have a turnip and wish to calculate its potential energy. Well for every atom i of mass m_i in the turnip, the potential energy, is just $m_i g$ times the vertical position of the atom y_i. To get the total potential energy, we sum over all atoms:

$$U = \sum_i m_i g y_i = g \sum_i m_i y_i \qquad (8.31)$$

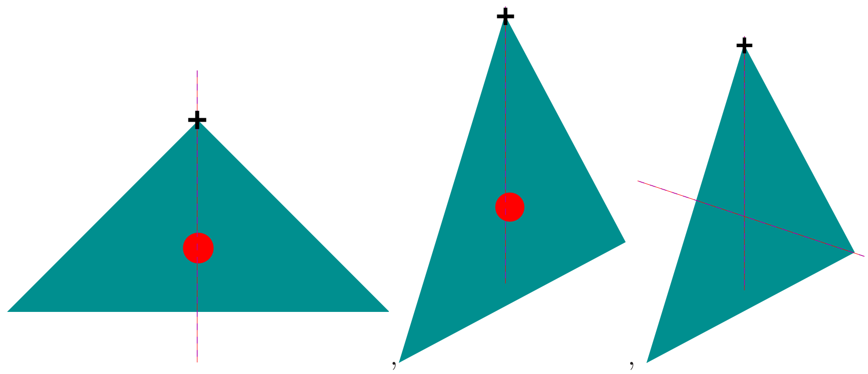


Figure 8.4:

By some fancy foot work we see this is

$$U = \left(\frac{\sum_i m_i y_i}{\sum_i m_i}\right) \sum_i m_i \ g \tag{8.32}$$

But $\sum_i m_i$ is just the total mass M_t, and the term in parenthesis is just the definition of the center of mass in the z-direction, y_c. So finally we obtain the simple formula:

$$U = M_t g y_c \tag{8.33}$$

In other words, to compute the potential energy of a turnip, we just have to find its center of mass, and its total mass. If we *replace* the turnip with a point of mass M_t located at the center of mass, it will have the same potential energy.

This gives us a nifty way to experimentally determine the center of mass. Take the right icoscoles triangle we looked at earlier, see Eq. 8.2.1.

If we hang it from the right angled corner, the hypotenuse will lie horizontally, see the first panel in Fig. 8.4.

Why? We can replace the triangle by the center of mass (the red spot), and we're hanging it from the top (the + symbol). A single mass will always dangle vertically (dashed purple line) from where it's being held, because it wants to minimize its potential energy (i.e. height of the vertical component of the center of mass).

We could also hang it from another corner and the same thing should happen. The red spot should dangle vertically from the point its being held, see the second panel in Fig. 8.4.

If we *didn't know where the center of mass was*, this method would allow us to determine its position. It's just the intersection of the two purple lines as shown in the third panel in Fig. 8.4.

This is a neat way of determining the center of mass of a body.

8.3 F=ma revisited

We can differentiate Eq. 8.18 with respect to time.

$$\frac{d\mathbf{r}_c}{dt} = \frac{d}{dt}\frac{\sum_{i=1}^{N} m_i\mathbf{r}_i}{\sum_{i=1}^{N} m_i} \tag{8.34}$$

Since the m's are constant, the differentiation only acts on the r's, giving

$$\frac{d\mathbf{r}_c}{dt} = \frac{\sum_{i=1}^{N} m_i\mathbf{v}_i}{M} \tag{8.35}$$

or

$$\mathbf{v}_c = \frac{\sum_{i=1}^{N} m_i\mathbf{v}_i}{M} = \mathbf{p}_{tot}/M \tag{8.36}$$

This implies that if the momentum $p_{tot}$is conserved, so is the velocity of the center of mass v_c.

So we have the equation

$$\mathbf{p}_{tot} = M\mathbf{v}_c \tag{8.37}$$

This says that the total momentum of a system is the same as when all the mass is concentrated at one point, at the system's center of mass!

If you can keep your eyes open a little longer, we can go further and differentiate this again:

$$\frac{dp_{tot}}{dt} = M\mathbf{a}_c \tag{8.38}$$

So from 8.12 we finally have that

$$\mathbf{F}_{net} = M\mathbf{a}_c \tag{8.39}$$

This says that the net force acting on an object is just the total mass times the acceleration of the center of mass.

Didn't we use this already? Isn't this just Newton's second law?

Yes, we did use this already! When analyzed blocks going down inclined planes, which of course are non-particle-like objects. And we did apply F=ma to them. But err, we shouldn't have done this so cavlierly! Newton's second law really should have been defined for point objects. What we just showed is that it also works for blocks, turnips, camels, or what ever system you care to choose. You just have to use the total mass of the system for "m" and the center of mass acceleration for "a". The force $\mathbf{F}_{net}$ is the net *external* force acting the object.

I could have gone through all this math a few chapters ago, but it wasn't really necessary to solve problems. I'm including this to show you why what you did was OK. I also figured you might be in need of a good nap.

Now it's time to wake up and do some more physics problems!

8.3.1 Example

An astronaut of mass 80 kg drifts away from a spacecraft with a center of mass velocty of 2 m/s. The astronaut is very silly and starts dancing, trying hard to do the "hustle". Describe the velocity of the center of mass as a function of time.

Assume, as is usual in this situation that no external forces act on the astronaut.

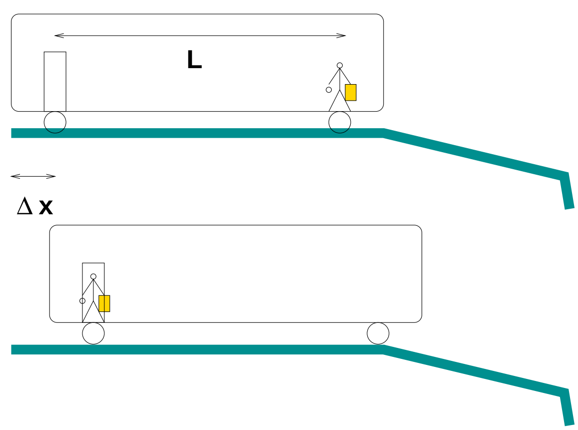
8.3.2 Solution

The momentum of the astronaut is conserved. So as you see from Eq. 8.37 that the velocity of the center of mass of the astronaut is constant.

How can that be? If you're out there in free space, you can certainly move around, I've seen pictures of the space shuttle and things like that. So can't you change your velocity? Well you can change the velocity of your hand, or your foot, but since total momentum is conserved, it can't change, hence your center of mass velocity will remain the same. If it starts out as 2 m/s, it'll stay that way. If it starts out as zero, it'll stay that way too. If you have a rocket pack and turn on the jet thrusters, you can change the velocity of your center of mass, but this is compensated by the velocity of the gas ejected by the jet thrusters; the center of mass of the ejected gas and you *together* moves at constant veolocity.

8.3.3 Example

A bad guy of mass 100 kg has just robbed a bank and is carrying 200 kg of gold bullion. The problem is that he's inside of a stationary freight car with nearly frictionless bearings, and the freight car is on the verge of a precipice as pictured.



If he tries to move to the left with his gold all the way to the door $L = 30$ ft away, the freight car, which weighs $M_f = 2700$ kg will move to the right, very possibly far enough so that he'll plunge with the car down the precipice. The freight car has to move only 2 ft to the right for this to happen. Will he make it to the door?

8.3.4 Solution

If the bearings are frictionless, the freight car plus robber has a conserved momentum of zero, meaning from Eq. 8.37 that the center of mass velocity is zero. That implies that the center of mass x_c is constant. The robber could be pounding against the wall or running around in a violent rage, and this would not alter the position of the center of mass of the complete system one iota. So if the robber goes to the left, the freight car must go to the right to compensate.

Let's then make use of center of mass conservation to solve the problem. Let's choose a coordinate system where the center of mass of the freight car (right in its middle) is at the origin.

Initially the robber is at a distance $x_r = L/2$, and the mass of robber plus gold is $M_{rg} = 300$ kg.

The center of mass of the system is then

$$x_c = \frac{M_{rg} x_r}{M_{rg} + M_f} = \frac{M_{rg} L/2}{M_{rg} + M_f} \tag{8.40}$$

After moving to the door, the freight car has shifted an amount Δx and therefore the center of mass of the the car is now at Δx. That's what we're interested in finding. The position of the robber is now $-L/2 + \Delta x$. The center of mass is then

$$x_c = \frac{M_{rg}(-L/2 + \Delta x) + M_f \Delta x}{M_{rg} + M_f} \tag{8.41}$$

Equating the center of mass before with the center of mass after the robber has moved gives:

$$M_{rg} L/2 = M_{rg}(-L/2 + \Delta x) + M_f \Delta x \tag{8.42}$$

Moving the first term on the right hand side to the left and factoring the remaining terms of the right hand side gives

$$M_{rg} L = \Delta x (M_{rg} + M_f) \tag{8.43}$$

or

$$\Delta x = \frac{M_{rg}}{M_{rg} + M_f} L \tag{8.44}$$

Plugging in numbers we have

$$\Delta x = \frac{300 \text{ kg}}{300 \text{ kg} + 2700 \text{ kg}} 30 \text{ ft} = 3 \text{ ft}. \tag{8.45}$$

This is greater than 2 ft, so the robber would end up as fertilizer for some weeds.

8.3.5 Example continued

But is there a way for our poor robber to leave the freight car and become a changed man? He promises to be a good boy, keep off the street and study physics. What can we do to help him?

8.3.6 Solution

Perhaps leaving the gold at the right hand side of the freight car and walking over to the door. How can we solve this problem without doing much more work?

So the trick is to regard the gold as *part of the freight car*. Note that we didn't have to assume a symmetrical mass distribution for the freight car (try changing its center of mass from $L/2$ to some other value, say X, in the previous solution). So this should work fine. Eqn. 8.44 should be changed a litte. M_{rg} is no longer the combined mass of robber plus gold, since the robber is now separated from the gold. This term should be M_{rg}. The denominator $M_{rg} + M_f$ is altered two ways. M_{rg} is reduced by the mass of the gold, but the mass of the freight car M_f is increased by the same amount. So the denominator is unaltered.

We get then

$$\Delta x \;=\; \frac{100kg}{300kg + 2700kg} 30ft \;=\; 1ft. \tag{8.46}$$

So he'll make it if he abandons the gold. Of course he could still take $100kg$ of gold with him and get out alive, but we're all honest citizens and won't tell him that. His new career as a physicist will be enough of a reward.

8.4 Collisions

Collisions are when two objects bang into each other. An example is when you play pool. If you're really good, you'll be able to actually aim the stick well enough so that the white ball hits another ball on the table. When that happens, you get a collision. There are two extreme versions of collisions. Totally inelastic collions, and elastic collisions.

Totally inelastic collisions are when the two objects that hit each other stay in contact. Think about two pieces of chewing gum being thrown at each other and sticking. On the other hand don't. It's too disgusting.

When a dropped superball hits the ground, it bounces up to almost its original height. That's the opposite case of an elastic collision. In both cases the actual collision process is quite short. Often of the order of milliseconds. We'll try to understand the process of the collision now in a little more detail.

8.4.1 The Impulse

Suppose we look at a superball hitting a hard table with some high speed camera. What will we see? Well, we'll see the ball getting kind of smushed against the table. The more smushed, the greater the force the table exerts on the ball. So like a brontosaurus, the normal force of the table starts off small, gets bigger, and then gets smaller again.

So the external force acting on the superball, namely the normal force of the table is some function of time pictured below.

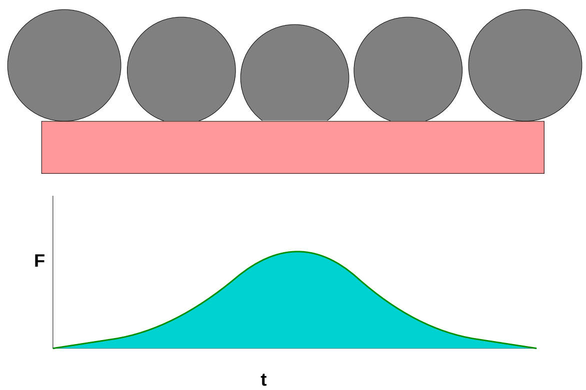


Figure 8.5:

Now how do we relate this to the momentum? Let's use Eq. 8.12.

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{8.47}$$

The time derivative of the momentum of the ball **p** is equal to the normal force of the table **F**

You can think of this as saying that normal force of the table is decelerating the ball, causing it to slow down, so the momentum decreases. Eventually the ball completely changes direction because the force continues to act on the ball even after it has become maximally squashed. So the ball accelerates away from the wall until it looses contact with the table.

So if the normal force is responsible for the change in momentum, how is the total change in momentum related to this force?

Well we just integrate Eq. 8.12 from the initial time of contact t_i to the final time of contact t_f. This gives an expression for the change in momentum

$$\Delta\mathbf{p} = \mathbf{p}(t_f) - \mathbf{p}(t_i) = \int_{t_i}^{t_f} \mathbf{F}(t)dt \tag{8.48}$$

This is just the area under the force curve pictured in figure 8.5, and is called the "**impulse**". We can get an idea of the magnitude of the force by calculating the forced averaged over the interval from t_i to t_f.

$$\bar{\mathbf{F}} = \frac{\int_{t_i}^{t_f} \mathbf{F}(t)dt}{t_f - t_i} \tag{8.49}$$

From the last equation this means

$$\bar{\mathbf{F}} = \frac{\Delta \mathbf{p}}{\Delta t} \tag{8.50}$$

Ok, the derivation of this seemed like a lot of work, but the final result is quite interesting. The average force felt during a collision is the ratio of the change in momentum to the change in time.

So what happens if a collision is really fast? The average force during that time will be really large. Let's think about this in everyday life. When you drop a glass on a tile surface it breaks, where as on a shag carpet it won't even chip. What's the difference? About $10 if was a fancy glass like the kind my mom likes to keep around (but never uses except for when they have "sophisticated" company). But seriously, the difference is that the carpet "gives" more than the tile. It's softer. This means that the collision lasts a lot longer than on the tile. So the average force will be quite a bit larger on the tile, causing the glass to break.

With a lot of collisions, baseballs against bats, pool balls against each other, glass against tile, and countless other situations, the time of the collision is very short, of order milliseconds and the change in momentum is quite large. In this case you get quite humongous forces during a collisions. Let's do an estimate. Assume a time of contact of 10^{-3} s a mass of 1 kg, and a change in velocity of 1 m/s. You get an average force of (1 kg $\times$ 1 m/s)/10^{-3} s which is 10^3 N. The weight of a 1 kg mass is about 10 N, so that the force during the collision is 100 times greater than the force of gravity. No wonder people try to avoid getting hit by baseball bats!

What that means is that during one of these fast collisions, you can ignore the effect of gravity, friction, springs, or any other force acting. The collisional force will exceed these other forces.

With this in mind, let's discuss collisions in more detail.

8.4.2 Totally inelastic collisions

Bobby is cycling east at $5m/s$, and Lisa is cycling north at $10m/s$, when they happen to run into each other, and not in a good way. They and their bikes get all tangled up and start moving together. What's the velocity of the Bobby-Lisa-bicycle glob right after the collision? Let's take the combined mass of Bobby and his bike to be $m_B = 100kg$ and Lisa and her bike to be $m_L = 50kg$. (Don't worry, Bobby and Lisa are quite fine. This is not a real example.)

To figure this out, we'll use conservation of momentum, as we claimed that to a good approximation, we can ignore other forces during the collision.

Before: The total momentum $p_{tot} = m_B\mathbf{v}_B + m_L\mathbf{v}_L$

After: The $p_{tot} = (m_B + m_L)\mathbf{v}_f$. where $\mathbf{v}_f$ denotes the final velocity, right after the collision.

Equating these two expressions (since momentum is conserved) we have

$$\mathbf{v}_f = \frac{m_B \mathbf{v}_B + m_L \mathbf{v}_L}{m_B + m_L} \tag{8.51}$$

Now $\mathbf{v}_B = 5\hat{i} m/s$ and $\mathbf{v}_L = 10\hat{j} m/s$ so we get

$$\mathbf{v}_f = \frac{100i \times 5\hat{i} + 50 \times 10\hat{j}}{150} m/s = 3.33(\hat{i} + \hat{j}) m/s \tag{8.52}$$

The combined mass goes shooting off in the north-east direction.

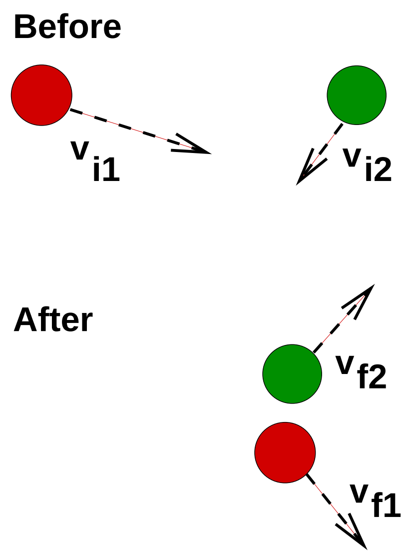
This is an example of a totally inelastic collision. When the two masses hit, they stick together. The final velocity is just the center of mass velocity of the system, since the center of mass velocity is constant for any process obeying conservation of momentum.

Momentum is conserved but in general, energy is not. You could calculate the change in kinetic energy during this collision and would find that it is negative. What happens to this energy? It mostly goes into heat, some of it goes into sound waves. The scrunching sound that you hear is powered by the initial kinetic energy of our two unfortunate bicyclists, (who of course are not real).

8.4.3 Elastic collisions in one dimension

With elastic collisions, energy is also conserved. The only really good examples of elastic collisions involve atomic particles and the like. But you can get pretty close in ordinary life, say with pool balls or a superball.

Let's say we start with two balls with masses m_1 and m_2 having initial velocities $\mathbf{v}_{i1}$ and $\mathbf{v}_{i2}$ respectively. They collide and end up with velocities $\mathbf{v}_{f1}$ and $\mathbf{v}_{f2}$



Before: The total momentum $\mathbf{p}_{tot} = m_1\mathbf{v}_{i1} + m_2\mathbf{v}_{i2}$. The total energy is $E = \frac{1}{2}m_1 v_{i1}^2 + \frac{1}{2}m_2 v_{i2}^2$.

After: The total momentum $\mathbf{p}_{tot} = m_1\mathbf{v}_{f1} + m_2\mathbf{v}_{f2}$. The total energy is $E = \frac{1}{2}m_1 v_{f1}^2 + \frac{1}{2}m_2 v_{f2}^2$.

Equating the total momentum and energy before and after the collision

$$m_1\mathbf{v}_{i1} + m_2\mathbf{v}_{i2} = m_1\mathbf{v}_{f1} + m_2\mathbf{v}_{f2} \quad (8.53)$$
$$m_1 v_{i1}^2 + m_2 v_{i2}^2 = m_1 v_{f1}^2 + m_2 v_{f2}^2 \quad (8.54)$$

Let's try to analyze the collision of two balls in one dimension.

So now we just have to solve for the final velocities v_{f1} and v_{f2} and we're done. But how do you do that? One equation is linear and the other quadratic? If you try to substitute one into the other, you get a big mess. You can get around this by being a little clever, but there is a better way to think about all this which gives you more physical insight into what's going on with the physics of this problem.

Center of mass reference frame

What we'll do is change to a moving coordinate system, which goes with the velocity of the center of mass! That's sounds weird. It makes the problem sound even harder. But actually you'll see how simple things look in this frame.

Remember we said that if momentum is conserved, the center of mass velocity of the system v_c is also. As the collision is taking place, it doesn't alter the motion of the center of mass a bit. It just plods along at a constant velocity. If we were coasting along on a bike at this center of mass velocity, watching the collision, what would we see?

Well in this reference frame, the center of mass velocity, by definition, is zero. And therefore by Eq. 8.37 the total momentum is also zero. I'll notate all variables in this new system the same as the old, but just to remind ourselves that we're in this new frame I'll also add " ' " to them. So for example, the initial momentum of the first particle is denoted p'_{i1}.

So let's play before and after again, but this time in the center of mass reference frame.

Before: The total momentum $p'_{tot} = 0 = p'_{i1} + p'_{i2}$. The total energy is $E' = \frac{1}{2m_1}{p'}_{i1}^2 + \frac{1}{2m_2}{p'}_{i2}^2$.

After: The total momentum $p'_{tot} = 0 = p'_{f1} + p'_{f2}$. The total energy is $E' = \frac{1}{2m_1}{p'}_{f1}^2 + \frac{1}{2m_2}{p'}_{f2}^2$.

This looks a lot simpler. The momentum equations say that the particles have equal and opposite momenta, $p'_{i1} = -p'_{i2}$ and $p'_{f1} = -p'_{f2}$ Using this, equating energy is almost as easy

$${p'}_{i1}^2/m_1 + {p'}_{i1}^2/m_2 = {p'}_{f1}^2/m_1 + \frac{1}{2}{p'}_{f1}^2/m_2 \quad (8.55)$$

Factoring the masses and cancelling gives ${p'}^2_{i1} = {p'}^2_{f1}$. There are two solutions to this. One is kind of boring, $p'_{i1} = p'_{f1}$. It means that before and after, nothing changes. This certainly obeys conservation of energy and momentum, but means that the particles haven't bounced off each other. So what's the other more interesting solution? It's $p'_{i1} = -p'_{f1}$. From conservation of momentum, that means $p'_{i2} = -p'_{f2}$. In terms of velocity this gives

$$v'_{i1} = -v'_{f1} \tag{8.56}$$

$$v'_{i2} = -v'_{f2} \tag{8.57}$$

This says that after the collision, the two balls have reversed their initial velocities. That's it. This satisfies both momentum and energy conservation.

Back to the old frame

What did going to the center of mass velocity frame do for us? It told us the two balls reverse their velocties in that particular frame. But we want to know what happens in our original frame. Can we express what we just learned in a way that'll help us solve our original problem?

If we ask what happens to the *difference* in velocities before and after the collision, that'll tell us something independent of reference frame. The difference in the velocity of the two balls is always the same irrespective of frame. So what happens to the difference in velocities?

In the center of mass frame we see from Eq. 8.57 that $v'_{i1} - v'_{i2} = -(v'_{f1} - v'_{f2})$. But as was just stated, this is true in our original frame

$$v_{i1} - v_{i2} = -(v_{f1} - v_{f2}) \tag{8.58}$$

That is, the difference in velocities just reverse sign after the collision. You could also say that for any one dimensional elastic collision, *the relative velocity of approach equals the relative velocity of recession.* This is true in any reference frame. This equation is nice because it's linear. *It is key to figuring out 1d problems with elastic collisions.*

So we now have two linear equations that we have to solve

$$m_1 v_{i1} + m_2 v_{i2} = m_1 v_{f1} + m_2 v_{f2} \tag{8.59}$$

$$v_{i1} - v_{i2} = v_{f2} - v_{f1} \tag{8.60}$$

Multiply the second equation by m_1, and add it to the first equation to eliminate v_{f1}

$$m_1 v_{i1} + m_2 v_{i2} + m_1 v_{i1} - m_1 v_{i2} = (m_1 + m_2) v_{f2} \tag{8.61}$$

Solving for v_{f2}

$$v_{f2} = \frac{2m_1}{m_1 + m_2} v_{i1} + \frac{m_2 - m_1}{m_1 + m_2} v_{i2} \tag{8.62}$$

We can get v_{f1} most easily by interchanging the indices 1 and 2

$$v_{f1} = \frac{2m_2}{m_1 + m_2} v_{i2} + \frac{m_1 - m_2}{m_1 + m_2} v_{i1} \tag{8.63}$$

Let's do a couple of examples now.

8.4.4 Example

A 1000 kg lead ball traveling at 1 m/s hits a .01 kg superball that is at rest. Calculate the final velocity of the two objects after the collision, if it is elastic.

8.4.5 Solution

In this example we have $v_{i1} = 1$ m/s, $m_1 = 1000$ kg, $v_{i2} = 0$ and $m_2 = 0.01$ kg. Plugging these into Eqs. 8.62 and 8.63 we have to a good approximation

$$v_{f2} = \frac{2}{1} 1 \text{ m/s} + 0 = 2 \text{ m/s} \tag{8.64}$$

and

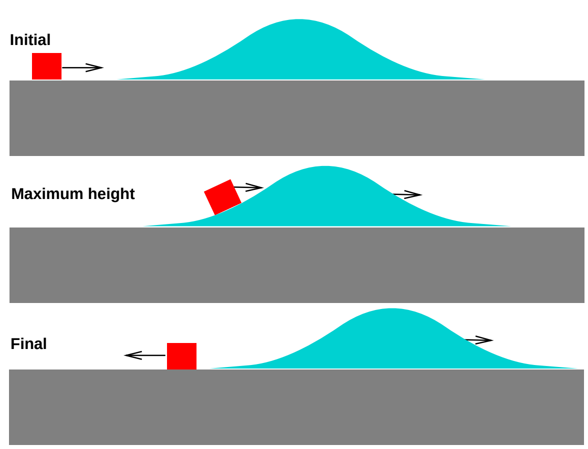
$$v_{f1} = 1 \text{ m/s} \tag{8.65}$$

So what does this say? The heavy ball keeps on going pretty much with its same initial velocity. That's not too surprising. However the lighter ball gets propelled to *twice* the initial velocity of the heavy ball. That might seem a but unexpected, but you can understand this by going to the reference frame of the heavy ball.

In this frame the heavy ball sees this piddling little ball coming towards it at 1 m/s and then bounces off it. The lighter ball will reverse it's velocity traveling away at 1 m/s. But the heavy ball will just keep on truckin, its velocity hardly altering due to the collision. Now let's look back in the reference frame of the ground. If you add up the 1 m/s of the heavy ball to the 1 m/s of the lighter ball relative to the heavy ball, you get 2 m/s.

8.4.6 Example

Consider a block going towards a mobile hump. Mobile hump? Is this getting esoteric or what! Well, look why not? This could be an oddly shaped piece of ice that slides friction-free on a surface. See the figure below.



Let's suppose the block and the hump have the same mass m and that the block, pictured in red, starts off with a velocity $v_i = 1$ m/s towards the hump which is initially stationary. The hump is of height 1 m.

a How far will the block rise?

b At that point, what is the velocity of the hump and the block?

c What is velocity of the hump and block after the block slides down again?

8.4.7 Solution

First of all, it might rise all the way to the top and back down the other side! We don't know that yet. Let's assume that it doesn't get all the way to the top and solve for what height we get. If this is greater than 1 m, then we know that it get's to the other side.

So what is the physics at the point where the block reaches it's maximum height. Well in the reference frame of the hump, the block rises up it and comes to a maximum as pictured above. At this point the block is stationary relative to the hump. That means that at this point, the block and the hump have the same velocity. We already considered this type of problem earlier! It's identical to the problem of inelastic collisions. The velocity in this case is just the center of mass velocity of the system which by Eq. 8.37 is

$$v_c = p_{tot}/M = (mv_i/2m) = v_i/2 = 0.5 \text{ m/s} \tag{8.66}$$

This is the answer to part b of the problem.

At that point the total kinetic energy of the system is

$$K_c = \frac{1}{2}mv_c^2 + \frac{1}{2}mv_c^2 = m(\frac{v_i}{2})^2 = m\frac{v_i^2}{4} \tag{8.67}$$

The inital kinetic energy was just $\frac{1}{2}mv_i^2$, so the difference in these two kinetic energies must be the potential energy mgh (by conservation of energy).

Therefore

$$mgh = K_i - K_c = m\frac{v_i^2}{2} - m\frac{v_i^2}{4} \tag{8.68}$$

or

$$h = \frac{v_i^2}{4g} = \frac{1}{9.81} \text{ m} = 0.102 \text{ m} \tag{8.69}$$

This is less than a meter so the block doesn't go over the top. This is the answer to part a of the problem.

Now how do we figure out c? We know that momentum is conserved and energy is conserved, so we this is an example of an *elastic collision.* Even though it is a little bizarre, it is a collision nevertheless.

So we can use our painfully derived formulae Eqs. 8.62 and 8.63 to calculate the final velocities of the objects. In this case $m_1 = m_2$, $v_{i1} = v_i$ and $v_{i2} = 0$.

Plugging these in, we see that $v_{f1} = 0$, and $v_{f2} = v_i = 1$ m/s. So the block stops completely and the hump moves off with a velocity of 1 m/s.

Notice how fiendishly difficult this problem would have been to solve using F=ma!

We could have solved this whole problem starting from Newton's three laws of motion. It would have been much much harder. Although conservation of momentum and energy are a consequence of Newton's laws, for many questions, they provide much insight to what's going on. You shouldn't forget however that they are a consequence of Newton's laws. They contain no extra information.

8.4.8 Elastic collisions in two dimensions

Now let's figure out what happens when objects collide elastically in higher dimension. Two should be enough for us don't you think? Let's ask what we can learn from Eqs. 8.54, which are the equations for energy and momentum conservation. If we're given the initial velocities of the two objects before impact, we'd like to know what they'r velocities are after the collision. That's what we'd like but I'm afraid we're not going to be able to get those completely. What??? How can that be? In one dimension, we saw that applying these equations gave you the final velocities, so what's going wrong?

Well we applied Newton's laws directly, we're certain to get the complete trajectory of the particles, before, during, and after the collision. But that's quite tough. Instead we're using conservation laws. However even though conservation laws are derived from Newton's laws, they aren't equivalent. In general they contain less information so we shouldn't be surprised if they give an incomplete description of what's going on.

So back to the problem at hand. If we want the final velocities of the objects, that means we want to know four quantities since each velocity has both a magnitude and a direction that we wish to know. The problem is that we have only three equations. (It's three equations because the momentum equation is a vector equation, so each component is an equation in its own right. Two components means two equations.) So we have four unknowns and three equations. So we don't have enough equations to find the final velocities.

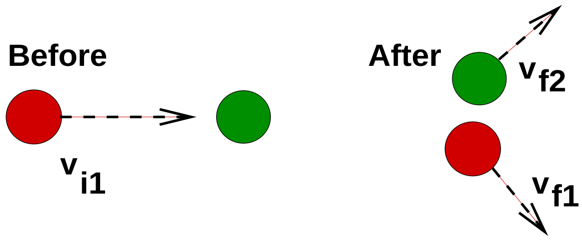
Does that make sense physically? Sure! If two balls collide with each other, they go off in different directions depending on how they collide. Think about playing pool. Depending on where the moving ball hits the stationary one, it goes off in a different direction. So without more detailed knowledge of the situation, it makes sense that we aren't going to know what the outcome of the collision will be.

Let's consider a particular case, where the masses of the balls are equal $m_1 = m_2 = m$. Let's also take mass m_2 to be initially stationary, that is $v_{2i} = 0$.

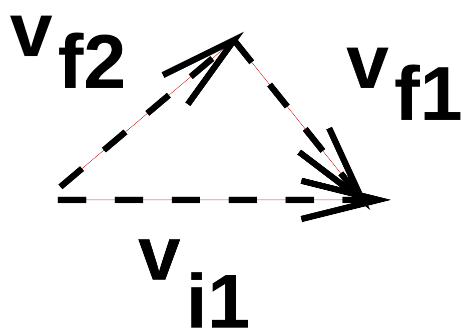
Then cancelling out the m's Eqs. 8.54 become

$$\mathbf{v}_{i1} = \mathbf{v}_{f1} + \mathbf{v}_{f2} \quad (8.70)$$

$$v_{i1}^2 = v_{f1}^2 + v_{f2}^2 \quad (8.71)$$



The first equation says the vector sum of the final velocities is the initial veloicity. If you represent the two final velocity vectors $\mathbf{v}_{f1}$ and $\mathbf{v}_{f2}$ as the sides of a triangle, then $\mathbf{v}_{i1}$ will be the hypotenuse. The second equation looks kind of like the Pythagorean theorem. What kind of triangle obeys this theorem? A right triangle. So this says that $\mathbf{v}_{f1}$ and $\mathbf{v}_{f2}$ are *perpendicular*!



If you don't follow this reasoning you can get the same result by taking $\mathbf{v}_{i1} = \mathbf{v}_{f1} + \mathbf{v}_{f2}$ and squaring it

$$\mathbf{v}_{i1}^2 = \mathbf{v}_{i1} \cdot \mathbf{v}_{i1} = v_{f1}^2 + v_{f2}^2 + 2\mathbf{v}_{f1} \cdot \mathbf{v}_{f2} \tag{8.72}$$

Subtracting $v_{i1}^2 = v_{f1}^2 + v_{f2}^2$ gives $\mathbf{v}_{f1} \cdot \mathbf{v}_{f2} = 0$. So the final velocities are perpendicular.

So you can see the momentum and energy conservation imply an interesting result. Of course we still don't know what final direction the balls will be traveling in, because you need more information, i.e. precisely where they hit each other. However we do know that they'll always be going off perpendicular to each other.

8.5 Rockets

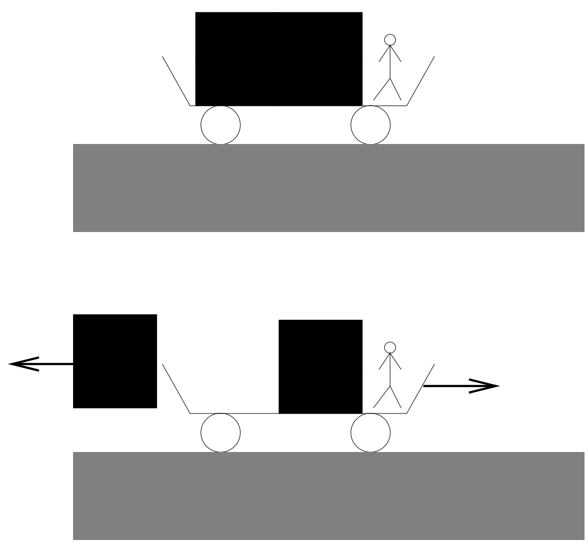
We've all seen majestic pictures of big rockets blasting off from the Earth, burning millions of dollars a second, and carrying interesting payloads into space, such as experiments on how spiders build webs in space. Fascinating.

But there are some general things about this picture that might strike you as odd. One is that payloads are pretty small. Also, the size of these big rockets is absolutely enormous. Is there a reason for this? Yes I know what you're thinking: it's girls and boys playing with their toys. But no, that's not the only reason. They're big because they have to be. We're now in a position to understand rockets, and you'll see why this happens.

What is a rocket? It's this thing that spews out lots of fuel at high velocity. The expulsion of fuel speeds up the rocket. The more fuel expelled, the faster the thing goes. We want to know how fast the rocket will be going after the total mass of the rocket has been reduced from it's initial mass M_i to its final mass M_f. This will clearly have to depend on the velocity that fuel is expelled at. Let's call that U.

Let's try to understand this with a simple low tech example. A small child called Cindy sits on top of her little red wagon loaded up with a huge pile of bricks. She's a strong girl and can hurl a large number of these out in one go.

Suppose she starts at rest with 128 bricks each weighing 1 kg. We'll ignore Cindy and the wagon's mass. She hurls out 64 of these bricks, with velocity of $U = 1$ m/s. What will be the final velocity of the wagon?



Well the initial momentum was zero, so assuming a frictionless wagon, the final momentum must be the same. So if half the bricks are travelling to the left at 1 m/s the other half must be travelling to the right at the same speed, 1 m/s.

OK, now they're 64 bricks left in the wagon. Cindy hurls out half of these, again with the same velocity relative to her. Now what will the final velocity of the wagon be? To figure this out, go to the reference frame of the wagon, which was initially travelling at 1 m/s. In this frame, it looks like the initial problem, where the wagon was stationary, but with 64 bricks inside it. After 32 bricks are hurled, by the same reasoning as before, the wagon will be going an additional 1 m/s, so that relative to the ground the wagon is going 2 m/s.

Now let's ask what happens when Cindy again hurls out half of these 32 bricks, at the same velocity she did before. Going to to a reference frame of 2 m/s. we see as we did before, that the wagon will be going 1 m/s faster than before.

You get the idea. Every time half the mass is thrown out, the velocity increases by 1 m/s. So if the mass decreases by 2^n, the velocity will increase by n m/s. In this case there is a *logarithmic* dependence of mass on velocity. You can check that the final velocity

$$v_f \ = \ U log_2(M_i/M_f) \tag{8.73}$$

Here U is 1 m/s, and M_i/M_f is the ratio of initial to final masses. What does this say? It's pretty hard to speed up a rocket. If you want the speed to go up by U, you have to expel half half your mass. We can rewrite the above equation as

$$M_i \ = \ M_f 2^{v_f/U} \tag{8.74}$$

So if you have a payload of mass M_f, the amount of mass you have to start with, depends exponentially on the final velocity. Exponentials are very rapidly increasing functions. So your initial mass normally has to be very large.

Now that we understand this simplified problem, we can ask what happens in a real rocket. Now we don't have a little girl hurling bricks, but a rocket called Cindy. Cindy expels fuel at a velocity $-U$ (to the left), but does so continuously, not in the punctuated way we just analyzed. So initially, we choose a reference frame where the rocket is at rest. What happens after it expels a little fuel of mass dM? The initial mass of the rocket is M. After expulsion, the mass will have gone down to $M - dM$. Let's call the increase in velocity dv. So conservation of momentum says that

$$-dM \ U + (M - dM) \ dv \ = \ 0 \tag{8.75}$$

This says that the velocty after this is

$$dv \ = \ \frac{dM}{M - dM} U \tag{8.76}$$

Since $dM << M$ (formally its an infinitessimal, we can safely ignore it in the denominator. So we get the equation

$$dv \ = \ \frac{dM}{M} U \tag{8.77}$$

This is the increase in velocity after a very short time. If we now add up all these velocity increases, we'd formally be integrating the left hand side:

$$v \ = \ \int dv \ = \ \int \frac{dM}{M} U \ = \ U \ln M + \text{const.} \tag{8.78}$$

Alternatively, we can treat this as an inspired guess and verify that $d \ln M/dM = 1/M$, i.e. $d(lnM) = dM/M$, confirming that the previous equation is satisfied.

The constant is fixed by requiring that $v = 0$ when $M = M_i$, so that

$$v_f \;=\; U \ln \frac{M_i}{M_f} \tag{8.79}$$

This is almost the same form as our previous analysis except now we get a logarithm base e instead of base 2. Qualitatively, it acts the same way.

We can obtain the same result without calculus. Suppose Cindy expels a fraction f of the present mass of the rocket (including fuel), at a velocity U. Then $-fMU + (1-f)M\Delta v = 0$, i.e.

$$\Delta v = \frac{f}{1-f} U. \tag{8.80}$$

Repeating this process N times, the mass of the rocket is reduced by a factor of $(1-f)^N$, and the velocity is increased by $NfU/(1-f)$. Taking the limit $f \to 0$ with Nf held constant, $v_f = U(Nf)$ and $M_f = \exp[-(Nf)]M_i$. Therefore $\ln(M_i/M_f) = v_f/U$.

Chapter 9

Rotational Motion of Rigid Bodies

Rotational motion is very common. Spinning objects like tops, wheels, balls and pianos are all examples of rotational motion that we would like to understand. We'll concentrate on rotation of rigid bodies, so keep in mind that what we say does not apply to jellyfish.

9.1 Basic quantities

So if you have a rigid body, say a wheel, how do you describe the position of various points? The wheel rotates about some fixed axis. Points on the axis don't move but points off axis do. A point on the wheel can be described as being some distance r from the axis. If you watch it as it rotates, you see that this point moves along the arc of a circle (Fig. 9.1). If the wheel rotates by an angle θ (measured in radians), the arc is of length

$$s = \theta r. \tag{9.1}$$

Is this right? Think about a few case. If $\theta = 2\pi$, you get $s = 2\pi r$ which is the circumference of a circle. If $\theta = \pi$, you get half the circumference. So it makes sense.

So we now know how to relate the distance traveled by a point on the wheel, to how much it rotated.

Now let's introduce another concept, the *angular velocity*. This is the rate the angle θ changes with time. In other words, how fast the wheel is spinning. Suppose the wheel rotates one revolution in a second. That means it rotates 2π radians in a second. The average angular velocity over this time interval is 2π rad/s. So in general, we can define the average angular velocity as

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} \tag{9.2}$$

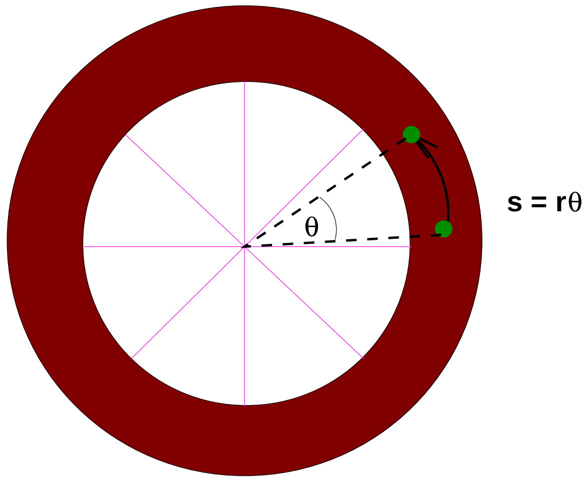


Figure 9.1:

This is similar to the definition of average velocity, except in that case you were concerned with distance traveled, not angle. In analogy with regular velocity, we can write the instantaneous angular velocity as

$$\omega = \frac{d\theta}{dt} \tag{9.3}$$

As we mentioned in our discussion on circular motion, the angle θ is measured in radians and ω has units of rad/s or (since radians are really dimensionless s^{-1}).

We can define *angular acceleration* in analogy to acceleration, the average acceleration being

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} , \tag{9.4}$$

and the instantaneous angular acceleration as

$$\alpha = \frac{d\omega}{dt} \tag{9.5}$$

How do we relate the the velocity of a point to its angular velocity? Well the distance traveled is $s = r\theta$, so the speed of the point

$$v = \frac{ds}{dt} = \frac{d\ r\theta}{dt} = r\frac{d\theta}{dt} \tag{9.6}$$

or

$$v = \omega r \tag{9.7}$$

The direction of velocity is along a circular arc. That was pretty straighforward, and we've talked about this before. What's a little trickier is the acceleration.

Remember the centripetal acceleration. Even when the wheel is rotating at constant speed, a point at radius r feels a radial acceleration $a_r = v^2/r = \omega^2 r$. But what if the angular velocity of the wheel is changing? Then there's also a *tangential acceleration* in the direction of the velocity

$$a_t = \frac{dv}{dt} = \frac{dr\omega}{dt} = r\alpha \tag{9.8}$$

This is perpendicular to the centripetal acceleration, and the two accelerations when combined (in vector form) give the total acceleration of a point on the wheel.

9.1.1 Example

A potter's wheel starts of at rest but after a second is going 10 rad/s. What is the average angular acceleration of the wheel?

Solution

Using Eq. 9.4 we have

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{10 \text{ rad/s}}{1 \text{ s}} = 10 \text{ rad/s}^2 \tag{9.9}$$

9.2 Constant angular acceleration

Now we can, understand the motion of a wheel with constant angular acceleration α.

The math of this is identical to one dimensional kinematics with constant acceleration. So we can just crib what we did before changing the name of variables. With one dimensional motion you've got

$$v = \frac{dx}{dt} \tag{9.10}$$

$$a = \frac{dv}{dt} \tag{9.11}$$

So lets change variable names:

$$x \to \theta \tag{9.12}$$

$$v \to \omega \tag{9.13}$$

$$a \to \alpha \tag{9.14}$$

With these name changes we get the back the same equations we just defined above namely

$$\omega = \frac{d\theta}{dt} \tag{9.15}$$

$$\alpha = \frac{d\omega}{dt} \tag{9.16}$$

So rotation with constant angular acceleration α is mathematically the same as motion with constant acceleration a. Doing the naming substitution we have

$$\omega = \alpha t + \omega_0 \tag{9.17}$$

$$\theta = \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0 \tag{9.18}$$

$$\theta = \frac{1}{2}(\omega + \omega_0)t + \theta_0 \tag{9.19}$$

$$\omega^2 - {\omega_0}^2 = 2\alpha(\theta - \theta_0) \tag{9.20}$$

9.2.1 Example

A fan starts off at rest but after two complete rotations is going at 10 revolutions per second. What is the angular acceleration of the wheel? Assume that the angular acceleration is constant.

Solution

Let's note the units given for ω, it's in revolutions per second. But we want to work in *radians*. So we should first do the appropriate conversion using 1 rev $= 2\pi$ rad:

$$10 \text{ rev/s} = 10\ 2\pi \text{ rad/s} = 20\pi \text{ rad/s} \tag{9.21}$$

Our problem gives the initial and final angular velocity, and the number of revolutions that the fan rotated. That looks like a job for Eq. 9.20. Setting $\omega_0 = 0$ and solving for α we have

$$\alpha = \frac{\omega^2}{2(\theta - \theta_0)} = \frac{(20\pi)^2}{4\pi} = 100\pi \text{ rad/s}^2 \approx 314 \text{ rad/s}^2 \tag{9.22}$$

9.3 Rotational Kinetic Energy

A "flywheel" is a device that spins around real fast on an almost frictionless bearing so that it can maintain a high angular velocity. The velocity at the tips of the flywheel can be really high and so the kinetic energy of the system can be quite enormous. This device is used to store energy. It'd be nice to be able to figure out the total kinetic energy of a flywheel as a function of its angular velocity, its mass and its shape. So let's try to figure out the kinetic energy in a rotating rigid body.

The way we're solve this problem is to divide and conquer. We know what the kinetic energy of a point particle of mass m moving at velocity $\mathbf{v}$. It's $\frac{1}{2}mv^2$. Now what we have is a collection of particles. For N particles, the kinetic energy K is

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 \tag{9.23}$$

but now the velocities of each particles are related because they're part of the same rigid body, $v_i = \omega r_i$. r_i is the the distance between a point on the body and the axis of rotation. Plugging this in we have

$$K = \sum_i^N \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2}\left(\sum_{i=1}^N m_i {r_i}^2\right)\omega^2. \tag{9.24}$$

The term in the parentheses is called the "moment of inertia"

$$I = \sum_{i=1}^N m_i {r_i}^2 \tag{9.25}$$

It doesn't depend on the angular velocity of the object, but just the masses that it's made up of, and their distance away from the axis of rotation. It might be complicated to compute, and we'll discuss what it is for some simple shapes pretty soon. It's useful in rotational dynamics because it plays the same role that we've seen mass plays in our previous studies of dynamics. In terms of the moment of inertia the kinetic energy is

$$K = \frac{1}{2} I \omega^2. \tag{9.26}$$

If we replace I by m and ω by v we get the good old formula $K = \frac{1}{2}mv^2$.

So we can add this to our list of equations (Eqs. 9.17-9.20) that are similar to what we've already seen, but with a few name changes.

9.3.1 Example

We have four masses of $1kg$ that lie at the coordinates

$$\mathbf{r}_1 = 1m\ \hat{i},\ \ \mathbf{r}_2 = -1m\ \hat{i},\ \ \mathbf{r}_3 = 1m\ \hat{j},\ \ \mathbf{r}_4 = -1m\ \hat{j} \tag{9.27}$$

Calculate the kinetic energy when the object spins at $1rad/s.$ around

(a) The x axis.

(b) The y axis.

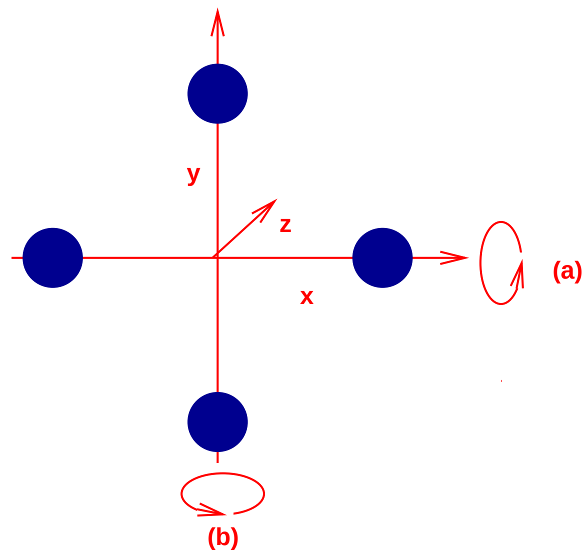
(z) The z axis.

Solution

Let's compute the moment of inertia of the object for all three cases.

For (a), masses 1 and 2 are zero distance from the axis of rotation, but masses 3 and 4 are both at a distance of $1m$, so the moment of inertia is

$$I = (1\text{ kg } 0^2 + 1\text{ kg } 0^2 + 1\text{ kg } (1\text{ m})^2 + 1\text{ kg } (1\text{ m})^2) = 2\text{ kg m}^2 \tag{9.28}$$



Therefore the kinetic energy is

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}2 \text{ kg m}^2(1 \text{ rad/s})^2 = 1 \text{ J} \tag{9.29}$$

Notice that the radians in ω, although shown for clarity, are really dimensionless, and are dropped at the next step when we use $1 \text{ kg m}^2 \text{ s}^{-2} = 1 \text{ J}$.

For (b) we notice that it's really the same situation as (a) but with a relabeling of the axis. So we get the same answer.

(c) has all the masses at a distane $1m$ from the axis of rotation so we get

$$I = (1 \text{ kg } (1 \text{ m})^2 + 1 \text{ kg } (1 \text{ m})^2 + 1 \text{ kg } (1 \text{ m})^2 + 1 \text{ kg } (1 \text{ m})^2) = 4 \text{ kg m}^2 \tag{9.30}$$

Therefore the kinetic energy is

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}4 \text{ kg m}^2(1 \text{ rad/s})^2 = 2 \text{ J} \tag{9.31}$$

9.4 The moment of inertia for continuous bodies

This section is similar to the calculation of the center of mass for a continuous body, so I won't repeat all the boring steps. You can easily fill those in if you understand how to do multiple integration and you understand how to get the formula for the center of mass of a continuous body.

If you have a continuous object with a density that varies with position $\rho(\mathbf{r})$, then you can write the formula for the moment of inertia as

$$I = \int r^2 dm \tag{9.32}$$

A less mysterious way of writing this is

$$I = \int r^2 \rho(\mathbf{r}) d^3 r \tag{9.33}$$

Let's examine what the formula says.

1. The integral is like a sum over a lot of tiny little cubes. We divide the object into little cubes of volume $d^3 r$ and sum up the moment of inertia dI of all the cubes $I = \int dI$.
2. The moment of inertia dI of each cube is just the mass in that volume dm times r^2 where r is the distance away from the axis of rotation. Don't confuse this r with the distance away from some point. It's the distance away from the axis of rotation. That is the shortest distance between the point and the axis of rotation, just like in the last example. For instance, if the axis of rotation is the x-axis, then r^2 for a mass at $\mathbf{r}_0$ is $y_0^2 + z_0^2$, not $x_0^2 + y_0^2 + z_0^2$. So we have $dI = r^2 dm$.
3. dm can be written in terms of the density $dm = \rho(\mathbf{r}) d^3 r$.

Let's now to some examples.

9.4.1 Example of a ring

Suppose we have something like a hoola hoop, that is a thin hoop or ring that rotates about the central axis perpendicular to the hoop. What's the moment of inertia in terms of the total mass M of the hoop and its radius R?

9.4.2 Solution

This problem is so simple, we can think about it in terms of discrete masses.

$$I = \sum_i m_i r_i^2 \tag{9.34}$$

but all the r_i's are the same and equal to R. So we can factor out the radii,

$$I = R^2 \sum_i m_i \tag{9.35}$$

But $\sum m_i$ is just the total mass M. Therefore

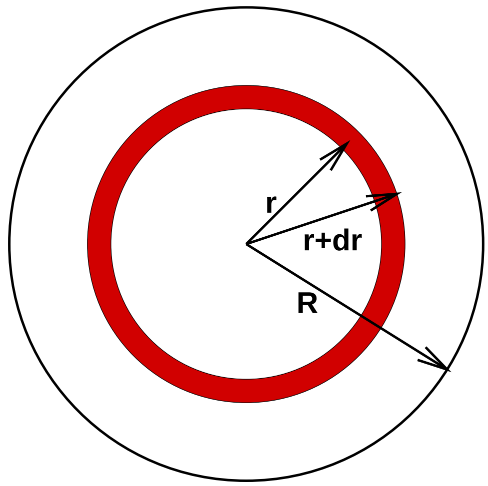
$$I_{hoop} = MR^2 \tag{9.36}$$

9.4.3 Example of a disk

Now we know the moment of inertia of a ring, let's calculate what it is for a disk of uniform density that rotates about the central axis perpendicular to the disk. Call the total mass of the disk M, and its radius R.

9.4.4 Solution

The trick here is to think of the disk as a collection of a bunch of concentric rings. Call the mass per unit area of the disk σ. Then we have a bunch of rings, the inner radius of one of these rings is r and outer radius $r + dr$.



Then the mass of this ring dm is the surface dA times σ. So

$$dm = \sigma dA \quad . \tag{9.37}$$

But dA is just the circumference times dr, $dA = 2\pi r\ dr$. So

$$dI = dmr^2 = \sigma 2\pi r\ dr\ r^2 \tag{9.38}$$

Now we want to integrate dI over all radii to get I so

$$I = \int_0^R dI = \int_0^R \sigma 2\pi r\ r^2\ dr = \sigma 2\pi \int_0^R r^3 dr = \sigma 2\pi \frac{R^4}{4} \tag{9.39}$$

Let's write σ in terms of M and R:

$$\sigma = \frac{M}{\pi R^2} \tag{9.40}$$

So finally we get

$$I_{disk} = \frac{1}{2} M R^2 \tag{9.41}$$

You can get the same result indirectly, using only differentiation, if you are not familiar with integration. By considering the units of different quantities, it is clear that $I \propto \sigma R^4$. We write this as $I = a\sigma R^4$. If R is increased to $R + dR$, the moment of inertia is increased by $4a\sigma R^3 dR$. But increasing R is equivalent to adding a hoop just outside the original disk, with a moment of inertia $(\sigma 2\pi R dR)R^2$. Equating the two, we see that $a = \pi/2$, so that $I = \frac{1}{2}(\pi\sigma R^2)R^2 = \frac{1}{2}MR^2$.

9.4.5 Example of a spherical shell

Calculate the moment of inertia of a spherical shell of mass M and radius R that rotates through an axis that goes throught the center of the sphere.

9.4.6 Solution

So now that you've seen how to make a disk out of a bunch of hoops, we could instead make a spherical shell out of a bunch of then also. It's kind of like a technique in pottery where you slowly add little rings of clay of different sizes, until you have a beautiful vase! Oh shut up!

So we can do that here too. Pottery and physics meet. On the other hand I was never good much at pottery. You get the size of a ring off by a factor of two and it ends up looking like a moldy lump of clay. The same is true of the math involved in this example. I could go through and do it, but it's a bit tedious. There's a much much more elegant way of calculating the moment of inertia in this example. It requires you to think a lot more, but it requires you to write a lot less.

It uses the symmetry of sphere. Let's write things out in terms of discrete masses because it's easier to understand

$$I = \sum_i m_i r_i^2 \tag{9.42}$$

If we rotate about the z axis, then r_i is the distance between the point and the z axis, so $r_i^2 = x_i^2 + y_i^2$. So

$$I = \sum_i m_i x_i^2 + m_i y_i^2 \tag{9.43}$$

We could instead compute what I'll call I_x

$$I_x = \sum_i m_i x_i^2 \tag{9.44}$$

or

$$I_y = \sum_i m_i y_i^2 \tag{9.45}$$

Because of the symmetry of a sphere we can replace x by y and nothing should change so

$$I_x = I_y \tag{9.46}$$

I could also calcluate

$$I_z = \sum_i m_i z_i^2 \tag{9.47}$$

That should also be the same as I_x, again because of symmetry. There is nothing special about the choice of axis. We could call x y, y z, and z x, and we'd get the same answers.

Now lets calculate $3I_x = I_x + I_y + I_z$. That's

$$\sum_i m_i x_i^2 + \sum_i m_i y_i^2 + \sum_i m_i z_i^2 = \sum_i m_i(x_i^2 + y_i^2 + z_i^2) \tag{9.48}$$

But since we have a sphere, we know that $x_i^2 + y_i^2 + z_i^2 = R^2$. So we can pull that out of the sum and then we just have a sum over the m_i's which just equals M. So $3I_x = MR^2$. But $I = I_x + I_y = 2I_x$. So

$$I_{shell} = \frac{2}{3}MR^2 \tag{9.49}$$

9.4.7 Example of a solid sphere

What's the moment of inertia of a solid sphere through an axis that passes through its center? The sphere is of uniform density.

9.4.8 Solution

Now we have the moment of inertia of a spherical shell, we can sum up all these shells to get what it is for solid sphere. This is a lot like the example of the disk.

So what's the mass dm of a shell of inner radius r and outer radius $r + dr$? Call the density ρ. Then $dm = \rho dV$. What's the volume dV? It's the surface area of a sphere of radius r times dr. The surface area of a sphere is $4\pi r^2$ so

$$dm = \rho 4\pi r^2 dr \tag{9.50}$$

And from the last example, that $dI = (2/3)\ dm\ r^2$. So the moment of inertia is

$$I = \int dI = \int \frac{2}{3} dmr^2 = \int_0^R \frac{2}{3}(\rho 4\pi r^2) drr^2 = \frac{2}{3}\rho 4\pi \int_0^R r^4 dr = \frac{2}{3} 4\pi\rho \frac{R^5}{5} \tag{9.51}$$

Let's write ρ in terms of the M and R. The volume a sphere is $\frac{4}{3}\pi R^3$, so

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} \tag{9.52}$$

Plugging that in to the formula for I

$$I = \frac{2}{3} 4\pi \frac{M}{\frac{4}{3}\pi R^3} \frac{R^5}{5} \tag{9.53}$$

or

$$I_{sphere} = \frac{2}{5}MR^2 \tag{9.54}$$

As with the disk, we can obtain the same result indirectly, without integration. Considering the units of various quantities, $I = a\rho R^5$, where a is some (unknown) constant. If the radius of the sphere is increased by dR, the moment of inertia is increased by $5a\rho R^4 dR$. The moment of inertia of the added spherical shell is $\frac{2}{3}(4\pi\rho R^2 dR)R^2$. Equating the two, we obtain $a = 8\pi/15$, and the moment of inertia of the sphere is $\frac{2}{5}(4\pi\rho R^3/3)R^2 = \frac{2}{5}MR^2$, which is the expression we just obtained.

9.5 Cool Theorems about I

We've now seen quite a few examples of the moment of inertia for a lot of different objects. But notice we have to specify the axis of rotation. If we change the axis, in general so will the moment of inertia. It'd be nice if we didn't have to recompute the moment of inertia every time we chose a different axis. For example, a disk has a moment of inertia of $\frac{1}{2}MR^2$ through an axis going through the center and perpendicular to the disk. What if the axis was still perpendicular but didn't go through the center? What if the axis went throught the center but was in the plane of the disk? How would you figure out these cases?

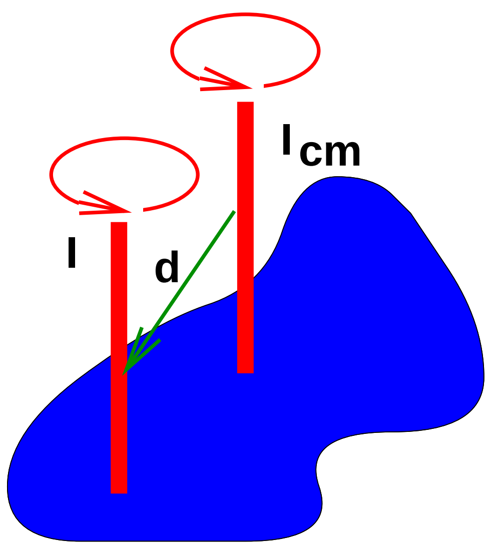
Fortunately we can do these problems without much extra effort thanks to two cool theorems, the *perpendicular axis theorem* and the *parallel axis theorem*.

9.5.1 Parallel axis theorem

Suppose we know the moment of inertia when the axis goes through the center of mass in a certain direction. Call it I_{cm}. If this axis is displaced by a distance d but is still parallel to the original axis, then the moment of inertia through this axis is

$$I \;=\; Md^2 + I_{cm} \tag{9.55}$$

where M is the mass of the object.



9.5.2 Proof:

Say the object is composed of N pieces with masses $m_1, \ldots, m_N$. Call the displacment vectors between these pieces and the axis distances $\mathbf{r}_1, \ldots, \mathbf{r}_N$ between the pieces and the axis, then

$$I_{cm} = \sum_{i=1}^{N} m_i r_i^2 \tag{9.56}$$

When the axis is displaced by a vector $\mathbf{d}$, then we want to compute

$$I = \sum_{i=1}^{N} m_i {r'_i}^2 \tag{9.57}$$

where $\mathbf{r}'_i$ is the displacment vectors between these pieces and the new axis.

Let's relate the $\mathbf{r}'_i$ to $\mathbf{r}_i$

$$\mathbf{r}' = \mathbf{r} + \mathbf{d} \tag{9.58}$$

And since $r^2 = \mathbf{r} \cdot \mathbf{r}$ (dropping the subscript for convenience)

$$r'^2 = (\mathbf{r} + \mathbf{d}) \cdot (\mathbf{r} + \mathbf{d}) = r^2 + d^2 + 2\mathbf{d} \cdot \mathbf{r} \tag{9.59}$$

Now plugging this into 9.57 we have

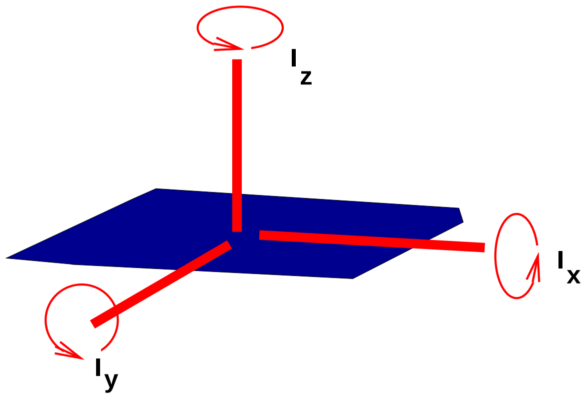
$$I = \sum_{i=1}^{N} m_i r_i^2 + (\sum_{i=1}^{N} m_i) d^2 + 2\mathbf{d} \cdot \sum_{i=1}^{N} m_i \mathbf{r}_i \tag{9.60}$$

The last term contains $\sum_{i=1}^{N} m_i \mathbf{r}_i$. Dividing this by M, this would be $\mathbf{r}_c$, the location of the center of mass projected into the plane perpendicular to the axis. Since the axis passes through the center of mass, by definition, this must be zero. That is, if you calculate the position of the center of mass of an object when the origin of the coordinate system is the center of mass, you get zero. So we only have the first two terms in the above equation, and we get Eq. 9.55.

9.5.3 Perpendicular axis theorem

Suppose you have *planar* object, that is, one that is flat. An example of this is a floppy disk, or a cdrom. Suppose the object is in the x-y plane. If you rotate it the z axis, the moment of inertia is I_z. If you rotate it by the x or y axes, the moment of inertia is I_x, or I_y respectively. Then

$$I_z = I_x + I_y \tag{9.61}$$



9.5.4 Proof:

It's a lot like the last proof.

$$I_z = \sum_{i=1}^{N} m_i r_i^2 = \sum_{i=1}^{N} m_i (x_i^2 + y_i^2) \tag{9.62}$$

But because the object is planar

$$I_x = \sum_{i=1}^{N} m_i y_i^2 \tag{9.63}$$

and

$$I_y = \sum_{i=1}^{N} m_i x_i^2 \tag{9.64}$$

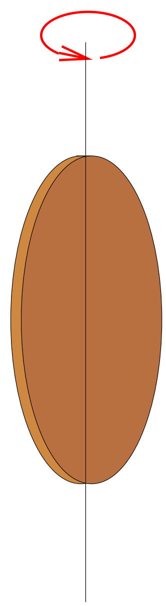
Combining these we obtain Eq. 9.61.

9.5.5 Example of spinning coin

What is the moment of inertia of a coin that you set spinning on a table? It spins through an axis in the plane of the coin. Treat the coin as a flat uniform disk of radius R and mass M.

9.5.6 Solution

If we put the disk in the x-y plane, then we saw that $I_z = \frac{1}{2}MR^2$. We want to know the answer when spun around the x axis I_x. That should be the same as I_y. So using the parallel axis theorem, Eqn. 9.61, we have that $2I_x = I_z$. Therefore $I_x = \frac{1}{4}MR^2$.



9.5.7 Example of an off center sphere

What is moment of inertia of a solid sphere of radius of R that we set rotating about an axis that just touches its surface? Take the mass the be M.

9.5.8 Solution

We use the parallel axis theorem, Eq. 9.55. Here the axis has been shifted by the radius of the sphere so $d = R$, and we already saw in example 9.4.7 that $I_{cm} = \frac{2}{5}MR^2$. So

$$I = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2 \tag{9.65}$$

9.5.9 Example of a rod

What's the moment of inertia of a uniform rod rotated about an axis perpendicular to the rod going through the middle. Take the length of the rod to be L and the mass to be M.

9.5.10 Solution

You can get the moment of inertia of a uniform rod by integration and we did for a solid disk, but let's do it in a more tricky way using the parallel axis theorem.

It's neat because it doesn't take much calculation, but you have to think about the problem!

We've seen a lot of examples, and we know the final answer has to be of the form

$$I_{cm} \;=\; C\ M\ L^2 \tag{9.66}$$

where we want to determine C. What if we rotate the rod about an end intead? Then the parallel axis theorem tells us that

$$I_{end} \;=\; I_{cm} + M(\frac{L}{2})^2 \;=\; C\ M\ L^2 + \frac{1}{4}ML^2 \tag{9.67}$$

We still don't know C so why does this help? Because we can get a different formula for this a different way. If instead of shifting the axis, we chop off the left half of the rod, then the new moment of inertia is

$$I_{chopped} \;=\; I_{cm}/2 \;=\; \frac{C}{2}\ M\ L^2 \tag{9.68}$$

So this is the moment of inertia of a rod of length $L/2$ and mass $M/2$ rotating on it's end.

What will be the moment of of a rod twice the length? We want to know this because it's the same as I_{end}. Well then $L \rightarrow 2L$ and $M \rightarrow 2M$. So from the last formula this is

$$I_{end} \;=\; \frac{C}{2}\ (2M)\ (2L)^2 \;=\; 4C\ M\ L^2 \tag{9.69}$$

Comparing this with our first formula for I_{end}, Eq. 9.67 we have

$$C\ M\ L^2 + \frac{1}{4}ML^2 \;=\; 4C\ M\ L^2 \tag{9.70}$$

Solving for C we can just cancel the ML^2's, obtaining $C \;=\; 1/12$. Therefore from Eq. 9.66 we have

$$I_{cm} \;=\; \frac{1}{12}\ M\ L^2 \tag{9.71}$$

Chapter 10

Angular Momentum and Torque

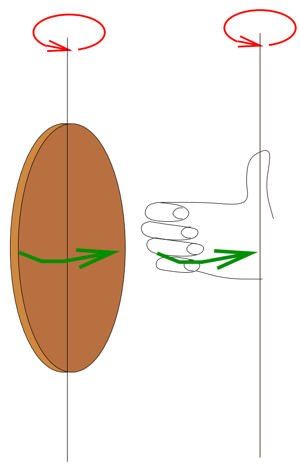
We just discussed the rotation of a rigid body. We defined the basic concepts and used them to solve simple problems. It was kind of a similar philosophy to what we did with kinematics a few chapters ago. But now it's time to forge ahead and try to understand better how physical laws allow you to figure out the motion of rotating bodies. This is analagous to using Newton's laws to find the motion of blocks and things like that in response to gravity and friction. But now we will include rotational motion also. For example, instead of a block sliding down a plane, we could try to figure out how fast a log rolls down a hill.

Let's summarize what we've got so far. We have quantities analogous to linear motion: angle is like distance, angular velocity is like velocity, angular acceleration is like acceleration, moment of inertia is like mass.

If we want to understand mechanics better, what else can we crib from what we've done before? Well how about momentum? We'll see there's something analogous to that called angular momentum. How about force? Yes we're lucky again, there's something called torque that behaves in a similar way. But one thing is a bit odd. The above linear quantities, velocity, momentum, and force, are all vectors, yet so far we haven't talked about how to "vectorize" the corresponding angular concept. Let's see how to do that.

10.1 The direction of a rotation

Let's think about angular velocity. We could make it a vector by giving it a direction. That seems a bit tough at first sight. There seem to be a lot of directions to the velocity of a rotating coin. Different parts are going in totally different directions. But suppose we wanted to uniquely define a direction so that everyone could tell what axis the object was rotating from only this one direction. What would we use? Well the axis of course! Like take the earth. It rotates around an axis and knowing that, you uniquely have the direction of rotation. All except, you don't know if it's spinning clockwise or counter-clockwise (as seen looking down from the north pole). Well that's ok because though we have the axis, we don't yet have a direction for the axis. It could be an arrow pointing up, or one pointing down. So we define something called the *right hand rule* that says that if you take your right hand and have your fingers follow the direction of rotation, your thumb will define the direction of the angular velocity vector.



So to make a respectable vector out of the angular velocity, we say we take it with the magnitude as defined before, and direction given by the axis of rotation and the right hand rule.

10.1.1 Relation with velocity

Now remember we learned that the velocity at a point on the object $v = \omega r_{perp}$, where r_{perp} is the distance between the point and the axis. It is the distance measured perpendicularly from the axis to the point. Now let's try to express this relation as a vector relation.

If we define a vector $\mathbf{r}$, measuring the moving point in relation to an origin that is on the axis, we'd like to say $\mathbf{v} = \boldsymbol{\omega}\mathbf{r}$, except that makes no sense since we haven't defined how to multiply to vectors and get a vector! Well it's about time we do that. So we have a vector $\boldsymbol{\omega}$ pointing up, and a vector r pointing onto some point on the object as pictured in Fig. 10.1.

r_{perp} in this picture is $r\sin\theta$. θ is the angle between $\mathbf{r}$ and $\boldsymbol{\omega}$. so

$$v = \omega r \sin\theta \tag{10.1}$$

What about it's direction. It's not hard to see that $\mathbf{v}$ is perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{r}$.

This is what's called a *cross product* or *vector product*. If you have two vectors $\mathbf{A}$ and $\mathbf{B}$, then we can define a vector $\mathbf{C} = \mathbf{A}\times\mathbf{B}$. It has a magnitude

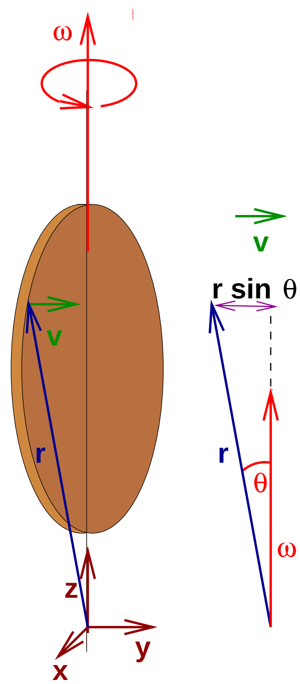


Figure 10.1:

$AB \sin\theta$, with θ being the angle between **A** and **B**. And it's perpendicular to both **A** and **B**.

So we see that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. Could I have said $\mathbf{v} = \mathbf{r} \times \boldsymbol{\omega}$? The answer is no! We have to be careful about getting our signs right, and again we use a right hand rule to make that clear.

In Fig. 10.2, we show how you can get the direction of the cross product $\mathbf{A} \times \mathbf{B}$. You first hold the fingers of your *right hand* straight to point in the direction of **A**, shown in red. Then you bend them to point in the direction of **B**, shown in blus. So if the first vector being multiplied, **A** is represented by the stretched fingers, and the second, **B**, by the bent ones, then the *thumb* gives the direction of the resulting cross product. Notice that if the order was reversed, so would the directions so that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$!

An alternative statement of the right hand rule for cross products is the "screw rule": if you consider the plane containing the vectors **A** and **B** and find that you have to rotate clockwise to go from **A** to **B**, the cross product will point inwards. If you have to rotate anticlockwise, the cross product will point outwards. (You have to rotate in the direction where you will reach **B** faster; of course, you will eventually reach **B** even if you go in the wrong direction.) This is like standard right-handed screws, which go in when they are tightened clockwise and out when they are loosened anticlockwise.

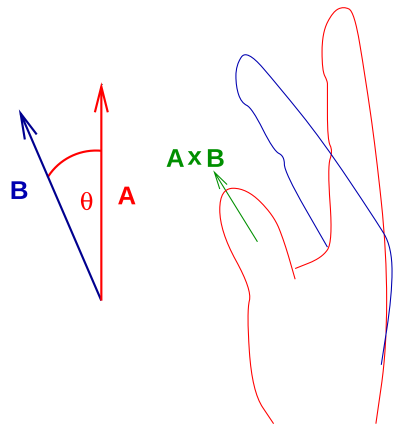


Figure 10.2:

10.1.2 Properties of the cross product

Like the dot product, the cross product has some nice properties. It's easy to see that $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) \;=\; \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$.

Also if two vectors are going in the same direction, the cross product is zero. If the vectors are perpendicular then $|\sin\theta| \;=\; 1$ so that the magnitudes just multiply. Let's work out some of the cross products between unit vectors:

$$\hat{\imath} \times \hat{\jmath} \;=\; \hat{k},\;\; \hat{k} \times \hat{\imath} \;=\; \hat{\jmath},\;\; \hat{\jmath} \times \hat{k} \;=\; \hat{\imath} \qquad (10.2)$$

So now we should be able to work out the cross product in cartesian coordinates by multiplying out all of the components. It's a lot of work and the answer is

$$\mathbf{A} \times \mathbf{B} \;=\; (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k} \qquad (10.3)$$

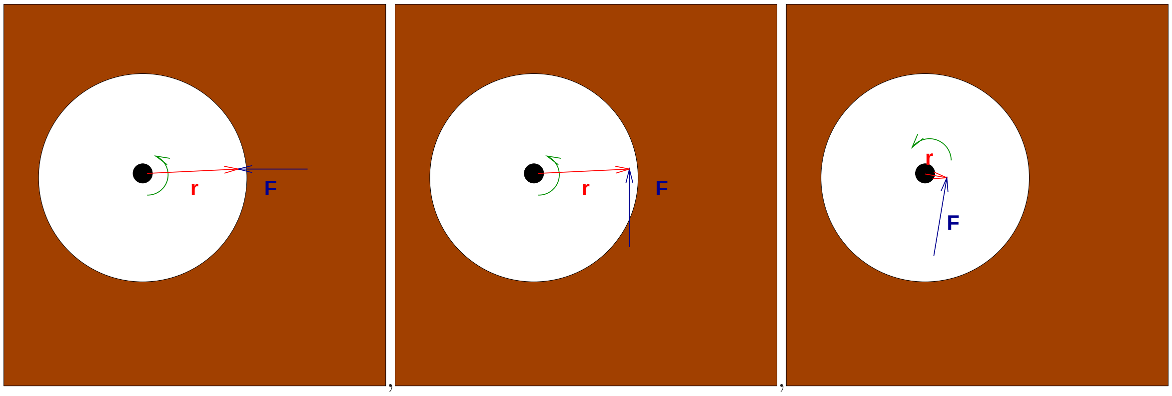
The component formula is often the simplest way to work out problems involving cross products.

10.2 Angular Momentum and Torque

So now we see how to make our rotational quantities vectors. So let's ask what is the analog of momentum and force. Let's start with force.

10.2.1 Torque

First think about what a force does. If you apply a force to an object, it'll have a tendency to go in that direction. Now you want to know what kind of thing you apply to *twist* an object. For example, if you have a merry-go-round, you can apply a force to the rim of it. If the force is towards the center, it won't



move at all. If it's tangent to the rim, it'll move a lot more easily. Also if you apply a force close to the center, it won't move much.

Given this, we see that what causes a rotating object to twist is not just the force, it depends on where the force is being applied and in what direction. So now we define the *torque* to be

$$\boldsymbol{\tau} \;=\; \mathbf{r} \times \mathbf{F} \tag{10.4}$$

where **r** is the vector representing the point where the force is being applied. Does this make sense? Well, the dependence on r means that the larger r is, the more effective the force will be in twisting the object. The cross product makes sense too. If **r** and **F** are in the same direction, you get no twist, if they're perpendicular, you get the maximum effect. The whole thing seems pretty sensible. Now the direction of $\boldsymbol{\tau}$ is a bit odd at first sight. It's perpendicular to both **r** and **F**. Think about appling torque with a wrench to undo a screw. The direction of the torque is in the direction of the bolt. It defines the axis that you're attempting to twist the object.

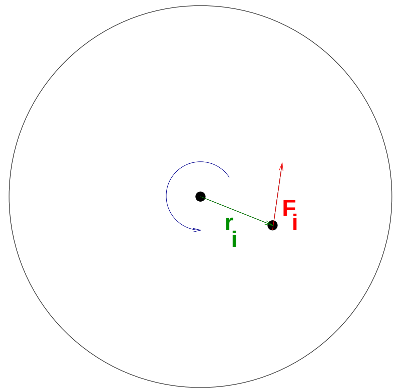
But does our analogy with translational motion hold? If I is like the mass m, and α is like a, then the analog to $f = ma$ should be $\tau \;=\; I\alpha$. Is this true?

Let's try to see if this works out. The strategy is as follows. We'll consider some object like a disk rotating about a fixed axis. We'll consider it to be two dimensional for simplicity. That way we don't have to worry too much about the vector nature of the problem. We'll divide it up into lots of tiny masses and understand each one individually. Then we'll sum up the effect from all of them.

First of all consider just one point mass circling at a distance r from an axis. Let's say we apply a force **F** to this particle. **r** and **F** are both in the x-y plane, so their cross product always points in the z direction, so τ here will represent the z-component of $\boldsymbol{\tau}$.

$$\tau \;=\; rF\sin\theta \;=\; rF_t \tag{10.5}$$

Here $F_t \;=\; F\sin\theta$ is the tangential component of the force.



But $F_t \;=\; ma_t \;=\; mr\alpha$ So plugging this in we have

$$\tau \;=\; rmr\alpha \;=\; mr^2\alpha \tag{10.6}$$

The moment of inertia for a single particle is just mr^2, so we get $\tau \;=\; I\alpha$ for this one particle. Now let's say you've got a jillion particles. To each one you apply a different torque τ_i, then the sum of all these torques is just

$$\tau_{net} \;=\; \sum_i mr_i^2\alpha \;=\; I\alpha \tag{10.7}$$

This is analogous to $F = ma$.

Now we've understood the analogous quantity to the force, let's do the same for momentum.

10.2.2 Angular Momentum

Let's define the angular momentum for a point object similar to the way we did it for the torque

$$\mathbf{L} \;=\; \mathbf{r} \times \mathbf{p} \tag{10.8}$$

If this is really analogous, the first thing we should check is if the analogy to $p \;=\; dF/dt$ holds:

$$\frac{d\mathbf{L}}{dt} \;=\; \frac{d(\mathbf{r}\times\mathbf{p})}{dt} \;=\; \frac{d\mathbf{r}}{dt}\times\mathbf{p} + r\times\frac{d\mathbf{p}}{dt} \;= \tag{10.9}$$

$$\mathbf{v}\times(m\mathbf{v}) + r\times\frac{d\mathbf{p}}{dt} \;=\; \boldsymbol{\tau} \tag{10.10}$$

Here we used the fact that the cross product of a vector with itself is zero, and the last equality used $\mathbf{F} = d\mathbf{p}/dt$.

So it does seem like this definition of $\mathbf{L}$ makes sense from a mathematical point of view. If we have many forces acting on the system, it is straightforward to extend this so that in general

$$\boldsymbol{\tau}_{net} \;=\; \frac{d\mathbf{L}_{tot}}{dt} \tag{10.11}$$

For a continuous object let's look at the two dimensional case considered above where we showed that $\tau = I\alpha$, where $\boldsymbol{\tau}$ points perpendicular to the rotating object. Since I is constant in time this says that

$$\frac{dL}{dt} = \frac{dI\omega}{dt} \tag{10.12}$$

So

$$L = I\omega \tag{10.13}$$

(note that if $\omega = 0$ all velocities are zero so $L = 0$).

10.2.3 Note on $\tau = I\alpha$

We derived $\tau = I\alpha$ and $L = I\omega$ for a two dimensional continuous object rotating in an axis perpendicular to its plane, but we can apply it to non-pancake-like objects with a few caveats.

First consider some symmetric object that's rotating about a symmetry axis in zero gravity as shown in Fig. 10.3. there is no torque on the bearing.

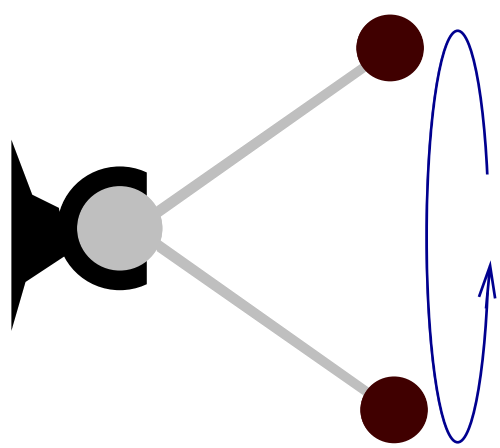


Figure 10.3:

Now take off one of the arms as shown in Fig. 10.4. Now the object will not continue to spin around. It is "off balance". To ensure that it continues to rotate around as it did before, you need to change the construction of the bearing as shown in Fig. 10.5. Now the bearing provides a torque to hold the object at the right angle. If we apply an *external* torque to the red ball, it will cause a torque to be applied to the bearing. The bearing torque keeps the objects rotating about in the horizontal direction. In this case we can still write $\tau = I\alpha$, but keep in mind that additional torques are being generated to keep the object rotating around like we want it to.

Now what happens to **L**? In the first picture, the angular momentum vector points along the axis of rotation. In the second it does not. Measured from

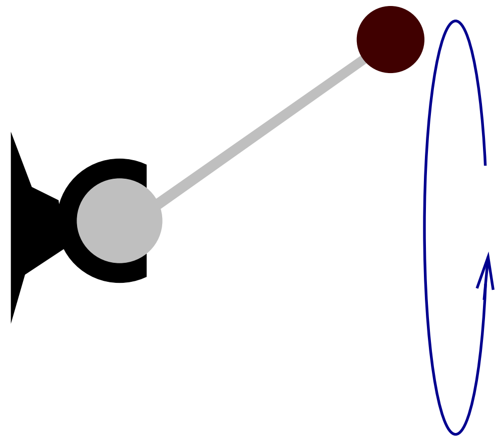


Figure 10.4:

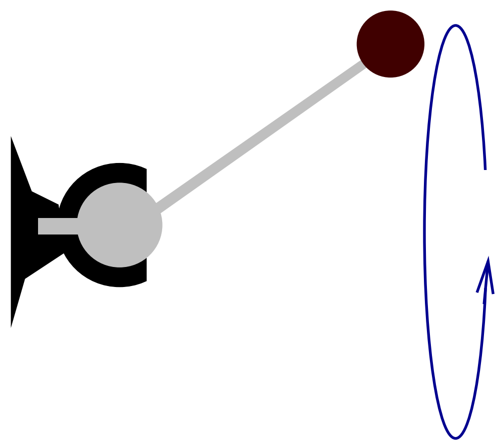


Figure 10.5:

the bearing, $\mathbf{r} \times \mathbf{p}$ for each arm of the object has a vertical and a horizontal component, but when both arms are present, the horizontal components add up and the vertical components cancel. The angular velocity is horizontal, and $L = I\omega$. With just one arm, there is a vertical component to the angular momentum, but the angular velocity remains horizontal. Moreover, as the object rotates, the vertical component of the angular momentum also rotates about the axis of rotation: downwards, into the paper, upwards, then out of the paper. Because the angular momentum is changing with time, a torque has to be applied to keep the object steady.

Our derivation of $L = I\omega$ assumes the former case. We have to be careful to only use this formula when rotation is along an axis of symmetry.

We can obtain the same result more formally. For a collection of particles, or a continuum object,

$$\mathbf{L} = \sum_i \mathbf{r}_i \times (m_i \mathbf{v}_i) = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \sum_i m_i r_i^2 \boldsymbol{\omega} - \sum_i m_i \mathbf{r}_i (\mathbf{r}_i \cdot \boldsymbol{\omega}) \quad (10.14)$$

so that, if the z-direction is chosen to be along $\boldsymbol{\omega}$, then $L_x = -(\sum_i m_i x_i z_i)\omega$, $L_y = -(\sum_i m_i y_i z_i)$ and $L_z = [\sum_i m_i (r_i^2 - z^2)]\omega = [\sum_i m_i r_{i,perp}^2]\omega$. If the rotation is about an axis of symmetry, L_x and L_y are zero. Note that all the other formulae we have derived are always true, regardless of whether the rotation is about an axis of symmetry:

$$\begin{aligned} \mathbf{v}_i &= \boldsymbol{\omega} \times \mathbf{r}_i \\ \mathbf{L} &= \sum_i \mathbf{r}_i \times \mathbf{p}_i \\ d\mathbf{L}/dt &= \sum_i \mathbf{r}_i \times \mathbf{F}_i = \boldsymbol{\tau} \\ K &= \frac{1}{2}\sum_i \mathbf{v}_i \cdot \mathbf{p}_i = \frac{1}{2}\sum_i [\boldsymbol{\omega} \times \mathbf{r}_i] \cdot \mathbf{p}_i = \frac{1}{2}\sum_i \boldsymbol{\omega} \cdot [\mathbf{r}_i \times \mathbf{p}_i] = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2} I\omega^2 \end{aligned} \quad (10.15)$$

where the sum over i is a sum over all the particles. It is *only* the equation $\mathbf{L} = I\boldsymbol{\omega}$ (and, therefore, $\tau = I\boldsymbol{\alpha}$) that requires the rotation to be about an axis of symmetry.

10.2.4 Example

Consider two masses connected by a massless string slung over a pulley as shown. The moment of inertia of the pulley is I, and its radius is R. Calculate the acceleration of the masses.

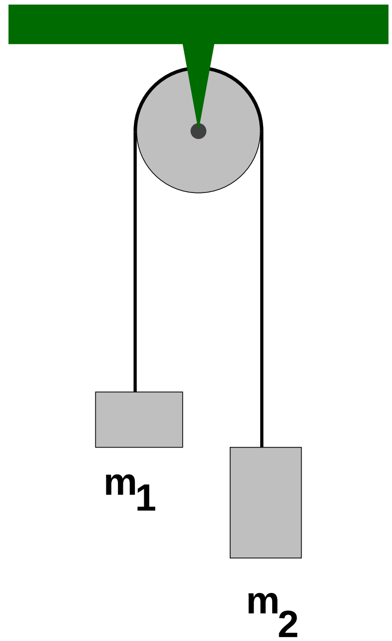
10.2.5 Solution

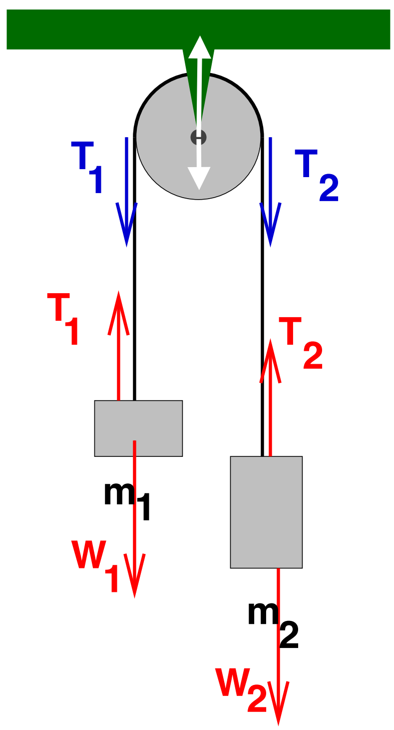
First of all we identify all forces acting on the masses and the pulley. The force of gravity acts on the two masses, the tension pulls up. Now the pulley has four forces acting on it: the tensions T_1 and T_2, and two more forces acting at the center. Because the strings are massless, these tensions are the same in magnitude as the ones acting on the weights.

Now we draw free body diagrams for the pulley and the two masses.

Free body diagram for pulley

The are four forces acting on the pulley, the two tensions pulling down, and two forces on the bearing at the center due to the weight of the pulley and the normal force, but these last two don't contribute to the torque (why?). The two tensions are shown in Fig. 10.6. What's the net torque? The torque about the center of the pulley from T_2 is the cross product of the radius vector with $\mathbf{T}_2$.


m_1
m_2


T_1
T_2
T_1
T_2
m_1
W_1
m_2
W_2

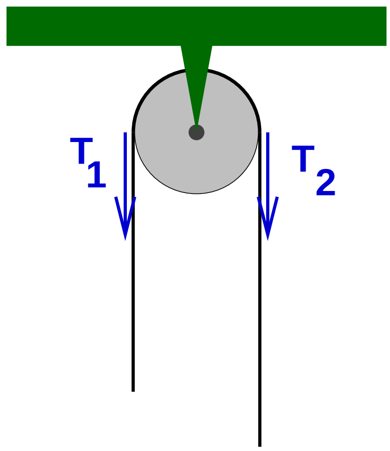


Figure 10.6:

This points into the page, which is the negative $\hat{k}$ direction. The torque from T_1 points the opposite way, in the positive $\hat{k}$ direction. So in the z-direction

$$\tau_{net} \;=\; RT_1 - RT_2 \;=\; R(T_1 - T_2) \tag{10.16}$$

Now we apply $\tau_{net} \;=\; I\alpha$ in the z-direction.

$$R(T_1 - T_2) \;=\; I\alpha \tag{10.17}$$

Free body diagram for masses

The free body diagrams for each mass is shown in Fig. 10.7. The net force on the first mass is

$$F_{net,1} \;=\; T_1 - m_1 g \tag{10.18}$$

The net force on the second mass is

$$F_{net,2} \;=\; T_2 - m_2 g \tag{10.19}$$

Applying $F_{net,1} \;=\; m_1 a_1$ and $F_{net,2} \;=\; m_2 a_2$ we have

$$m_1 a_1 \;=\; T_1 - m_1 g \tag{10.20}$$

and

$$m_2 a_2 \;=\; T_2 - m_2 g \tag{10.21}$$

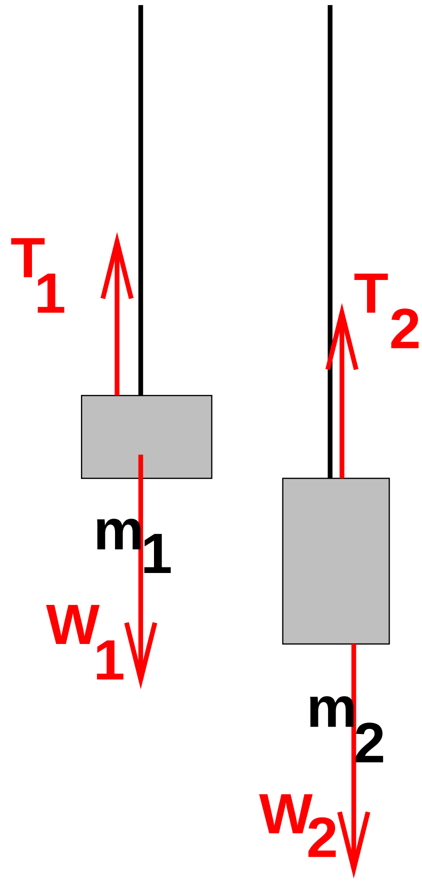


Figure 10.7:

Additional relations

Now we need to figure out what are the relations between a_1, a_2, and α. a_1 and a_2 have to have the same magnitude. This was discussed earlier when we considered examples with forces. If m_2 moves up one inch, m_1 must move down the same distance. But their directions are opposite. Therefore

$$a_1 = -a_2 \tag{10.22}$$

If a_2 is positive then the pulley spins around in the counterclockwise direction. By the right hand rule, if you curl you right hand around the pulley in the direction of motion, then your thumb sticks out of the page, which is the positive $\hat{k}$ direction. Therefore

$$a_2 = \alpha R \tag{10.23}$$

or

$$a_1 = -\alpha R \tag{10.24}$$

Solution to equations

Substituting for the acceleration and α in the above equations 10.17, 10.20, and 10.21 we have

$$R(T_1 - T_2) = -I\frac{a_1}{R} \tag{10.25}$$
$$m_1 a_1 = T_1 - m_1 g \tag{10.26}$$
$$-m_2 a_1 = T_2 - m_2 g \tag{10.27}$$

Eliminating the tensions from the first equation using the other two equations

$$m_1 a_1 + m_1 g - (-m_2 a_1 + m_2 g) = -I\frac{a_1}{R^2} \tag{10.28}$$

and solving for a_1

$$(m_1 + m_2 + \frac{I}{R^2})a_1 = (m_2 - m_1)g \tag{10.29}$$

or

$$a_1 = \frac{(m_2 - m_1)g}{m_1 + m_2 + \frac{I}{R^2}} \tag{10.30}$$

One could also get the same result from conservation of energy. If h_1 is the height of the first mass, the potential energy is equal to $(m_1 - m_2)gh_1$ plus some constant, because the second mass is lowered when the first mass is raised. Therefore

$$E = \frac{1}{2}[m_1 v_1^2 + m_2 v_2^2 + I\omega^2] + (m_1 - m_2)gh_1 = \frac{1}{2}[m_1 + m_2 + I/R^2]v_1^2 + (m_1 - m_2)gh_1 \tag{10.31}$$

where we have used $v_2 = -v_1$ and $(R\omega)^2 = v_1^2$. Differentiating with respect to time,

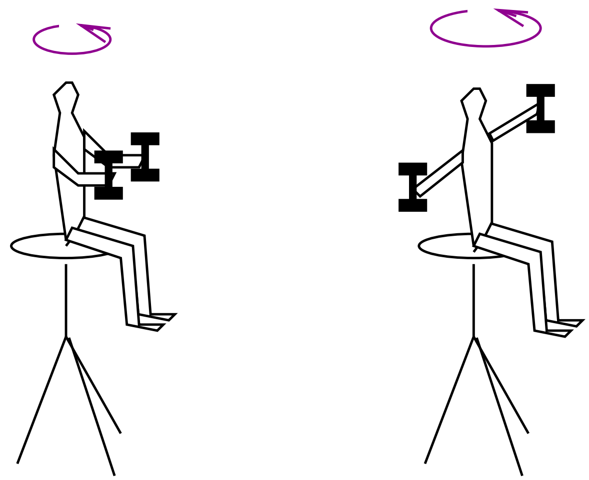
$$0 = [m_1 + m_2 + I/R^2]v_1 a_1 + (m_1 - m_2)gv_1 \tag{10.32}$$

from which one can obtain a_1.

10.3 Conservation of angular momentum

If the net torque acting on a system is zero then

$$\frac{d\mathbf{L}_{tot}}{dt} = 0 \tag{10.33}$$



which says that the total angular momentum is conserved. As with conservation of energy and conservation of momentum, this is a very useful law. Let's consider someone rotating around on a stool with nearly frictionless bearings.

If the person is holding some weights in their hands then they can change their moment of inertia by stretching out their hands. It increases from I_i to I_f. If the initial angular velocity is ω_i, what is the final angular velocity ω_f? Well if the stool is frictionless, then the net torque on the person plus weights is zero, so angular momentum is conserved. So

$$L \;=\; I_i\omega_i \;=\; I_f\omega_f \tag{10.34}$$

and therefore

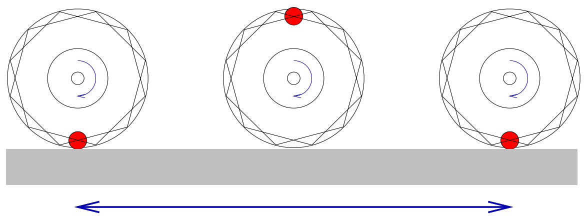
$$\omega_f \;=\; \frac{I_i}{I_f}\omega_i \tag{10.35}$$

So when the moment of inertia increases, the angular velocity decreases.

10.4 Rolling

Wheels are a marvelous invention that allow us to zip around a lot faster than we would normally with our own two legs. Whatever cave-creature invented the wheel should certainly be given the Nobel prize. On the other hand, they did choose a pretty stupid spelling for their invention.

A lot of mechanics problems involve wheels. They're interesting because they couple translational motion with rotational motion. If a wheel is rolling with a velocity v, what is it's angular velocity? We can compute that by considering the figure below.



The red dot marks a fixed spot on the rim of the wheel. In the first snapshot, the red dot is next to the ground. In the second, it has rotated 180 degrees and is now at the top. Finally at time T, it's back at the ground again. The wheel has done one revolution and in that same time, the center of the wheel has gone the wheel's circumference $d = 2\pi r$, where r is the radius of the wheel. Now the angular velocity $\omega = 2\pi/T$. That is, the wheel has rotated 2π radians in the time T it takes to make one revolution. So the velocity

$$v = \frac{d}{T} = \frac{2\pi r}{T} = \omega r \tag{10.36}$$

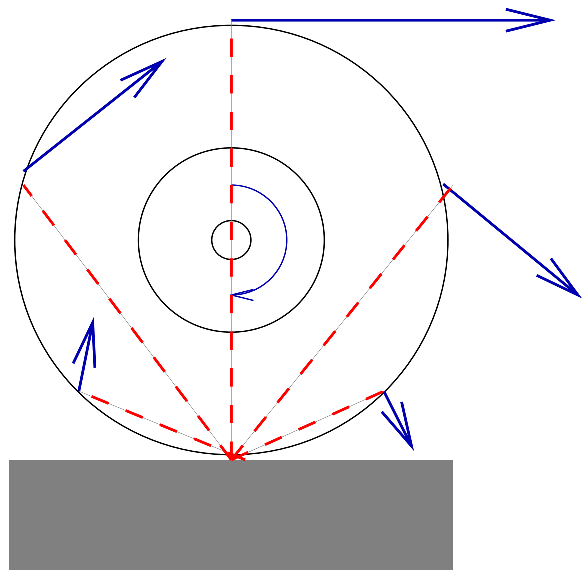
Imagine that! It's our old friend $v = \omega r$. But this means something quite different here. The v here is the center of mass velocity of the wheel. Our old friend refers to a different v, the velocity of the rim of the wheel when the center isn't moving. Ok, so what do we make out of all of this? We can think of rolling as combining a pure translation of the wheel at center of mass velocity $v_c = \omega r$ with rotational motion about the center of mass with angular velocity ω. This then tells us what the speed of the wheel is at various places on it's rim. For example, at the top of the wheel, we add $v_c = \omega r$ to the velocity due to rotational motion ωr. This gives $2\omega r = 2v_c$. At the bottom of the wheel, right next to the ground, these two effects cancel, and so the velocity of the wheel at the ground is zero! This makes sense, because if it were different from zero, the wheel would be sliding along the ground, which we are not allowing. The following is a sketch of the velocity vectors for several points along the rim of the wheel.

Notice that it is identical to the velocity if the wheel was *pivoted* at the bottom. So instantaneously a rolling wheel *looks as if it's pivoted*.

10.4.1 Kinetic energy

Now we can calculate the kinetic energy of a rolling wheel. One way is to consider the motion as a combination of rotation about the center of mass and translation. The kinetic energy is then

$$K = \frac{1}{2}\sum_i m_i|\mathbf{v}_c+\boldsymbol{\omega}\times\mathbf{r}_i|^2 = \frac{1}{2}(\sum_i m_i)v_c^2+\frac{1}{2}I\omega^2+(\mathbf{v}_c\times\boldsymbol{\omega})\cdot(\sum_i m_i\mathbf{r}_i) = \frac{1}{2}mv_c^2+\frac{1}{2}I\omega^2 \tag{10.37}$$



because, by definition, $\sum m_i \mathbf{r}_i = 0$ at the center of mass. But for rolling motion, $\omega = v_c/r$, so

$$K = \frac{1}{2}(mv_c^2 + I(\frac{v_c}{r})^2) = \frac{1}{2}(m + \frac{I}{r^2})v_c^2 \tag{10.38}$$

So this says that the kinetic energy is increased by the moment of inertia of the wheel.

Another way of getting the kinetic energy is to use the fact that the wheel appears to be instantaneously pivoted about the bottom. So

$$K = \frac{1}{2}I_b\omega^2 \tag{10.39}$$

But I_b is the moment of inertia of the wheel when rotated about an axis passing through the rim. By the parallel axis theorem we can relate this to the moment of inertia about the center of mass:

$$I_b = I + mr^2 \tag{10.40}$$

So

$$K = \frac{1}{2}(I + mr^2)\omega^2 \tag{10.41}$$

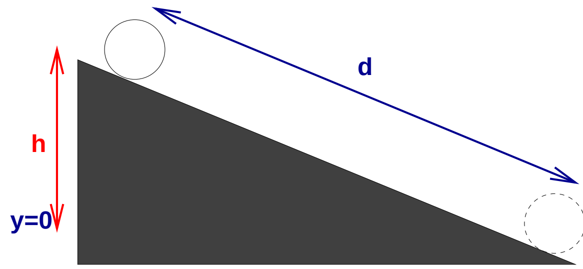
Giving the same result as above.

10.4.2 Example

Suppose a cylinder with moment of inertia I is let go from rest so that it slides down an inclined plane making an angle of θ with the horizontal. Calculate

a The velocity after the cylinder has gone down a vertical distance h.

b The acceleration along the inclined plane.



Solution

We will use conservation of energy to solve the problem. You can also solve it using torques and forces.

Initially the potential energy, measured from y as shown is $U = mgh$ and the kinetic energy is zero. So the total energy is $E = mgh$.

Finally, the potential energy is zero, and from Eq. 10.38 we have the kinetic energy. Therefore

$$E = mgh = \frac{1}{2}(m + \frac{I}{r^2})v^2 \tag{10.42}$$

Therefore the answer to (a) is

$$v^2 = \frac{2gh}{1 + \frac{I}{mr^2}} \tag{10.43}$$

To relate this to the distance travelled along the incline. $h = d\sin\theta$. Therefore

$$v^2 = 2\frac{g\sin\theta}{1 + \frac{I}{mr^2}}d \tag{10.44}$$

To get the acceleration a, note that $v^2 = 2ad$. Comparing with the above, we have that answer to (b):

$$a = \frac{g\sin\theta}{1 + \frac{I}{mr^2}} \tag{10.45}$$

Now let's look at what happens for different kinds of cylinders. If the cylinder is hollow, then $I = mr^2$, so $a = \frac{1}{2}g\sin\theta$. If it's solid then $I = \frac{1}{2}mr^2$, so $a = \frac{2}{3}g\sin\theta$. Therefore it accelerates faster if the cylinder is solid.

10.5 Gyroscopes

A gyroscope is a fascinating mechanical device with a lot of technological applications. You can easily make a gyroscope from a wheel with a handle put

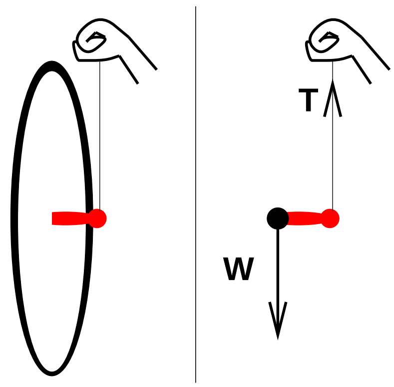


Figure 10.8:

through the middle. A picture of what this looks like from the side is shown in Figure 10.8.

If you dangle the end of the handle from a string, the wheel doesn't flop over put stays up straight! This might seem a bit odd because the weight **W** is pointing straight down. Instead it slowly moves around in a circle, that is it *precesses*. I kid you not! This is quite counterintuitive but can easily be understood by using the concepts of angular momentum and torque.

In Figure 10.8, the forces on the gyroscope are its weight, and an opposite force from the tension in the string. The torque around the center of mass comes from the tension, and points out of the paper. The angular momentum of the wheel points along its axis of rotation, i.e. to the left. The same situation is shown in Figure 10.9, looking down from above. The state of the gyroscope in Figure 10.8 and at a later time are both shown in this figure. The angular momentum and torque are shown too, and rotate with the gyroscope.

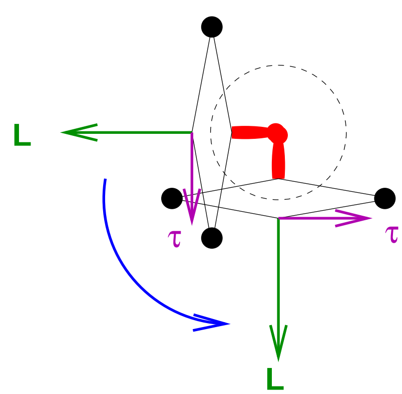


Figure 10.9:

In Figure 10.10, the angular momentum is shown for two times t and $t+dt$, where t corresponds to the first snapshot in Figure 10.9. The difference between them is the vector **dL**. Does this make sense yet? As seen from the figure, the

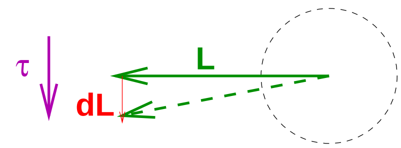


Figure 10.10:

torque is in the same direction as $d\mathbf{L}$. Therefore the equation

$$\boldsymbol{\tau} \;=\; \frac{d\mathbf{L}}{dt} \tag{10.46}$$

can be satisfied if the gyroscope is precessing at a suitable rate. Here we're assuming that the rate of rotation of the wheel ω is much greater than the rate of precession Ω, so that we can properly use $L \;=\; I\omega$. So let's compute the rate of precession.

If the length from the string to the center of mass of the wheel is called d, then the torque has magnitude

$$\tau \;=\; mgL \tag{10.47}$$

What is dL? It's $dL \;=\; d\theta L$, where $d\theta$ is the angle by which the gyroscope precesses and therefore **L** rotates (remember that $d\theta$ is in radians). Therefore

$$\tau \;=\; mgd \;=\; \frac{dL}{dt} \;=\; L\frac{d\theta}{dt} \;=\; I\omega\frac{d\theta}{dt} \tag{10.48}$$

But by definition

$$\Omega \;=\; \frac{d\theta}{dt} \tag{10.49}$$

So we have

$$\Omega \;=\; \frac{mgd}{I\omega} \tag{10.50}$$

The bigger ω, the slower the wheel precesses. The larger d, the distance between the center of mass and the axis of rotation, the faster the object precesses.

Let's test how well we understand this

10.5.1 Example

(a) Suppose you tap the handle downwards, in what direction will the wheel move?
(b) Suppose you tap the handle in the horizontal direction, in what direction will the wheel move?

Solution

(a) Well you've now added additional torque to the torque already there from gravity. So τ will stay in the same direction, but increase in magnitude. Therefore dL/dt will increase, causing the wheel to momentarily rotate around more quickly.
(b) If you tap in the horizontal direction, you add a component to the torque in the vertical direction. So $\mathbf{dL}$ is now no longer in the horizontal direction and the wheel will start moving up, or down, depending on the direction you tap it.

Chapter 11

Gravity

Now we're going to talk about one of a problem that has fascinated humans since we were cave dwellers. No, not how to make a decent club to thump our neighbors with, but what we always do in our more contemplative moods, look up at the sky at night, and wonder about all those little tiny lights up there.

The more we looked and looked the more that emerged. We pretty soon discovered that some of those lights up there were different than others. We call those planets. We started trying to see if we could predict where they were going to be. It was a big problem for thousands of years to figure out what those darned planets were doing.

It seems easy today since we know the answer. But if you didn't know the Earth was spinning, while rotating about the sun, the motion of the planets would seem pretty darned complex. People got really keen on finding the answers to where the planets would be. Lots of religions are based on these things. So surprise surprise, people became a little hot headed and started calling each other heretics and such nasty things. All over these silly spots of light in the sky.

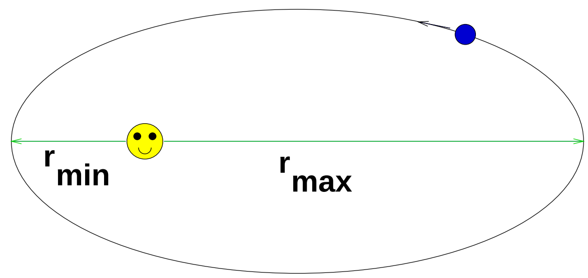
Eventually technology progressed to the time of Tycho Brahe, where he convinced the king of Denmark, in the sixteenth century to help him set up a big observatory to make accurate observations of the positions of the planets. He hoped to use this data to predict eclipses and the like, things that most humans thought would portend various unsavory events.

His disciple Johannes Kepler, spent a couple of decades analyzing the data and trying to make sense out of it all. The problem is that he seemed by present day standards to be a bit of a lune-tune, and quite literally. He had this notion that the planets were singing tunes as they merrily spun around the sun.

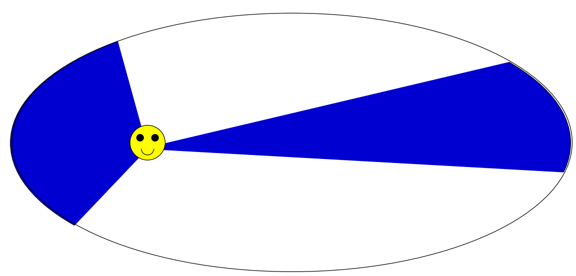
Galileo himself being a musician wasn't very impressed by Kepler's tunes and didn't take the man all that seriously. But buried in some long rambling treatise, are Kepler's three laws that have stood the test of time, and provoked Newton to give serious consideration to this problem.

11.1 Kepler's Three Laws

1. The *planets move in elliptical orbits*. The sun sits at one of the foci.



2. The radius vector, measured from the sun to the planet has a very neat property. The area swept out by the radius vector, *covers equal areas in equal times*.



3. The period T of orbit is simple related to $R = (r_{min} + r_{max})/2$. ($R$ is half of the major axis of the ellipse). It is that T^2 *is proportional to* R^3.

These laws seemed to describe the motion of the planets very accurately but people were at a loss to come up with a reasonable explanation. In fact, Newton's three laws are not enough to explain Kepler's. You still have to figure out what the force is between the sun and a planet.

Newton was a mighty smart fellow and once a form of this *gravitational* force was proposed, he could easily whip through pages of his newly invented calculus to figure out if it was consistent with Kepler's laws. It is debatable whether he actually came up with the correct form. Some say it was Robert Hooke, who has been largely overlooked by historians because of poor political skills, and just plain bad luck. But whatever the truth, the gravitational force has become known as Newton's law of gravity, or Newton's inverse square law, or Newton this, or Newton that. Next you might be wondering who really invented those cookies with figs inside of them.

11.2 The Law of Gravity

The force **F** between two point objects of masses m and M, separated by a distance r has a magnitude

$$F = G\frac{Mm}{r^2} \tag{11.1}$$

The force is attractive and points along the line between the two objects. This gravitational force is very fundamental. It can't be explained in terms of other forces such as the forces between charged particles. It doesn't matter what the material is made up of, old pillows, blueberries, lungfish, you name it, as long as the mass is the same, the force is the same. This has been thoroughly tested.

The symbol G is a constant known as *the gravitational constant*. It was first determined a long time ago , 1798, by Lord Cavendish. It was quite impressive how he managed to do it, see Fig. 11.1.

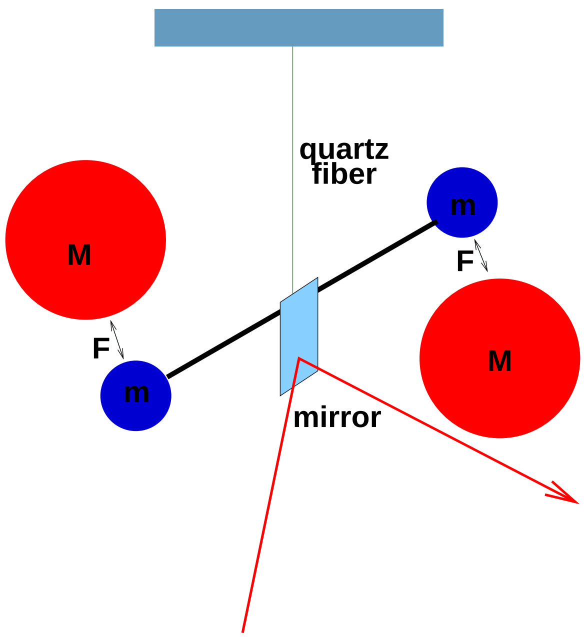


Figure 11.1:

Essentially he took two small masses and hung them from a very thin quartz fiber. The fiber exerts a tiny torque if twisted away from its equilibrium position. So it makes a very sensitive force meter. He then took two large masses and put them close to the hanging ones, measuring the twist of the fiber. He did this by attaching a little mirror to the fiber and shining a beam of light at it. It then bounces off the mirror to a wall quite far away. Measuring the displacement of the dot on the wall, you can calculate the force between to two masses. Knowing their distance and their masses, you have enough information to calculate G. Cavendish did a jolly good job and even today, G is only known to about four decimal places! Its value is

$$G = 6.674 \times 10^{-11} \frac{\mathrm{Nm}^2}{\mathrm{kg}^2} \tag{11.2}$$

11.3 Weighing the Earth

Essentially Cavendish used one mass to weigh another and get G. But we weigh masses every time we go to buy fish. We're using the Earth to weigh the fish. Could we use this to get the mass of the Earth? But wait, there are a couple of things I should say to make this clearer.

It is by no means obvious that the gravity that we feel sitting quite comfortably in our chairs is related to the forces that drive the planets around the sun. It was quite bold for Newton to assume it was the same force. All the particles in the Earth are pulling on you, keeping you in your seat. Atoms that make up the core of the Earth, a sea slug off the coast of Tasmania, an old smelly sock sitting in some dorm room in Amherst. All these atoms add up using Eq. 11.1 to give you total gravitational force acting on you. But what will that be? Clearly the Earth is not a point particle, and the law of gravity stated above only applies to point particles.

Using some math, one can show, as Newton did, a quite amazing result. The force between a uniform sphere and a point object is the same as if all the mass of the sphere was concentrated at its center! The easiest way to show this is to use something called "Gauss's Law" which you'll learn about when you study electricity and magnetism. You can either wait until then, or show it the brute force way, by doing some multiple integration.

So let's get back to the mass of the Earth. We know that when you weigh a fish of mass m, the force the Earth exerts on the fish is $F = mg$. But from what we've just said, the law of gravity gives us the same thing a different way

$$F = G\frac{M_e m}{R^2} \tag{11.3}$$

Where M_e is the mass of the Earth, that we're concentrating at the Earth's center. Then R, is the distance between the fish and the center of the Earth. Unless your fish is horribly confused, R will be the radius of the Earth. So now we can solve for the mass of the Earth

$$M_e = \frac{FR^2}{Gm} = \frac{mgR^2}{Gm} = \frac{gR^2}{G} \tag{11.4}$$

But we know from various means what the radius of the Earth is. It's about $6.37 \times 10^6 m$. Taking $g = 9.8m/s^2$, we get the mass of the Earth is about $6 \times 10^{24} kg$. That's a big number even in a state like Texas.

From this we can calculate the average density of the Earth since the volume of the Earth is $V = \frac{4}{3}\pi R^3$. It's

$$\rho = \frac{M_e}{V} \approx 5.5 \frac{\text{g}}{\text{cm}^3} \tag{11.5}$$

That's about twice the density of rock. So that gives you the idea that things underneath us are under an awful lot of pressure!

Now let's see if we can understand Kepler's laws

11.4 Explanation of Kepler's Laws

11.4.1 Law 1

I'm afraid you'll have to wait until you take a course in upper division mechanics to see an explanation of this one. If you solve $\mathbf{F} = m\mathbf{a}$ you find that your result is either an ellipse, a parabola, or a hyperbola. It's straightforward, but involves a fair knowlege of differential equations. But wait a second: why don't we see hyperbolic motion for planets? Well if we did, they would stay around for very long would they? That's why the planets have to have elliptical orbits.

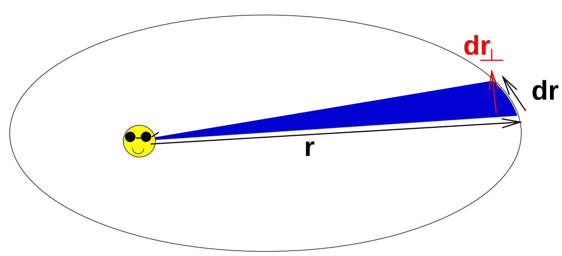
11.4.2 Law 2

This one we can deduce. First of all, what is the torque on a planet exerted by the sun? It's zero, since the direction of the force is in the same direction as the vector displacement between the planet and the sun. So $\tau = \mathbf{r} \times \mathbf{F} = 0$. In other words, the sun doesn't try to twist the planet.

So if the torque is zero, angular momentum is conserved. Let's figure out the angular momentum geometricaly. Now

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \frac{d\mathbf{r}}{dt}) \tag{11.6}$$

So what's $\mathbf{r} \times d\mathbf{r}$? This little picture should help



Here we have $\mathbf{r}$ at some point in time, and show how much area it sweeps out over a very short time dt. At the end of this time, $\mathbf{r}$ has moved by the vector

$d\mathbf{r}$. So let's compute $\mathbf{r} \times d\mathbf{r}$. It points out of the page and has a magnitude

$$|\mathbf{r} \times d\mathbf{r}| = r dr \sin\theta = r dr_\perp \tag{11.7}$$

This is just twice the area of the blue region (for small dr), because the area of this triangle is

$$dA = \frac{1}{2} r dr_\perp \tag{11.8}$$

So putting this all together

$$L = m|\mathbf{r} \times \frac{d\mathbf{r}}{dt}| = m\frac{2dA}{dt} \tag{11.9}$$

This says that the angular momentum, L, is proportional to the rate of change of the area swept out. And because L is constant, *the rate that area is swept out is constant*. Ths is equivalent to to saying that the planet sweeps out equal areas in equal times.

11.4.3 Law 3

We'll show this is true only for circular orbits. To derive this for general elliptical orbits requires more advanced math than we're using at present.

So let's apply $\mathbf{F} = m\mathbf{a}$ to a circular orbit. The acceleration points towards the sun and has a magnitude $\omega^2 R$. The gravitational force points in the same direction. Setting ma equal to 11.1 we have

$$F = G\frac{Mm}{R^2} = ma = m\omega^2 R \tag{11.10}$$

Solving for ω

$$\omega^2 = G\frac{M}{R^3} \tag{11.11}$$

The period T is related to ω through

$$\omega = \frac{2\pi}{T} \tag{11.12}$$

Substituting for T we finally have

$$T^2 = \frac{4\pi^2}{GM} R^3 \tag{11.13}$$

Well, this is Kepler's third law with the constant of proportionality explicity calculated.

11.4.4 Example

An orbiting satellite has a maximum distance from the Earth of r_p and a speed at that point of v_p. What is the speed half an orbit later when it is at a minimum distance from the Earth r_a?

Solution

Use conservation of angular momentum. At the maximum and the minimum distances, the radius vector is perpendicular to the velocity vector so that the angular momentum is

$$L = mr_pv_p = mr_av_a \tag{11.14}$$

Solving for v_a

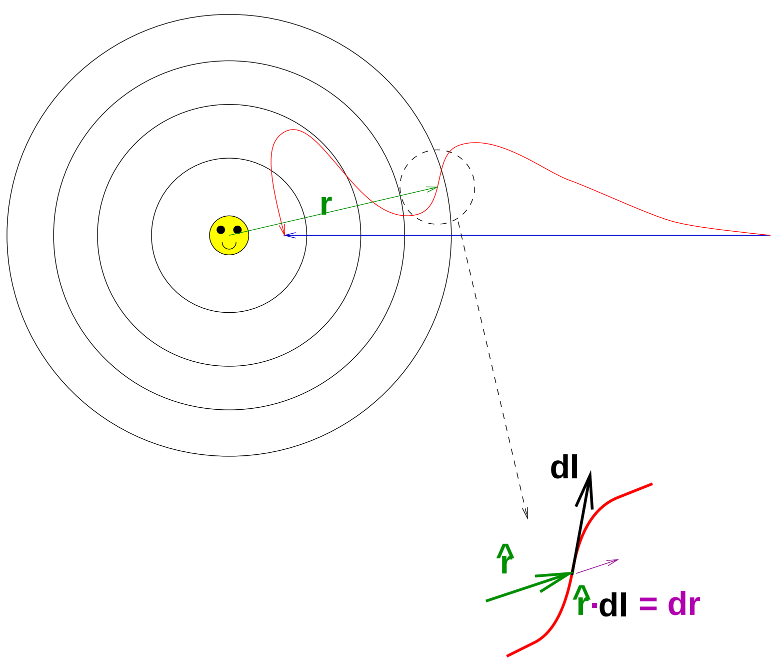
$$v_a = \frac{r_p}{r_a}v_p \tag{11.15}$$

11.5 Gravitational Potential Energy

The gravitational force is conservative. We can see that it only depends on the end points by considering the work it takes to go between two points

$$W = \int_A^B \mathbf{F}(\mathbf{r}) \cdot d\mathbf{l} \tag{11.16}$$

To be clear, we are *not* using the approximation of uniform gravitational acceleration that works for objects near the Earth's surface. Instead, we use the actual gravitational force from Newton's law of gravitation.



$\mathbf{F}$ points in the radial direction, that is in the negative $\mathbf{r}$ direction. The dot product $\mathbf{F}(r) \cdot d\mathbf{l} = |F(r)| dl \cos\theta$. But $\hat{r} \cdot d\mathbf{l} = dl \cos\theta$ is the component of $d\mathbf{l}$

in the radial direction. Call it dr. So

$$W = \int_{r_A}^{r_B} F(r)dr \tag{11.17}$$

In other words, the integral only depends on the radial coordinate r, and is therefore independent of any meanderings at different angles, that is it's independent of the path.

From the definition of potential energy, we know that the potential energy is defined in relation to some reference point, say at radius r_i. Let's set the potential U at radius r_i equal to zero so

$$U(r) - U(r_i) = U(r) = -\int_{r_i}^{r} F(r)dr \tag{11.18}$$

But $F(r)$ points towards the center, so it is

$$F(r) = -\frac{GMm}{r^2} \tag{11.19}$$

Plugging this in we have

$$U(r) = GMm\int_{r_i}^{r} \frac{dr}{r^2} = GMm(\frac{1}{r_i} - \frac{1}{r}) \tag{11.20}$$

To make life simple lets get rid of the first term by starting at infinity, so that $1/r_i = 0$. Then

$$U(r) = -\frac{GMm}{r} \tag{11.21}$$

So after all this, the final form for the potential energy is pretty simple. The potential energy decreases as the two objects get closer together. It is inversely proportional to r, unlike the force which is inversely propotional to r^2.

11.5.1 Escape velocity

"What goes up must always come down". Not necessarily. Consider this plot of the potential energy in red (Fig. 11.2).

Here we have a plot of the potential energy of an object, say a baseball as a function of distance from the Earth's center. Suppose we give it a kinetic energy of K, Then the total energy $U + K$ is given by the green horizonatal line. That means that the ball can increase r by turning some of its kinetic energy into potential energy. At the point where the red and green lines cross, it can go no higher. All its energy is potential and none kinetic. That's the maximum height that it can rise to.

OK, but suppose I give it enough kinetic energy so that $E > 0$? Now the green line is above the x axis and the object can keep going further and further away. It'll never have to worry about running out of kinetic energy. In this way it can escape to infinity and never come back down again!

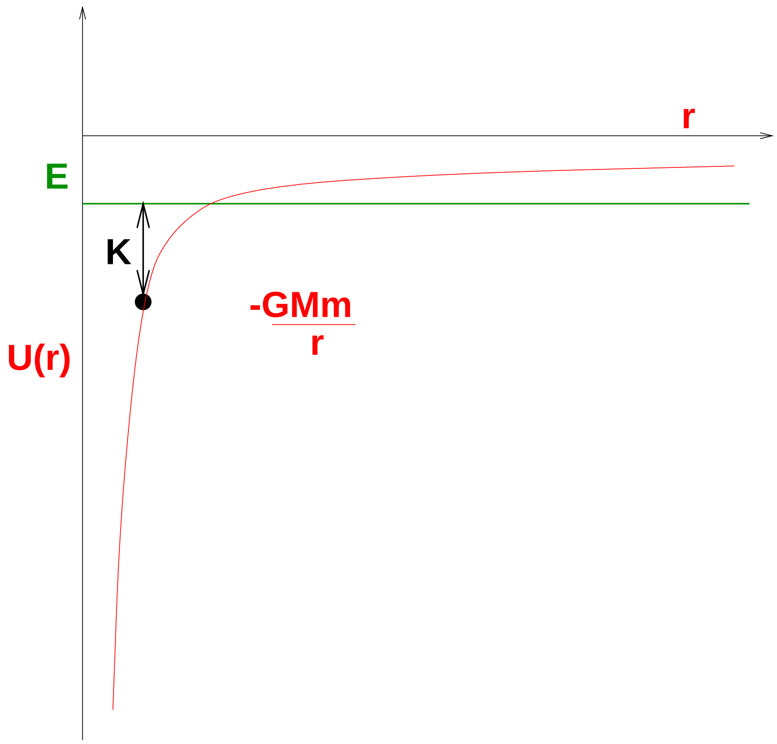


Figure 11.2:

How much velocity then, do we have to give a mass m that starts at the surface of the Earth, so that it can escape?

This threshold occurs when $E = U + K = 0$. So

$$\frac{1}{2}mv^2 = K = -U = \frac{GM_em}{R} \tag{11.22}$$

So

$$v^2 = 2\frac{GM_e}{R} \tag{11.23}$$

But we can express this in terms of g rewriting Eq. 11.4 as

$$gR^2 = GM_e \tag{11.24}$$

So

$$v^2 = 2gR \tag{11.25}$$

or

$$v = \sqrt{2gR} \tag{11.26}$$

Well, what is this?

With the radius of the Earth being $R = 6.37 \times 10^6$m and $g = 9.8$ m/s^2, we have that $v = 11.2 \times 10^3$ m/s. That's pretty fast!

But individual molecules at room temperature do cruise around at comparable speeds. You'll see later on when you study statistical mechanics, that the typical speed v of an atom depends on its mass m as $v \propto m^{-\frac{1}{2}}$. So the lighter molecules, such as hydrogen have enough velocity to escape from the Earth, whereas heavier ones such as nitrogen and oxygen don't. That's one of the reasons that you don't see much hydrogen or helium floating around in the air.

On other planets, the escape velocity is quite different. Jupiter has a much higher escape velocity and consequently has many more light elements in its atmosphere. Lighter planets like Mars have a lot less atmosphere than the Earth.

11.5.2 Example

A rocket is launched vertically from just outside the Earth's atmosphere (so there's no air resistance). What should it's initial velocity by so that it goes up by an earth radius $R = 6.37 \times 10^6$ m, from the Earth's surface? Compute your answer in terms of the escape velocity.

11.5.3 Solution

We use conservation of energy. Initially

$$E = K_i + U_i = \frac{1}{2}mv^2 - \frac{GM_e m}{R} \tag{11.27}$$

(where we have assumed that the atmosphere is sufficiently thin that the initial distance of the rocket from the center of the Earth is approximately R). At its maximum height of $2R$ from the center of the Earth, the kinetic energy is zero so that all the energy is potential. So

$$E = U_f = -\frac{GM_e m}{2R} \tag{11.28}$$

Equating initial and final energies, we have,

$$\frac{1}{2}mv^2 = \frac{GM_e m}{2R} \tag{11.29}$$

So

$$v^2 = \frac{GM_e}{R} \tag{11.30}$$

In terms of the escape velocity we have from Eq. 11.23 that

$$v^2 = v_{esc}^2/2 \tag{11.31}$$

or the velocity is $1/\sqrt{2}$ times the escape velocity.

11.5.4 Energy of orbits

Let's think a bit about the total energy of orbiting objects. Suppose an object with mass m doing a circular orbit around a much heavier object with mass M. Now we know its potential energy. It's

$$U = -\frac{GMm}{R} \tag{11.32}$$

How about it's kinetic energy? From Eq. 11.11 and the fact that $v = \omega R$ have

$$v^2 = \omega^2 R^2 = G\frac{M}{R} \tag{11.33}$$

so that

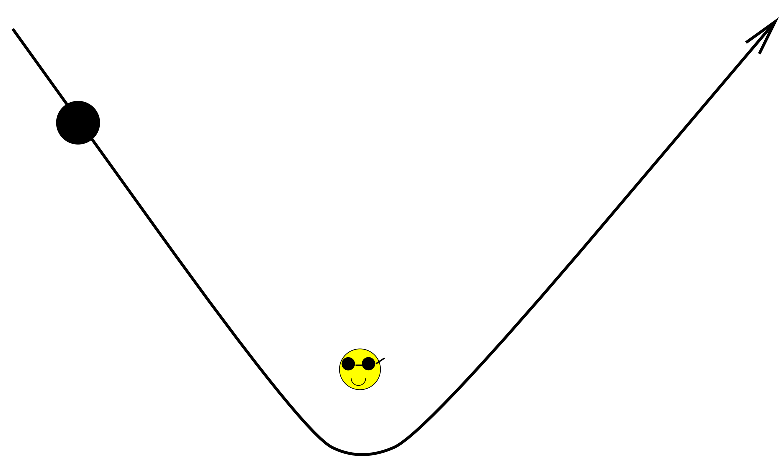
$$K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{R} \tag{11.34}$$

Notice that $K = -U/2$ and that

$$E = K + U = U/2 = -\frac{GMm}{2R} \tag{11.35}$$

So the total energy is always negative. In the same way that electrons in an atom are bound to their nucleus, we can say that a planet is bound to the sun. Its energy is negative, so it doesn't have enough energy to escape to infinity.

But what if the energy were positive? In that case the trajectories are no longer elliptical, and instead you get hyperbolic orbits! The object comes in from interstellar space (as they say on Star Trek) almost going in a straight line, and then cruises around the sun and is finally deflected in a straight line off into never-never land, never to be seen by us again!

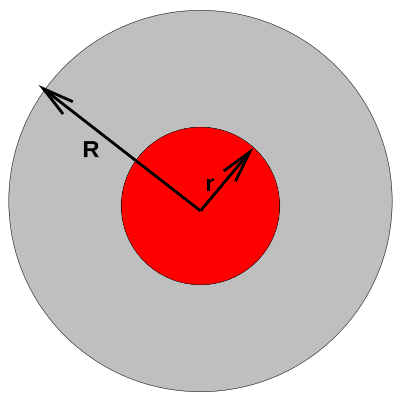


11.6 The hollow Earth

There are some crackpots out there that have written books claiming that the Earth is hollow and there a bunch of flying saucers inside it. Well, let's discuss what the force of gravity is like in this situation.

You have the forces from all the atoms in the shell tugging at you. If you're in the center, its pretty clear that they all cancel and you get zero. It turns out that the force inside is exactly zero even away from the center! As I mentioned before, this can be understood most easily using Gauss's law, so I won't try to prove it here.

Let's use this fact to figure out what the force of gravity is as we journey to the center of the Earth. Let's consider the Earth to be of uniform density. Call the radius of the Earth R. Let's say we're at a distance $r < R$ from the Earth's center. Then we can think about the Earth as being composed of a little sphere of radius r lying beneath you, and a hollow spherical cavity above you. The hollow spherical cavity is just a bunch of spherical shells and doesn't apply any force to you. So you're just left with the little sphere of radius r.



As we said before, for when you're outside a sphere, the force of gravity is the same as if all the mass were concentrated at the center. Call the mass of the sphere of radius r, $M(r)$. Call your mass m. So the force on you is

$$F = \frac{GM(r)m}{r^2} \tag{11.36}$$

What's $M(r)$? It's just the density ρ times the volume. So

$$M(r) = \rho V = \frac{M_e}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3 = M_e\frac{r^3}{R^3} \tag{11.37}$$

So using this we have that

$$F = \frac{GM_e(r^3/R^3)m}{r^2} = \frac{GM_e m}{R^3}r \tag{11.38}$$

So the force varies linearly with distance, just like one gigantic spring!

If the Earth wasn't so darned hot inside, and under such enormous pressure, it would suggest a neat means of transportation. Just dig really deep holes in the Earth that go all the way through, and jump in! Ignoring air resistance, you could jump in at San Francisco, fall for a while, and end up in Shanghai!

Chapter 12

Simple Harmonic Motion

In this chapter we explore a topic with huge implications for a lot of different subjects in physics. Simple harmonic motion describe the neatest form of oscillations. It has uses in everything from waves in solids and liquids, light waves, to atomic physics.

Simple harmonic motion is prototypically described by a mass attached to a spring, which is what we'll talk about now.

12.1 Mass and spring

Suppose you have a mass connected to a spring on a frictionless surface. The other end of the spring is attached to a wall. In equilibrium the mass just sits at the point $x = 0$. Now pull the mass and let it go as shown in Fig. 12.1.

You see that the whole thing oscillates with a period of T. That is after time T, the mass returns to its initial state. Without any friction this motion will persist indefinitely. After a while it gets a bit boring just starting at it, and you might want to know in more detail how the position of the mass varies with time, and how the period T depends on the spring constant k and the mass m.

Well let's try to figure that out. We start from $F = ma$. Here $F = -kx$ according to Hooke's law. So

$$a = -\frac{k}{m}x \qquad (12.1)$$

This says that the acceleration is proportional to the displacement.

The problem with this equation is that the acceleration is not constant, in fact it even changes sign when x changes direction. So we cannot apply $x = \frac{1}{2}at^2 + \ldots$ to this problem because that formula only applies for constant acceleration. What we'll first do is guess the solution and then after that I'll go through some other ways of getting it that at first might seem a bit tricky but turn out to be quite informative.

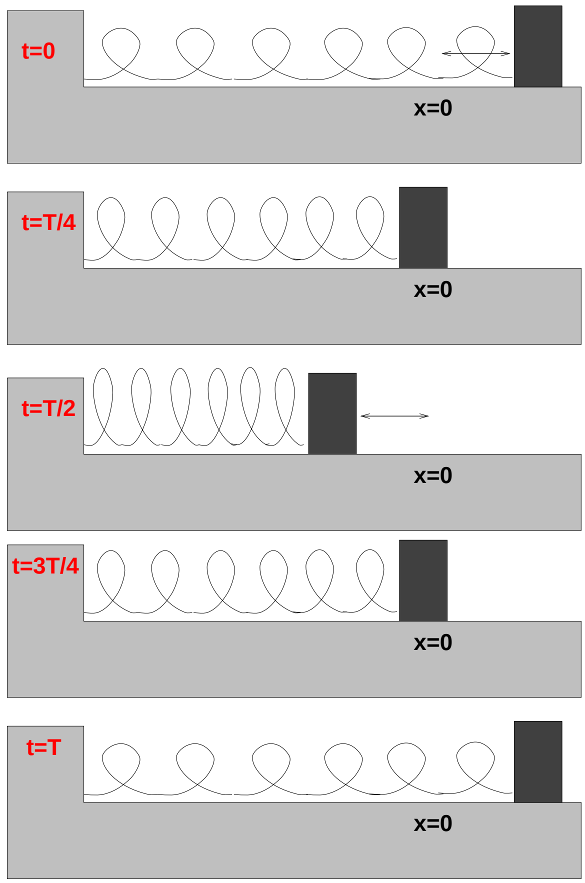


Figure 12.1:

12.1.1 An educated guess

Let's guess that the solution looks like a sine function. That seems sensible because a sine wave oscillates back and forth and back and forth just like our mass and spring as shown in Fig. 12.2.

The time it takes to complete one complete oscillation is called the *period* . Here we've stretched the sine wave so that its period is T instead of 2π. (Just as we did for circular motion and rotation, the argument of the sine is in radians, not degrees, so that we can use the standard results of differential calculus.) Mathematically, we can do that by writing it as $\cos(\omega t)$, where this new parameter ω is called the *angular frequency.* From it, you can get the period T as follows. You know that when the argument of the cosine is 2π you've gone a complete oscillation so that means that

$$\omega T = 2\pi \tag{12.2}$$

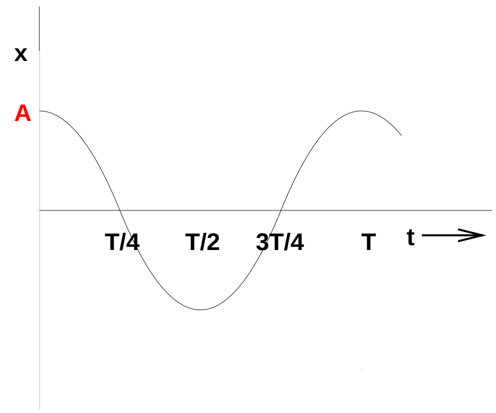


Figure 12.2:

or $\omega = 2\pi/T$. The angular frequency ω is measured in radians per second or, because radians have no units, in inverse seconds.

The *frequency* of oscillation means the number of complete cycles that are executed, per second. Mathematically, the frequency f is $1/T$, so

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{12.3}$$

Another thing that's important for this system is that the maximum displacement of x. Note that unlike a cosine function, its maximum is not 1. We'll call the maximum displacement the *amplitude* of oscillation A. We can take care of that by multiplying that cosine by A. So we have $x(t) = A\cos(\omega t)$

This describes oscillations pretty well. There's one additional thing though. We could defined $t = 0$ to be at some other time. This will then shift the argument of the cosine by some constant amount. Call it δ. This is often called *the phase lag*. So we have $\cos(\omega t + \delta)$. It shifts the maximum of the cosine to the left by a fraction $\frac{\delta}{2\pi}$ of a period. That corresponds to a time $\Delta T = \frac{\delta}{2\pi}T = \frac{\delta}{\omega}$.

So finally we write

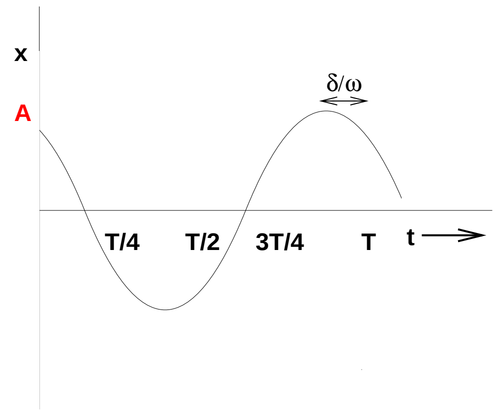
$$x(t) = A\cos(\omega t + \delta) \tag{12.4}$$

Note that because $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, we could write

$$x(t) = A(\cos(\omega t)\cos\delta - \sin(\omega t)\sin\delta) \tag{12.5}$$

But δ and A are arbitrary. So instead of this complicated expression, we could call $C \equiv A\cos\delta$ and $D \equiv A\sin\delta$, so

$$x(t) = C\cos(\omega t) + D\sin(\omega t) \tag{12.6}$$



This is an equivalent way of describing this kind of oscillation. In other words, adding a sine to a cosine gives a sine wave but shifted in phase, and altered in amplitude.

12.1.2 The solution

Now we take our educated guess 12.4 and check it satisfies $F = ma$, or more precisely, Eq. 12.1.

Let's compute the acceleration. To do that, we need to compute a. Differentiating once,

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t + \delta) \tag{12.7}$$

and again

$$a = \frac{dv}{dt} = -A\omega^2\cos(\omega t + \delta) \tag{12.8}$$

Well this is that same as

$$a = -\omega^2 x \tag{12.9}$$

So we have something similar to Eq. 12.1. We still haven't chosen ω, and now we can choose it so that everything works out. Choose

$$\omega^2 = \frac{k}{m} \tag{12.10}$$

and then it's clear that the acceleration satisfies Eq. 12.1.

So now we know the solution. It's a sine wave with an angular frequency of $\sqrt{\frac{k}{m}}$. Notice that A and δ can be anything that we want. In other words you can start of the spring with a small oscillation, or a large oscillation, and the angular frequency will be identical.

12.1.3 Easy example

Suppose a 1 kg mass attached to a spring is oscillating on a frictionless surface. You note that the period of oscillation is 6.28 s. What is the spring constant?

Solution

First let's compute ω. The period $T = 2\pi$ sec, so

$$\omega = \frac{2\pi}{T} = 1/\text{sec} \tag{12.11}$$

Now since $\omega^2 = k/m$,

$$k = m\omega^2 = 1 \text{ kg } 1^2/\text{s}^2 = 1\text{N/m} \tag{12.12}$$

12.1.4 More complicated example

Consider a mass m attached to a spring with spring constant k, sitting on a frictionless table at rest. A free mass, also with mass m, moves towards the first mass with velocity v and does a head-on elastic collision with it.

a What are the velocities right after the collision?

b What is the maximum compression of the spring?

c What is the time interval between the first collision and when the masses recollide?

d What are the final velocities of the masses?

Solution

Let's start by drawing some pictures

At first, the free mass has a velocity v and has a one dimensional elastic collision with the first mass. As we know, when equal masses have elastic collisions, they exchange velocities. Recall that collisions occur so fast that we don't have to worry about other forces while they are occuring: during the collision, we can forget about the spring.

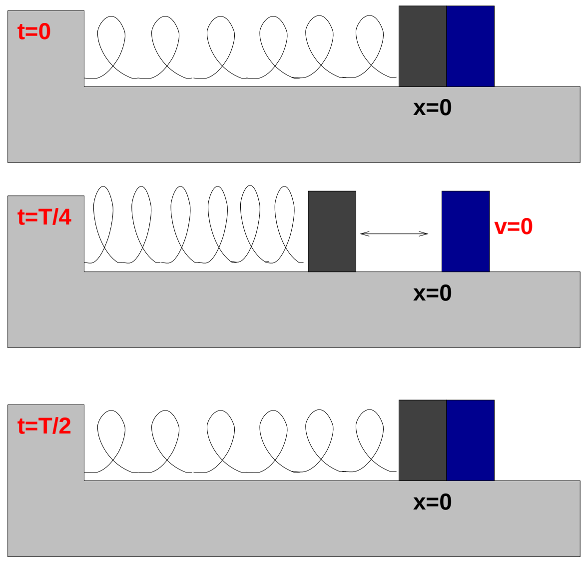
(a) In other words, after the collision, the free mass is at rest, and the mass on the spring starts moving with a velocity of v.

(b) We can use conservation of energy. The initial energy right after the collision, of the mass on the spring is $\frac{1}{2}mv^2$. At maximum compression all the energy is potential energy, so

$$\frac{1}{2}mv^2 = \frac{1}{2}kx^2 \tag{12.13}$$

Therefore the maximum compression is

$$x = \sqrt{\frac{m}{k}}v \tag{12.14}$$



(c) From the picture, you can see that the time interval is half a period. But $\omega = \frac{2\pi}{T}$ so

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{12.15}$$

So the time interval is $T/2 = \pi\sqrt{\frac{m}{k}}$. Note that this is independent of the initial velocity.

(d) When the first mass recollides with the second, they exchange velocities again. So you end up with the mass on the spring at rest, and the free mass flying off with velocity $-v$, the negative of the velocity it initially had before colliding with the mass on the spring.

12.1.5 Relation to circular motion

If you've been keeping your eyes open, you might have noticed a similarity between the lingo used here and that for rotational motion. This quantity ω the angular frequency, looks quite similar to the angular velocity.

Is this just a choice designed to maliciously mislead you, or a random choice, or a helpful hint? Well I'll let you decide. Read on!

A way of seeing the relation between two dimensional motion and simple harmonic motion can be made by rotating a wheel with a bit protruding from the rim, and looking at the shadow of this on a screen as shown in Fig. 12.3.

By defining the angle θ in the usual way as shown in Fig. 12.4, we see that

$$\theta = \omega t + \theta_0 \tag{12.16}$$

But let's call $\theta_0 \equiv \delta$ (can you guess why?). So if the radius of the wheel is r,

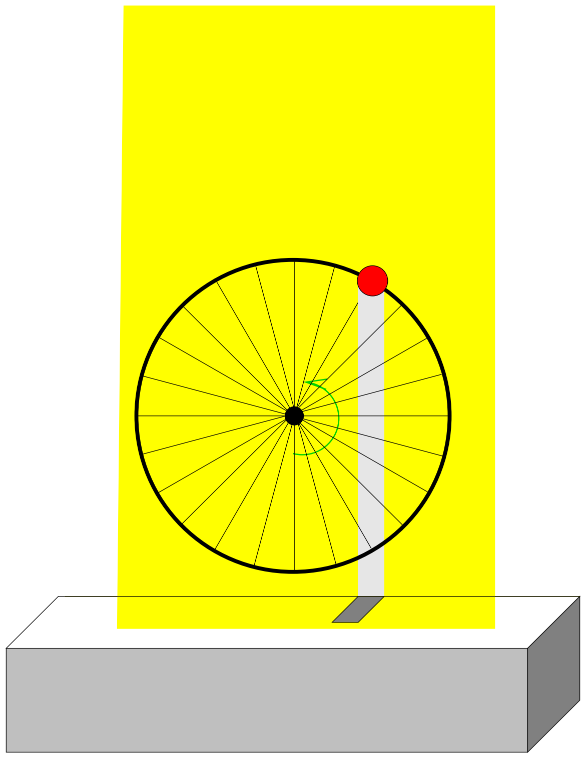


Figure 12.3:

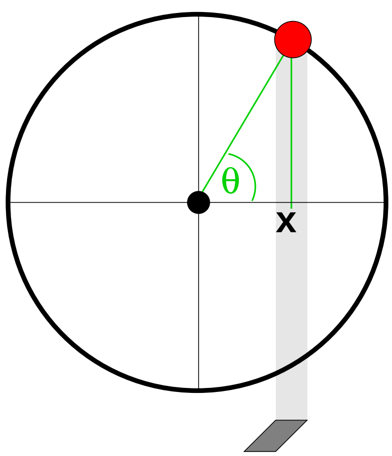


Figure 12.4:

then

$$x = r\cos\theta = r\cos(\omega t + \delta) \tag{12.17}$$

Let's call $A \equiv r$,

$$x = A\cos(\omega t + \delta) \quad (12.18)$$

Great! This is just what we guessed earlier, so you see that some how this simple harmonic motion is related to two dimensional motion. Let's try to figure out why.

Let's "two dimensionalize" the mass on the spring! We'll try to figure out how a mass tethered to a spring will move in two dimensions. Eqn. 12.1 can be written in terms of vectors.

$$\mathbf{a} = -\frac{k}{m}\mathbf{r} \quad (12.19)$$

Here **r** is the displacement vector and **a** is the acceleration. How do we interpret this? Instead of a spring going back and forth along a line, imagine a bungee cord with zero equilibrium length. You can describe the force that it exerts on you when you pull it, but saying $\mathbf{F} = -k\mathbf{r}$. So now imagine you have a bungee cord with one end tethered to the middle of a frictionless table. It's just like the one dimensional case, except now you can pull it in any direction. What happens if you start it off doing a circular "orbit"? Well we can figure out what happens just like we did for circular gravitational orbits. It's a really similar problem except we don't have Newton's law of gravity here, but rather Hooke's law for a spring. The acceleration is $-\omega^2\mathbf{r}$, so applying $\mathbf{F} = m\mathbf{a}$

$$-k\mathbf{r} = m\omega^2\mathbf{r} \quad (12.20)$$

Solving for ω

$$\omega^2 = \frac{k}{m} \quad (12.21)$$

Hmm, this looks just like what we found in the one dimensional case. So two dimenionalizing the problem didn't change this. Why not? Let's think about the problem now in component form. The x component of Eq. 12.19 is

$$a_x = -\frac{k}{m}x \quad (12.22)$$

This is the same as Eq.12.1, the one dimensional problem! So it's not surprising that the results should be the same, if it's the same equation. Right?

Well if we go further, we can derive what the x-component actually is. We just did this with the shadow and wheel example. We saw that this gave the correct general form for the solution, but from above we just got the correct value of ω. This time we derived the whole thing, with a bit of dimensional trickery. But we didn't actually need to do any calculus.

12.1.6 Imaginary numbers aren't so complex

There's a further way of seeing the solution for a harmonic oscillator. We can use *complex numbers*. This involves the use $i \equiv \sqrt{-1}$. An imaginary number is a multiple of i, like $3.1i$, $-2i$, etc. A complex number is the sum of a regular (real) number and an imaginary number. But actually these complex numbers aren't so complex and have a lot of really nifty properties.

The use of complex numbers is extremely important in physics and is used all the time in a wide variety of fields, so this should give you an introduction as to how to use them and what they mean.

Let's guess another functional form and see if it solves Eq. 12.1 This time guess

$$x(t) = A\, e^{\lambda t} \tag{12.23}$$

This doesn't seem right. An exponential form like this grows, or decays. It doesn't oscillate!. OK, but suppose that we're too stupid to see this and think that it might be a good solution all the same. Sometimes trying stupid things leads to new and interesting discoveries, and this is one of those times!

So let's calculate the velocity

$$v = \frac{dx}{dt} = \lambda A e^{\lambda t} \tag{12.24}$$

and

$$a = \frac{dv}{dt} = \lambda^2 A e^{\lambda t} \tag{12.25}$$

So

$$a = \lambda^2 x \tag{12.26}$$

Comparing this with Eq. 12.1, we have that

$$\lambda^2 = -\frac{k}{m} \tag{12.27}$$

Notice the negative sign! This says that

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm\sqrt{-1}\sqrt{\frac{k}{m}} = \pm\, i\sqrt{\frac{k}{m}} \tag{12.28}$$

So we get that λ is imaginary.

We therefore know that one solution to the harmonic oscillator problem is

$$x(t) = Ae^{i\sqrt{k/m}t} = Ae^{i\omega t} \tag{12.29}$$

where A can be any number, and another solution is

$$x(t) = Be^{-i\sqrt{k/m}t} = Be^{-i\omega t} \tag{12.30}$$

where here B is any number you like, and I have purposely made it different from A as the two numbers don't have to be linked.

It turns out that another solution to this problem would be to sum these two solutions together. So the general solution to this problem can be written as the sum of the two above solutions. If you don't want to keep writing all these pesky constants, you could say that the solution to this equation was

$$x(t) = \{e^{i\omega t}, e^{-i\omega t}\}, \quad \omega = \sqrt{\frac{k}{m}} \tag{12.31}$$

which means you take any linear combination of these two functions and it'll be a solution to the harmonic oscillator equation, Eq. 12.1.

Similarly, we learned previously that the general solution can also be written as

$$x(t) \;=\; \{\cos\omega t, \sin\omega t\} \tag{12.32}$$

They both solve the same equation 12.1 which can be rewritten

$$\frac{d^2x}{dt^2} \;=\; -\frac{k}{m}x \tag{12.33}$$

So there must be some relation between them! You don't need to know what it is to follow what comes later, but you might be curious. It's

$$e^{ix} \;=\; \cos x + i\sin x \tag{12.34}$$

That's called Euler's equation. Let's take $x \;=\; \pi$. This says the

$$e^{\imath\pi} \;=\; -1 \tag{12.35}$$

Pretty amazing!

12.2 Energy in oscillations

Let's figure out what the kinetic and potential energy is during different phases of an oscillation. The kinetic energy is $\frac{1}{2}mv^2$, and we calculated v in 12.7. Using that, and also that $\omega^2 \;=\; k/m$,

$$K \;=\; \frac{1}{2}mA^2\omega^2\sin^2(\omega t+\delta) \;=\; \frac{1}{2}kA^2\sin^2(\omega t+\delta) \tag{12.36}$$

The potential energy is $\frac{1}{2}kx^2$ so

$$U \;=\; \frac{1}{2}kA^2\cos^2(\omega t+\delta) \tag{12.37}$$

If we calculate $E \;=\; K+U$ we get

$$E \;=\; \frac{1}{2}kA^2\sin^2(\omega t+\delta) + \frac{1}{2}kA^2\cos^2(\omega t+\delta) \;= \tag{12.38}$$

$$\frac{1}{2}kA^2(\sin^2(\omega t+\delta)+\cos^2(\omega t+\delta) \;=\; \frac{1}{2}kA^2 \tag{12.39}$$

Which is constant, and equal to the potential energy when the spring is stretched to its maximum value. This seems correct.

In Fig. 12.5 we plot the both the position as a function of time and underneath that, kinetic and potential energy.

See how the energy shifts form, constantly transforming between kinetic and potential. For example, initially all the energy is potential, as the mass is at it's maximum and the velocity is zero. A quarter of the way through a cycle, all that potential energy has gone to kinetic as the mass passes through $x = 0$. Then it starts slowing down as it heads over to negative x. Eventually at half a cycle, the spring is maximally compressed and the energy is all potential again.

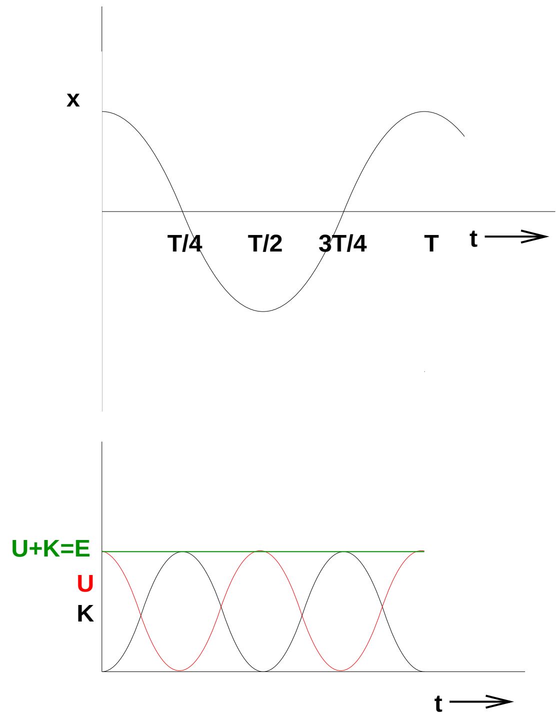


Figure 12.5:

12.3 Other systems

Now we'll look at a variety of systems all exhibiting the same phenomenon of simple harmonic motion, starting with a mass hanging vertically from a spring.

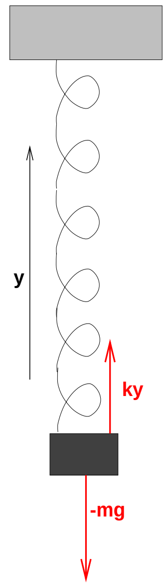
12.3.1 Vertical spring

In our previous example, we had to rely on a frictionless surface to keep the mass from from slowing down. But that's hard to do in reality. It's easier just to hang the spring

The problem is that this slightly complicates the situation, because even when the mass is hanging at rest, it is stretched due to gravity. We can calculate that stretch. In equilibrium the acceleration is zero so the net force is zero. So the two forces, the spring force, and force of gravity must balance each other

$$-ky_e - mg \;=\; 0 \tag{12.40}$$

or in equilibrium, $y_e \;=\; -mg/k$.



So with this in mind, let's write down $F_{net} = ma$

$$-ky - mg = ma \tag{12.41}$$

But let's change variables so that we measure distance from the equilibrium position y_e

$$-ky - mg = -k(y + mg/k) = -k(y - y_e) \tag{12.42}$$

Calling $x \equiv y - y_e$, we have

$$-kx = ma \tag{12.43}$$

This is identical to the equation for a horizontal spring, Eq. 12.1. So the angular frequency ω is $\sqrt{k/m}$. So gravity has no effect on the oscillation frequency, it just shifts the midpoint of the oscillations.

12.3.2 Pendulums

Pendulums are nothing more than objects than swing back and forth around some pivot point. Of course they do this under a gravitational field. They used to be fairly common, appearing in things like grandfather clocks. They're still used now by psychiatrists for being singularly good at putting their subjects into a deep deep sleep... Wait a gosh darn minute; you think that this book does a better job of that???

A couple of different examples of pendulums are discussed below. The simplest one is just a mass dangling from a string as shown in Fig. 12.6.

Let's apply $F = ma$ to this situation. The acceleration in the tangential direction a_t can be obtained by considering the components of the forces in that

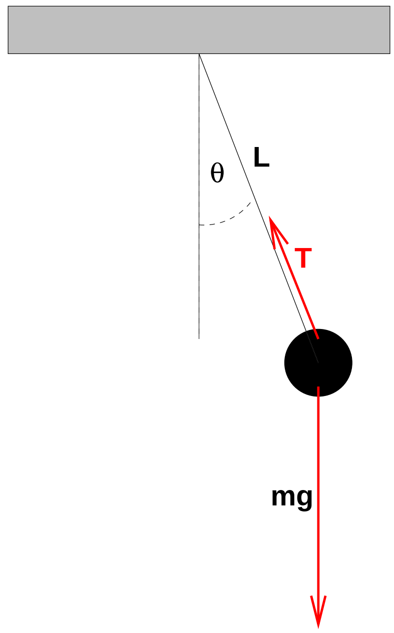


Figure 12.6:

direction. The easiest one is the tension T. That's just zero in the tangential direction because the string is perpendicular to this direction. The only one left over is the component of the weight. Well that's just $-mg\sin\theta$. So we have

$$ma_t \;=\; -mg\sin\theta \qquad (12.44)$$

But what is a_t? Well from what we figured out earlier, $a_t = L\alpha$. Here L is the length of the string. So

$$mL\frac{d^2\theta}{dt^2} \;=\; -mg\sin\theta \qquad (12.45)$$

Or

$$\frac{d^2\theta}{dt^2} \;=\; -\frac{g}{L}\sin\theta \qquad (12.46)$$

This is the equation we have to solve for this pendulum. Before we solve it, let's look at a more complicated situation, a "physical pendulum" as shown in Fig. 12.7.

Here you can see that we've pivoted this rigid body close to the top and it's going to swing back and forth. Because we're dealing with a rigid body we should use all this stuff that we learned about rigid body motion. That is, $\tau = I\alpha$. Here I is the moment of inertia reckoned around the pivot point.

What is τ? Well the force of gravity acts on the center of mass of the object, and let's say the distance between the pivot point and the center of mass is also

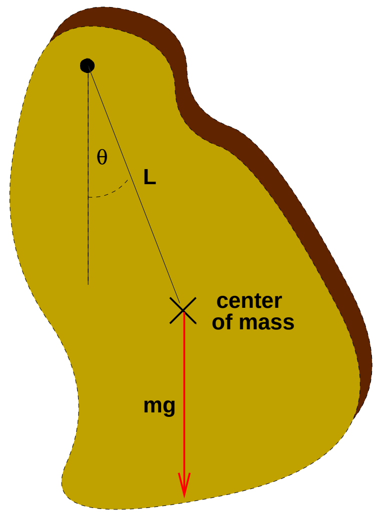


Figure 12.7:

called L. So we have

$$\tau \;=\; -\,Lmg\sin\theta \qquad (12.47)$$

So $I\alpha \;=\; \tau$ reads

$$I\frac{d^2\theta}{dt^2} \;=\; -\,mgL\sin\theta \qquad (12.48)$$

or

$$\frac{d^2\theta}{dt^2} \;=\; -\,\frac{mgL}{I}\sin\theta \qquad (12.49)$$

Both the equation for the simple pendulum and this equation are almost the same. The constant multiplying the sine function is the only difference. So how do we solve this? One method which is a commonly used technique, is to get the answer for very small, deviations in θ. You can check out on a calculator that

$$\sin\theta \approx \theta \qquad (12.50)$$

when $\theta \ll 1$. So if we stick to very small amplitude oscillations, we can safely use this approximation. Then Eq. 12.49 becomes

$$\frac{d^2\theta}{dt^2} \;=\; -\,\frac{mgL}{I}\theta \qquad (12.51)$$

This looks just like the equation for simple harmonic motion Eq. 12.33, except that we're using θ as a variable rather than x and our constant k/m has now change to $\frac{mgL}{I}$. Previously we got that $\omega^2 \;=\; k/m$ so the only difference now is that we have a different constant, so

$$\omega^2 \;=\; \frac{mgL}{I} \qquad (12.52)$$

Let's check out what this says for a simple pendulum. In that case $I = mL^2$ so

$$\omega^2 = \frac{mgL}{mL^2} = \frac{g}{L} \tag{12.53}$$

As usual the masses cancel, and the answer depends only on gravity and the length of the string. If you quadruple the length of the string, the oscillation frequency goes down by a factor of two, which means the period doubles.

Notice this is only true for small oscillations. Suppose we increase the angle θ, what happens to the period? It actually gets longer and longer. Eventually something catastrophic happens when you start the pendulum from $\theta = \pi$. Assuming we have a thin rod and not string, you can start off the pendulum from this point. That is, pointing directly up. If you start it pointing *exactly* up, it'll remain that way indefinitely. Of course that's impossible to do because this is an example of an unstable equilibrium. The point is though, that as you approach this point, the ball will stay at the top for a very long time, and it stays there longer the closer you can start it to $\theta = \pi$. So the period actually diverges in this limit. We have to keep in mind that our above formula for the oscillation frequency is only true in the limit of small angles.

12.3.3 Example

Suppose you hang a ring of radius R from some point on the rim. What is it's frequency of oscillation for small amplitudes?

Solution

Eqn. 12.52 can be used here. The center of mass of the hoop is in the middle. We're hanging it from the rim. So $L = R$. What's the moment of inertia? Be careful! We're *not* rotating it about the center. So $I = I_{cm} + MR^2 = 2MR^2$. Putting this together we have

$$\omega^2 = \frac{mgL}{I} = \frac{mgR}{2mR^2} = \frac{g}{2R} \tag{12.54}$$

12.3.4 Torsional oscillations

In the Cavendish experiment to measure gravity, we had a quartz fiber dangling from the ceiling. Attached to it was a rod with masses on it. The fiber exerts a torque when the rod is displaced from its equilibrium position.

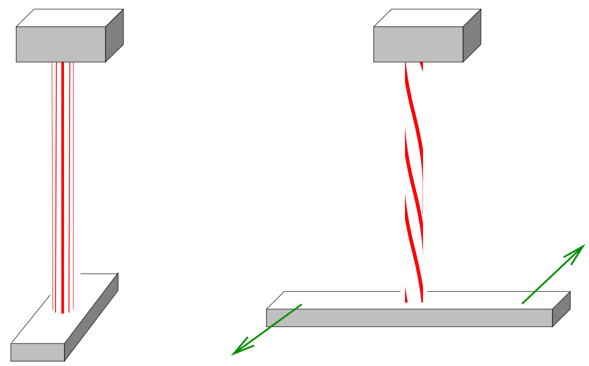
For small angles, you can say the torque exerted is proportional to the displacement from equilibrium

$$\tau = -\kappa\theta \tag{12.55}$$

This is just like $F = -kx$. κ is a constant having to do with the properties of the materials.

So applying $I\alpha = \tau$

$$I\frac{d^2\theta}{dt^2} = -\kappa\theta \tag{12.56}$$



or

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta \tag{12.57}$$

Again this is just like 12.33, so we have except of k/m we have here κ/I.

$$\omega^2 = \frac{\kappa}{I} \tag{12.58}$$

So the quartz fiber will oscillate back and forth at this angular frequency.

12.4 Damped oscillations

We know that in reality, a spring won't oscillate for ever. Non-conservative forces will diminish the amplitude of oscillation until eventually the system is at rest. If the mass at the end of the spring is sliding on a surface, one has to include the force of friction. But even if the spring and mass oscillate vertically, there is *air resistance* to slow them down gradually. This is due to "viscosity", a property of all fluids (i.e. gases and liquids). The magnitude of the force varies from one fluid to another. Because the effect of the viscosity is to slow the mass down, it is said to exert a "drag force."

We can increase the drag force on the mass and spring. Imagine that the mass is put in a liquid like molasses. Your lab instructor will not like it when they see their nice metal weight coated with a thick layer of ants in the morning. Be that as it may, when the mass is inside the molasses, it'll hardly oscillate at all. The viscosity of molasses is very high. On the other hand, a mass in air oscillates many times before it comes to rest.

A pretty good approximation for the drag force acting on a body moving at low velocity in a fluid is $f_r = -bv$. The constant b depends on the kind of liquid or gas the mass is in and the shape of the mass. The negative sign just says that the force is in the opposite direction to the body's motion. Notice that this is different from the friction at a surface that we discussed earlier: when an object moves across a surface, the force of (kinetic) friction is opposite to the motion, but is *independent* of its speed.

Let's add this drag force to the equation $f_{net} = ma$

$$-kx - bv = ma \tag{12.59}$$

In terms of derivatives

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{12.60}$$

This is a differential equation. We'll solve it using the guess we made in section 12.1.6.

But before diving into the math, what you expect is that the amplitude of oscillation decays with time. Let's say you have a spring oscillating pretty quickly, say 1 cycle/s. If the amplitude was 1 cm at t = 0, then suppose that the amplitude is half that, .5 cm, at t = 1 minute. What happens after another minute, at t = 2 min? Well we expect that it should halve again, and be .25 cm. After another minute at t = 3 min, it should halve again. This describes an exponential decay of the amplitude. Instead of the amplitude being constant, it's decaying with time:

$$A(t) = A_0 e^{-const\ t} \tag{12.61}$$

So

$$x(t) = A(t)\cos(\omega t + \delta) = A_0 e^{-const\ t}\cos(\omega t + \delta) \tag{12.62}$$

Fig. 12.8 shows a plot of an example of such a function, $x(t) = e^{-t}\cos(2\pi t)$

The green line is $A(t) = e^{-t}$. It is the envelope of the oscillation. Obviously, depending on the rate of decay of the amplitude, and the frequency, you'll get a different picture. But qualitatively, you'll see an oscillating function whose amplitude decays away to zero. This should describe weak damping. We don't expect this to work too well in molasses. To get a more quantitative understanding we'll have to do some more math.

We'll try sticking $x(t) = Ae^{\lambda t}$ into Eq. 12.60. Here again, A is just a constant. We already differentiated this function before in Eqs. 12.24 and 12.25 so we don't have to do it again. So we have

$$m\lambda^2 x + b\lambda x + kx = 0 \tag{12.63}$$

Cancelling the x's

$$m\lambda^2 + b\lambda + k = 0 \tag{12.64}$$

This is a quadratic equation for λ. Let's solve it:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \tag{12.65}$$

So we have two possible solutions for λ! They both solve the equation, and we have to have more information to figure out what to do with them. But for the moment, let's look at this equation more closely.

If the drag coefficient, b, is large, then the square root is real. However if $b^2 < 4mk$, then it becomes imaginary.

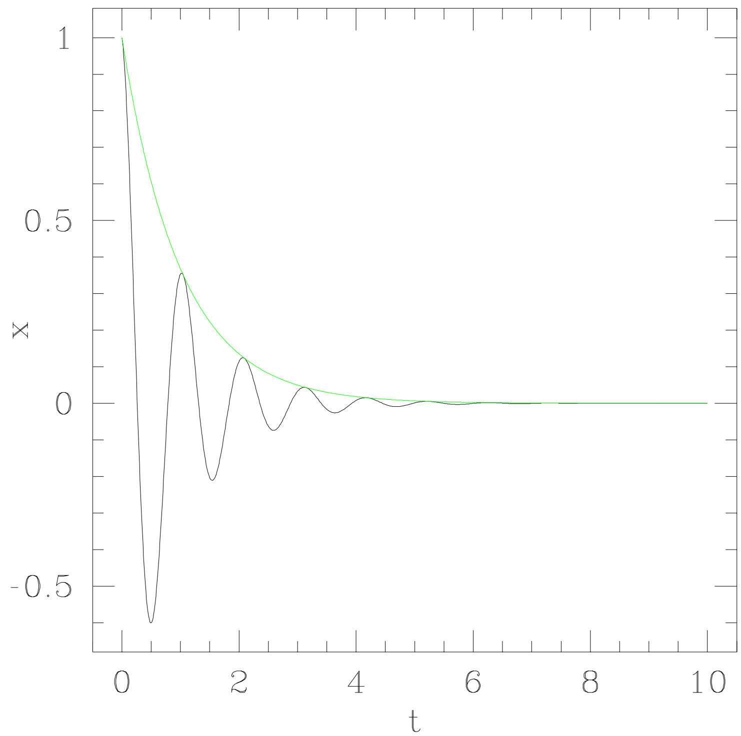


Figure 12.8:

12.4.1 Underdamping: $b^2 < 4mk$

Let's consider the latter case first. We you get oscillations, we call this *underdamping*. In this case λ is a complex number. It's got a real part and an imaginary part.

The real part is $-b/2m$ and we can figure out the imaginary part by writing $\sqrt{b^2 - 4mk}$ as

$$\sqrt{(-1)(4mk - b^2)} \;=\; i\sqrt{(4mk - b^2)} \qquad (12.66)$$

So we can rewrite the solution for λ as

$$\lambda \;=\; \frac{-b}{2m} \pm i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad (12.67)$$

The square root is the magnitude of the imaginary part. When $b = 0$, the square root just becomes $\sqrt{k/m}$, the normal frequency of oscillation, so it makes sense to interpret this as a frequency

$$\omega \;=\; \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad (12.68)$$

which decreases as b is increased.

So

$$\lambda \;=\; \frac{-b}{2m} \pm i\omega \tag{12.69}$$

and the solutions to the differential equation are

$$x(t) \;=\; Ae^{\lambda t} \;=\; Ae^{(\frac{-b}{2m} \pm i\omega)t} \;=\; Ae^{\frac{-b}{2m}t}e^{\pm i\omega t}. \tag{12.70}$$

The $\pm$ means there are two solutions here. This is just like our earlier use of imaginary numbers to solve the simple harmonic oscillator. Remember Eq. 12.31. The two solutions there were shown to be equivalent to the two solutions in Eq. 12.32. So the same is true here. The above two solutions are equivalent to

$$x(t) \;=\; \;=\; Ae^{\frac{-b}{2m}t}\{\cos(\omega t), \sin(\omega t)\} \tag{12.71}$$

This is what we guessed above before we plunged into all this math. You just have an exponential multiplying a sine wave. But now we know the expression precisely. The amplitude $A(t)$ decays as $e^{\frac{-b}{2m}t}$. What does this constant $-b/2m$ mean? If it's zero, there is no decay at all. If it's big it decays very fast.

Suppose we start with an amplitude of unity, and want to know the time τ it takes to decay to $1/e \approx .37$ of its original value. We have $A(t) \;=\; e^{\frac{-b}{2m}t}$ so at $t \;=\; 1$ we have

$$e^{\frac{-b}{2m}\tau} \;=\; e^{-1} \tag{12.72}$$

or

$$\tau \;=\; \frac{2m}{b} \tag{12.73}$$

This is often called the *the decay time*. We can rewrite the solution in terms of this

$$x(t) \;=\; Ae^{-t/\tau}\{\cos(\omega t), \sin(\omega t)\} \tag{12.74}$$

As the damping increases, τ decreases, that is, the oscillations damp out faster. But also note that as the damping b increases, ω decreases, finally hitting zero. Now we look at what happens past this point.

12.4.2 Overdamping: $b^2 \;>\; 4mk$

In this case both roots of λ are real.

$$\lambda_+ \;=\; \frac{-b + \sqrt{b^2 - 4mk}}{2m} \tag{12.75}$$

$$\lambda_- \;=\; \frac{-b - \sqrt{b^2 - 4mk}}{2m} \tag{12.76}$$

This means that both solutions decay exponentially

$$x(t) \;=\; \{e^{\lambda_+ t}, e^{\lambda_- t}\} \tag{12.77}$$

Fig. 12.9 shows an example of such a decay, $x(t) = \frac{1}{2}(\exp(-2t) + \exp(-0.5t))$.

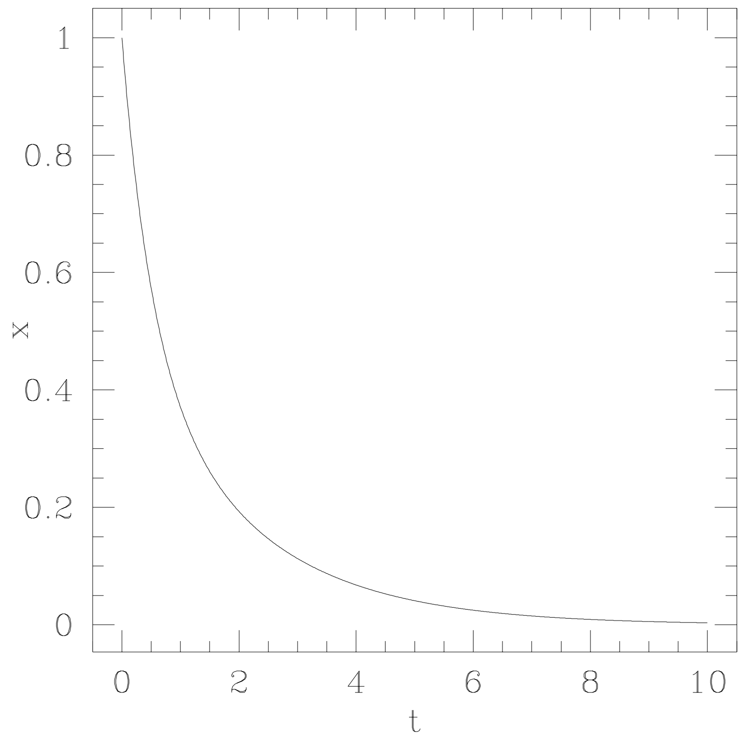


Figure 12.9:

Now as the damping b increases, the two solutions λ_+ and λ_- become very different. $\lambda_+ \to 0$ while $\lambda_- \to \infty$. When λ_+ is very small, that means the decay is very slow. So as you increase the damping b, the decay is slowed down. This is the opposite that happened in the above case of underdamping. We call this case *overdamping* because there are no oscillations, but the decay can be quite slow because the damping is so high that it's hard for the mass to move.

So let's ask the following question. What is the best value of b to choose so that the mass comes back to equilibrium most quickly. This is important if you were trying to design shock absorbers for a car. If b is too small it just oscillates back and forth for a long time without decaying in amplitude much. If b is too large, like in molasses, or tar, then it takes along time just to move the mass at all.

It turns out, that the best choice of b, is the *critically damped* case where $b^2 = 4mk$. It is at the point straddling the over and underdamped regimes. We won't solve this case but Fig. 12.10 shows a plot of the way it looks.

This plots the function $x(t) = \exp(-t)(1-t)$. The green line is a plot of $\exp(-t)$.

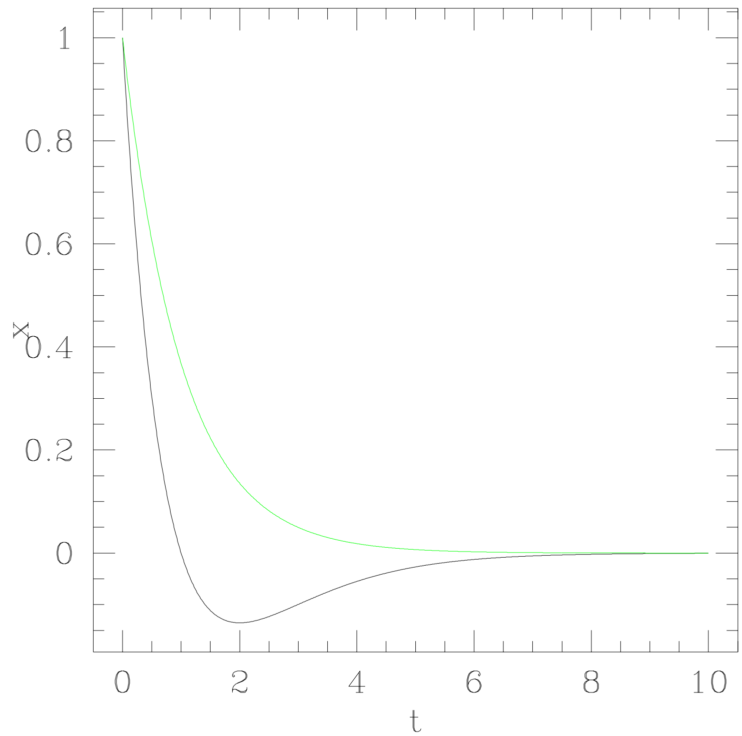


Figure 12.10:

12.5 Resonance

You've heard the story of the singer who can shatter a glass just by singing a particular note real loudly. Well that may actually be true, and it's an example of resonance. When you hear annoying buzzing sounds in your car, it is often something "resonating" with the car engine. When a child, or adult, is on a swing, they can increase their amplitude of oscillation by "pumping" the swing at just the right moments. That's also an example of resonance.

But resonance is even more common than that. Atoms is materials especially one with colors, resonate when light hits them. Light, by the way, applies a high frequency periodic force to electrons in a material. A microwave oven works on a similar principle. The microwaves apply periodic forces to water molecules causing them to resonate, and hence heat up. You might have seen some spectacular footage of some huge bridge collapsing right after it was built. That was an example of resonance. The wind was applying forces at just the right frequency to get the whole thing to swing wildly back and forth, ultimately collapsing.

So let's try to understand resonance. What you've got is a mass on a spring with some damping. You apply some periodic force at some frequency, and ask how much the system moves. If you apply it at just the right frequency and the damping isn't too big, you'd expect a small force would give a large response. Let's try to write down the equation we want to solve. F_{net} now includes, the spring $-kx$, damping $-bv$, and an external force $F_{ext} = F_0\cos(\omega t+\theta)$. We've added in a phase shift of θ because the applied force is not necessarily *in phase* with the harmonic motion. We'll talk more about that below. So

$$F_{net} = -kx - bv + F_0\cos(\omega t+\theta) \tag{12.78}$$

which equals ma so

$$ma + bv + kx = F_0\cos(\omega t+\theta) \tag{12.79}$$

Hmm, this looks even worse than the damped harmonic oscillator because now the right hand side isn't zero! Well fortunately there's a simple way of figuring this out without too much algebra. The hard way would be to try a solution of the form

$$x(t) = A\cos(\omega t) \tag{12.80}$$

and plug it into $F_{net} = ma$ and solve for A. That is a bit painful so let's do it the easier way. It involves the same trick we used earlier, to *two dimensionalize* the problem:

$$m\mathbf{a} + b\mathbf{v} + k\mathbf{r} = \mathbf{F}(t) \tag{12.81}$$

What does this problem look like? Well we already seen that it involves a bungee cord with $\mathbf{F}_{bungee} = -k\mathbf{r}$. But now we also have friction $F_{friction} = -b\mathbf{v}$. And now we have an external force $\mathbf{F}$. You can think of this external force as being applied to the object in some direction and the whole thing rotating around in a circle.

This is a lot like jetskiing in water with a bungee cord attached to a buoy shown in Fig. 12.11.

The external force is applied by the motor of the jetski and if it's pointing in just the right direction, it'll go round the buoy in a circular path. The angle between the jetski and the radius vector stays constant at θ.

We remember that if you consider just the x-component of the two dimensional problem, you should get the 1d problem. Let's check that it works. If we consider the x component of $m\mathbf{a}+b\mathbf{v}+k\mathbf{r}$ you get ma_x+bv_x+kx. That looks OK. How about $\mathbf{F}$? The angle that this makes with the x axis is $\omega t+\theta$, so you get an x component of $F\cos(\omega t+\theta)$. That is the same as $F_0\cos(\omega t+\theta)$ if you just make F_0 the magnitude of $\mathbf{F}$. So this two dimensional equation has as its x component the one dimensional equation we want to solve. The radius vector has an x component $x = R\cos(\omega t)$. So $R = A$ is the amplitude of oscillation of the 1d problem. This is all similar to what we did before for simple harmonic motion in section 12.1.5.

Let's try to figure out how the radius of the circle R, is related to the force that's applied and the angular velocity. Again we're applying $\mathbf{F}_{net} = m\mathbf{a}$. Now the free body diagram is shown in Fig. 12.12.

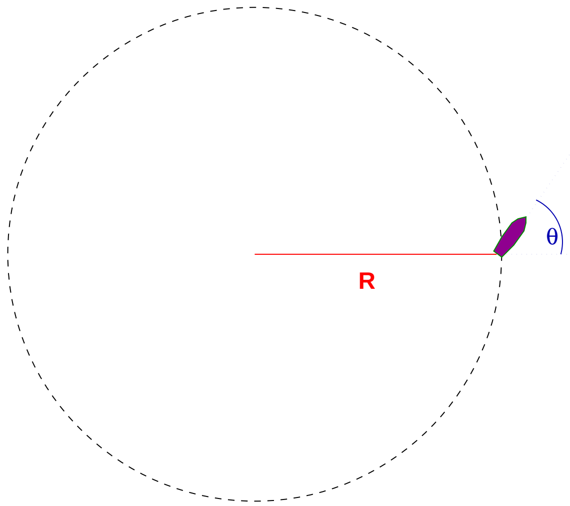


Figure 12.11:

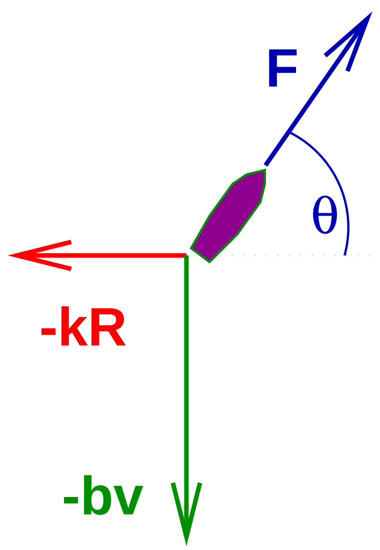


Figure 12.12:

So we have

$$\mathbf{F}_{net} \;=\; -kR\hat{i} - b\omega R\hat{j} + \mathbf{F} \;=\; -m\omega^2 R\hat{i} \tag{12.82}$$

Here we've used $v = \omega R$ and $a = \omega^2 R$. Solving for $\mathbf{F}$

$$\mathbf{F} \;=\; (kR - m\omega^2 R)\hat{i} + b\omega R\hat{j} \;=\; R\left((k - m\omega^2)\hat{i} + b\omega\hat{j}\right) \tag{12.83}$$

This tells us the direction and magnitude of the force necessary to keep the jetski going around the circle. It has a magnitude

$$F = F_0 = R\sqrt{(k - m\omega^2)^2 + b\omega} \tag{12.84}$$

or

$$A = R = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (b\omega)^2}} \tag{12.85}$$

and

$$\theta = \tan^{-1}(\frac{b\omega}{k - m\omega^2}) \tag{12.86}$$

Good, we've now understood the bungee cord - jetski problem, let's try to relate it back to the one dimensional resonance problem.

This says that if you apply a force with amplitude F_0, that the amplitude of resonance, A, will just be as given above in Eq. 12.85. Fig. 12.13 plots how this looks, taking $F_0 = k = m = 1$ and $b^2 = 0.1$.

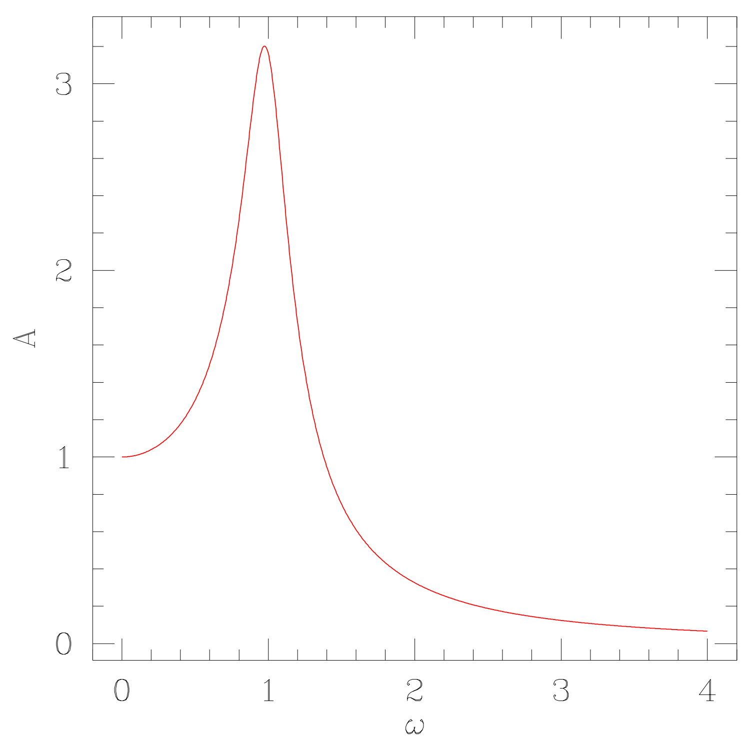


Figure 12.13:

You see that the amplitude has a strong maximum close to $\omega = \sqrt{k/m}$. Frequencies close to that are at resonance. The smaller the damping b is, the stronger the maximum, and the sharper the peak.

Now what about this mysterious angle θ? This says that if you apply a force, the response to it, $x(t)$, will not be instantaneous by will be shifted in phase by an angle θ as shown in Fig. 12.14.

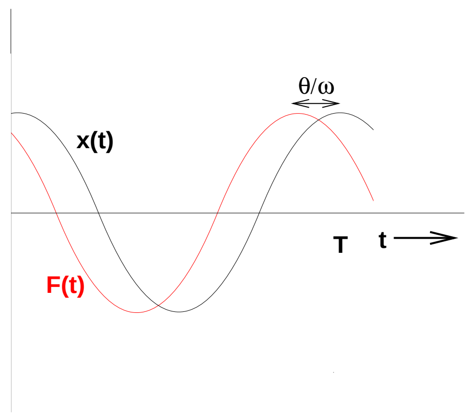


Figure 12.14:

Close to resonance when $\omega = \sqrt{k/m}$, you can see that $\theta = \pi/2$. There is a 90 degree phase shift between the applied force and the displacement.

Chapter 13

Liquids

So far we've been concerned with trying to understand systems where the answer involves just a few numbers, for example, the velocity of two balls after they collide. But what if you put zillions of balls in a box, where they'll collide with each other unbelievably fast. Let's try to figure out the velocities in this case, NOT! But putting a zillion balls in a box is something that we see all the time. That's what's happening in a glass of water, or molecules in the room you're in. Fig. 14.1 shows a simplified 2d version of what we'd like to understand, where all the circles are moving around in a hopelessly complicated way. Most

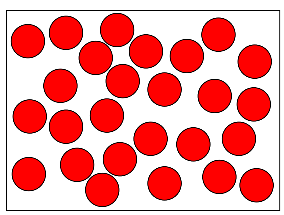


Figure 13.1:

of what we see in nature involve systems of this kind. In living organism, the environment is most often aqueous, so does that mean that we can't say anything sensible about them? As it turns out, applying physics to the zillions of molecules in a liquid, has been pretty successful. This in large part is due to the fact that energy and momentum are conserved despite the complexity of the microscopic motion. There are properties of liquids the *emerge* at a large, that is *macroscopic*, scale from all the frenetic motion at a microscopic scale.

When you think about it, liquids are pretty amazing things. You have an interface between a liquid and the air. If you were an extremely small ant diving into a glass of water, you'd notice that over just a few Angstroms the density of molecules increases dramatically as you enter the water from the air. So there is a pretty well defined *interface* between these gas and liquid phases, forming a nice smooth surface, at least on large enough scales.

Another thing you might take for granted, but is pretty amazing, is that if you let a glass of water sit around for a while, that its surfaces forms an extremely flat interface. You know that it must be pretty flat because you can see your partial reflection in it. Try to make a surface that flat by polishing glass. It's not easy! Liquids self-organize to be flat. This is just one of their amazing properties.

There are whole departments of scientists that study liquids, and still there are a lot of things about that them aren't completely understood. But there are loads of things that are, and I'll go through just a brief introduction to them. Even so, scratching the surface of liquids is not easy, as it were, but with a few important concepts, you'll be able understand them swimmingly.

13.1 Macroscopic quantities

If you were a million times smaller than Antman (and had x-ray vision), in a glass of water, you'd see a bunch of molecules moving around at the speed of rocket ships, and then crashing into other molecules, reversing direction in picoseconds. But at human scales, you see a relatively calm situation. Water looks like a continuous substance, because we're not able to see it on small enough scales. But that's not necessarily a bad thing. There is physics that emerges on a scale far larger than that of atoms, but still plenty small. A molecule of water is a few Angstroms in size, so if we consider a cube a micron on a side, that's still billions of molecules. We'll think about a liquid as being a continuous substance on this scale.

13.1.1 Density

In that case, we have a pretty well define *density* for this little cube of water. And it's gets to be an even more accurate representation as we increase the size of the cube. In this chapter, we'll be thinking about liquids on this scale most of the time. We recall that the density ρ is defined as the mass per unit volume

$$\rho = M/V. \tag{13.1}$$

The density of a liquid varies with external conditions. For example the density of water at the top of the ocean is about 0.3% smaller than a few miles down, because of the increase in pressure in the briny deep. It'll vary with temperature as well, by a few percent. So in both cases, the variation in the liquid phase, is pretty small and quite a bit less than the variation you'd get in a gas. The reason for this is that microscopically, the molecules are already strongly crowded together due to their attraction to each other. It's hard to squeeze them much further together. But to understand this further requires understanding thermal physics, and we'll postpone that subject until later.

13.1.2 Pressure

We just mentioned pressure actually, when we talked about the variation of density with depth in the ocean. Despite that fact the density of water doesn't change as you dive in water, anyone that's been diving in a swimming pool will realize that the pressure gets big pretty quickly. But what is the pressure mathematically speaking? You probably have already seen the pressure P, is a force per unit area

$$P = F/A. \tag{13.2}$$

But what is this force F, and area A?

If you put an object, say a thin slab, in water, what forces act on it? They'll be forces that act on all sides pushing in. This force is what gives rise to the pressure, see Fig. 13.2, in which we have two slabs at different orientations.

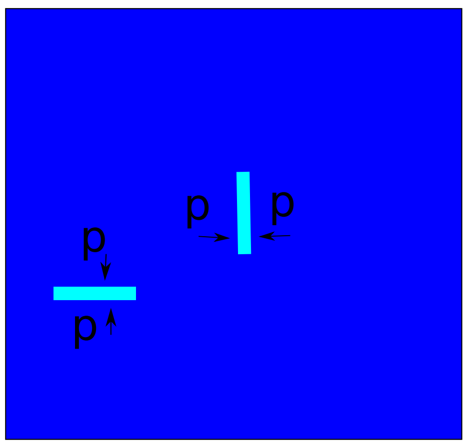


Figure 13.2:

We'll ignore gravity for the moment (we'll talk about that below), and assume that there are no currents in the water, so that everything is in equilibrium. In that case, force on one slide of a slab has to equal the force on the opposite side. Otherwise they'll be a net force and the slab will move, but in equilibrium, it shouldn't move at all, so those two forces have to be the same. For these thin slabs, the areas on opposite sides are equal, so that means that the pressure on the top and bottom of a slab are the same. This will be true irrespective of orientation. You could build a pressure meter by measuring the force on one of the surfaces of this thin slab.

So the pressure is the force per unit area acting on a piece of the surface. If you shrink the size of the slabs to essentially zero (yah yah, you can't because they're atoms, so it won't be quite zero), then you can think of the pressure at a point in the liquid. It should be a well defined quantity independent of the orientation of your slab.

Atmospheric pressure

The SI unit for pressure Nt/m^2 is referred to as a *pascal (Pa)*. How much pressure are you under at the moment? No, I'm not talking about this course, but physical pressure. Assuming you're not reading this on Mars, but on the surface of the Earth, the pressure at sea level is 1 atmosphere (atm), which is the same $1.01 \times 10^5 Pa$. Converting this to units we use in the U.S., this is the same as $14.5 lb/in^2$. That's pretty big, but only about (1/1000) of the pressure at the bottom of the ocean in the Marianas trench! We'll talk about how to calculate the pressure down there next.

13.2 Liquids in equilibrium

A lot about what's interesting about liquids is when they're currents, like what you see in a river, your blood stream, or water hose. But there is still a lot of interesting physics when the water looks completely still. This is especially true when we ask what are the effects of gravity because now the pressure will vary with depth.

13.2.1 Effects of gravity

As we mentioned above, when you're in the ocean, the pressure will increase as you dive down. Let's try to figure out how the pressure varies with depth. In Fig. 13.3, we depict a liquid under gravity. At the top is the surface, and the thin rectangle represents a cylinder of water of length z, starting at the water's surface. We want to know what the pressure p is, at the bottom of the cylinder, as shown by the little black arrow.

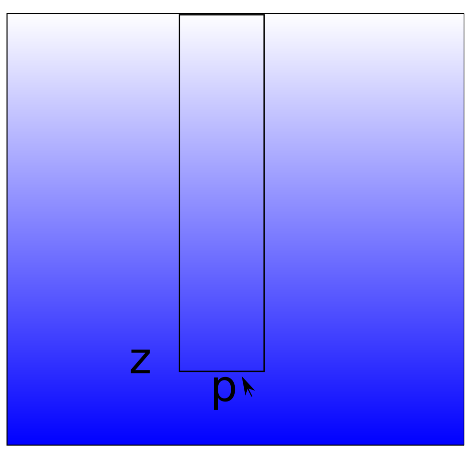


Figure 13.3:

We're assuming equilibrium, so they're no currents here. If we were to place our little slab shown in Fig. 13.2 at the bottom of the cylinder, then it'd feel a force equal to weight of the water above it plus the air above that. If you're miles down in the ocean, that's a lot of weight! So if the slab is of area A, the total force will be g times the total mass above it, the mass of the atmosphere M_a, plus the mass of the water M_w, $F = (M_a + M_w)g$. As we saw, the assumption of constant density is a pretty good approximation for the ocean. In that case because the volume of the cylinder is $V = Az$,

$$M_w = \rho V = \rho A z \tag{13.3}$$

We have

$$P = \frac{F}{A} = \frac{(M_a + M_w)g}{A} = \frac{M_a g}{A} + \frac{\rho A z g}{A} \tag{13.4}$$

The first term $M_a g/A$ is the pressure of air at the surface of the ocean, call it P_0, so

$$P = P_0 + \rho g z \tag{13.5}$$

So the pressure increases linearly as you go deeper into the ocean. Again, it's worth repeating that z *increases* as you go further down.

Note this would also work for a swimming pool, or a container of any shape. The pressure depends only on z and nothing else.

13.2.2 Buoyancy

Another way of seeing this is to consider a massless cylindrical can of cross sectional area A with its top at vertical distance below the water of z_2 and its bottom at z_1 as shown in Fig. 13.4. Consider the forces acting on this can

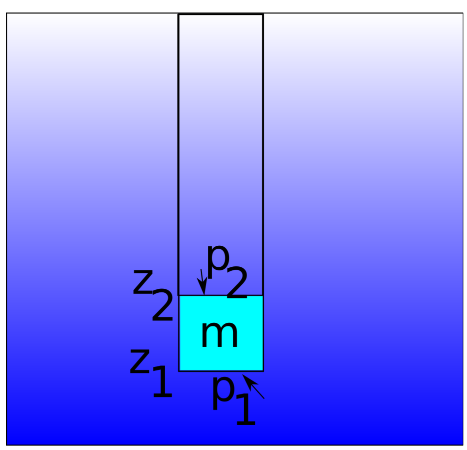


Figure 13.4:

due to the pressure of the water. The net force due to water pressure on the can sides cancel by symmetry. We're left with only the forces on the top and bottom. The pressure at z_1 and z_2 are different, giving a net force up on the can. This is called the force of buoyancy, first understood by the great scientist and mathematician Archimedes of Syracuse. If the can was hollow it'd feel a force pushing it up towards the surface, since the pressure at its bottom is greater than the pressure at its top. But if we fill the can with water, the can will just sit there because the can is massless and isn't doing anything. So we have a force balance equation that stays that the net force due to the external water pressure must equal the weight of water in the can

$$F_{net} = A(P_1 - P_2) = mg \tag{13.6}$$

And with $m = \rho V = \rho A(z_1 - z_2)$ this gives

$$P_1 - P_2 = \rho g(z_1 - z_2) \tag{13.7}$$

or using differences (i.e. Δ) notation:

$$\Delta P = \rho g \Delta z \tag{13.8}$$

Taking the difference of eqn. 13.5 gives the same thing!

Now we can extend this notion of buoyancy to objects of any shape, see Fig. 13.5. We now have pressure acting on the surface of the object in all sorts of

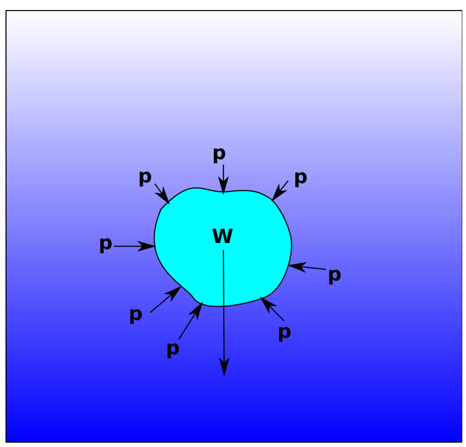


Figure 13.5:

different directions, but this pressure will be higher for points on the surface of the object that are further down in the water. What's the net buoyancy force from all of this? One way to calculate this would be to use vector calculus, but the easier way is to use the above argument. If the object is filled with water as well, the net force on it will be zero. So we can equate the weight W pushing down to the buoyancy force pulling up. This tells you that the force of buoyancy is the same as the weight, if the object was made out of water.

How how does this work? Suppose the object was hollow, so it weighed nothing. Then the force acting on it will be nothing but the buoyancy, that is the weight assuming the object was filled with water. Denoting the density of water to be ρ_W, and the volume of the object to be V, this buoyancy force F_B is

$$F_B = \rho_W V g \tag{13.9}$$

If the object is filled with water, the total force on it will be zero, as we discussed above. If the object is half filled with water, the buoyancy force will be half of that weight. If the density is half that of water, then the buoyancy force will still be half of that weight. In other words the total force on the object is the buoyancy force minus its weight: $F_{net} = F_B - W$. If the object is of uniform density ρ_O then its mass is $\rho_O V$ and the net force is

$$F_{net} = (\rho_W - \rho_O)V g. \tag{13.10}$$

This is why when you go into a swimming pool, you seem to weigh less. Your apparent density is no longer ρ_O, but $\rho_0 - \rho_W$, which is pretty small since we're largely made out of water.

So far we've talked about objects that are completely submerged. What about floating objects? At some point a buoyant object will rise to the surface. How far in the water does it float?

In equilibrium, the weight of the object must cancel the force of buoyancy. So if the object is of mass M_O, then a force of $M_O g$ will act on it, pointing down. What prevents it from sinking? The force of buoyancy. The further the object sinks, the more water it displaces. The way to get the buoyancy force in this case, is to consider the amount of the object that is now submerged. If that we filled with water, it would have some mass $M_{submerged} = \rho_W V_{submerged}$. The buoyancy force is $M_{submerged} g$ pointing up. The object will float at a point where this force $\rho_W V_{submerged} g$ equals the weight $M_0 g$.

The above is known as "Archimedes' principle": *Any floating object displaces its own weight of fluid*.

There's a slightly easier way to see how this works. Suppose we have a piece of wood floating the water as shown in Fig. 13.6 on the left. It floats because the density of wood is less than that of water. Now replace this with something equivalent for our purposes: A massless impermeable container whose inside has just enough water in it to give the object mass of the wood. This is shown on the right. Where will it float? Since the container isn't doing anything, we can get rid of it and since the water wants to for a flat surface, you can see where it floats. You could've figured this out without using any of the above math.

Example

Suppose you fill up a glass of water for a friend with a pitcher containing ice but you inadvertently pour it too far, so that the water is all the way up to the

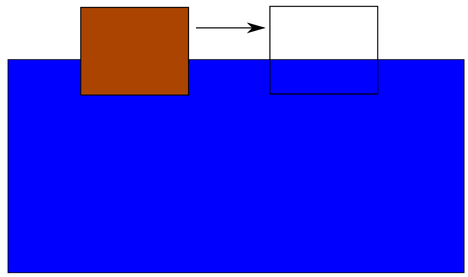


Figure 13.6:

rim, and the ice is now floating above the rim. You now watch in terror as the ice slowly melts. How much water will spill onto the table?

Solution

Maybe you're hoping that your friend is cool and will just slurp some of the water to bring the ice down to a comfortable level. But not to worry. Look again at Fig. 13.6. Gross, a brown ice cube. No matter, just pretend its transparent. This is your ice. You can think of the ice as shown on the right as being in a massless container with water inside. But wait! If the ice melts, it'll shrink in volume to be exactly the amount of water inside the massless container. Phew! The water stays at the same level!

13.2.3 Measuring pressure

Now we understand pressure in liquids a bit better, let's talk about a relatively simple way that people measure atmospheric pressure. You can use mercury, which is a liquid, but much denser than water. It's also not a good idea to play around with, so don't try this at home!

Take a glass tube (closed at the bottom), and fill it with mercury. Take a flat bowl and fill it with mercury as well. Turn the tube upside-down into the bowl. The mercury will spill out, but won't go out all the way because there is essentially a vacuum now at the top of the closed tubed (mercury vapor pressure is very low). This is shown in Fig. 13.7. How high up, h, will the mercury column rise above the mercury liquid in the bowl? Let's consider the forces on the column of mercury. Unlike the discussion above shown in Fig. 13.4, the pressure at the top of the column is zero is zero. Because pressure only depends on vertical height, the pressure at the bottom is the same as the pressure on the surface of the mercury, which is the atmospheric pressure p_A. So it needs to be atmospheric pressure p_A at the base of the column, shown in the figure as the height where $z = 0$. As we discovered, in Fig. 13.5 the pressure at this height (with $P_0 = 0$) is $\rho g h$. Therefore

$$p_a = \rho g h \tag{13.11}$$

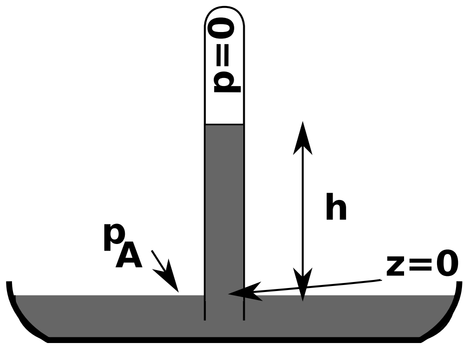


Figure 13.7:

Note that the shape of the tube is arbitrary since Fig. 13.5 is quite a general result. This has been a common way of measuring pressure, so much so, that there is a unit of pressure *torr* which is a millimeter's worth of pressure in this apparatus. 1 atmosphere is $760 torr$, meaning $h = 760 mm$ (with gravity and temperature taken to be precise and reasonable values).

13.3 Pascal's Principle

Let's ignore gravity and consider a cylindrical container filled with a liquid. At the top of the container, we place a water-tight piston and apply a force, as shown in Fig. 13.8. What happens to the pressure? Well we discussed this in Sec. 13.1.2. The pressure will increase, but be the same at all points in the liquid.

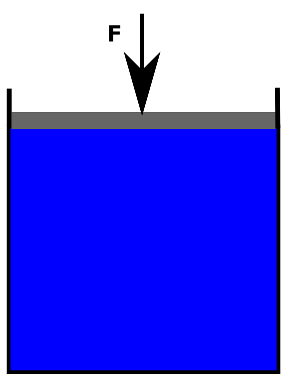


Figure 13.8:

Now let's include gravity. Our derivation of eqn. 13.8 remains unchanged.

The pressure still increases, the deeper you go down in the vertical direction. So the only thing that does change is that pressure goes up everywhere by the same amount. Eqn. 13.5 is still correct, but now the additive constant P_0 will increase when F increases. This is Pascals principle. If you increase the pressure by applying a force to some point of a closed system, the pressure everywhere in the system, even very far away, will go up by the same amount.

13.3.1 Hydraulics

There are many pieces of equipment that use hydraulics to amplify forces. These include, trucks, cars, cranes, planes, tractors, and elevators. The idea of a hydraulic lever is that you can apply a force at some point, and have it greatly amplified. In this way, it's similar to a pulley.

Fig. 13.9 shows how force amplification can occur. You have a sealed liquid-filled container.

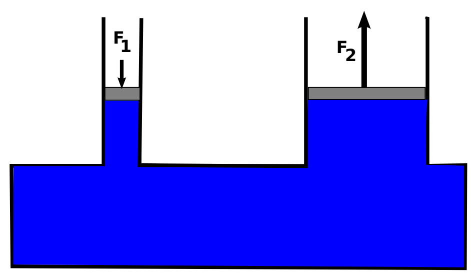


Figure 13.9:

You apply a force F_1 on the left piston, and this will then apply a larger force F_2 on the right. The arrows in figure denote direction of motion. You push down on the piston on the left, and it will move the piston on the right up. Because, for simplicity the two pistons are at the same heght, then the pressure on them must be the same. Denoting the two piston areas as A_1 and $A2$, we have $p_1 = F_1/A_1$ and $p_2 = F_2/A_2$. But $p_1 = p_2$. Solving fore F_2 we have

$$F_2 = \frac{A_2}{A_1} F_1 \tag{13.12}$$

So if $A_2 > A_1$, then $F_2 > F_1$. In this case, the force you apply F_1 is amplified.

But this seems like it's getting something for nothing. How can you amplify the force? Doesn't this violate conservation of energy. This is exactly the same situation as you get with a pulley, another contraption that can amplify forces. In both cases, the work you perform in moving piston 1 down is the same as the work moving piston 2 up. This is because although the force is higher, the distance it moves is less. This is because the fluid is incompressible. So if you move the first piston down by d_1, the volume change due to this motion is $d_1 A_1$.

This loss in volume must be compensated by a gain of volume of $d_2 A_2$ for the second piston. So $d_2 = (A_1/A_2)d_1$. So the work done by the second piston is

$$W_2 = F_2 d_2 = ((A_2/A_1)F_1)(A_1/A_2)d_2 = F_1 d_1 = W_1 \tag{13.13}$$

So the work done by the second piston is equal to the work supplied by the first one.

13.4 Fluid flow

The above discussions focused on fluid that had now currents. But most of the time fluids do have currents. Let's start by considering the case where the flow is in steady state, and *laminar*. Laminar flow can be understood as follows. If you inject tiny particles, or dye into one point in the flow, you'll see that all particles will move the same way. This is different than what you'd get if the fluid was turbulent. There the flow is constantly shifting around, so that each particle that you inject will follow a unique trajectory. Turbulent flow is quite common, but hard to understand, so we'll only consider laminar flow here. Also laminar flow is very important in biology and has a lot of interesting properties.

As discussed in the beginning, fluids are made up of zillions of atoms, and although they obey well established physical laws, they end up dancing around in an extremely complicated way. To understand what is happening on a macroscopic scale, we'll use conservation laws.

13.4.1 Conservation of mass

What goes in, must come out. To simplify the discussion, we will assume that the density of the fluid remains the same. Let's try to understand what happens when water flows into a pipe, and then the diameter of this pipe shrinks, as shown in Fig. 13.10. Here fluid enters the pipe with a velocity of v_1. The cross-

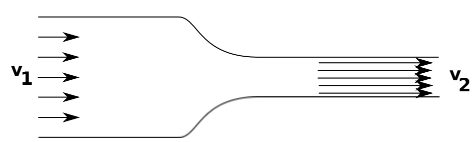


Figure 13.10:

sectional area of the pipe at that point is A_1. Then it exits at some velocity v_2 where the cross-sectional area is A_2. Here not only are we assuming laminar flow, but that the flow is the same at all point in these two cross section. That is actually a really really bad assumption for a real pipe, but later on, we'll see that this pipe is just another one of these silly mathematical constructions us physicists are so found of making.

So now we're going to use the fact that the liquid in the pipe *conserves mass*. That is, atoms don't magically appear or disappear out of thin air. This means that if we wait a time Δt, what's the volume of liquid ΔV_1, that passes into the pipe? In this time, a particle on the left, will have traveled a distance $\Delta x_1 = v_1 \Delta t$. So this says that $\Delta V_1 = A_1 \Delta x_1 = A_1 v_1 \Delta t$.

On the other side in the same time Δt, an equivalent volume must emerge since the density is constant and particles in steady state can't magically disappear. So $\Delta V_1 = \Delta V_2$ which means that

$$A_1 v_1 \Delta t = A_2 v_2 \Delta t \tag{13.14}$$

or

$$A_1 v_1 = A_2 v_2 \tag{13.15}$$

Another way of seeing this is to note that from the above discussion, the volume of water that flows per unit time is

$$I = A_1 v_1. \tag{13.16}$$

The flow rate must be constant independent of location. Therefore $A_1 v_1 = A_2 v_2$

This tells you that as the pipe narrows, the flow must speed up to compensate. You might have noticed this watching rivers and streams.

Example

When you turn on a cylindrical tap of radius r_0, suppose water flows out at a rate of $I = 1.0 cm^3/s$. What is the radius of the stream of water as a function of the distance from the tap?

Solution

The flow as a function of position must be constant, that is $I = Av$ doesn't change with height z. But the so $A = I/v$. What is v as a function of h? Going back to one dimensional kinetics, you might recall (or using conservation of energy) that

$$v^2 = v_0^2 + 2gh \tag{13.17}$$

and $v_0 = I/A_0 = I/(\pi r_0^2)$. So

$$A = \frac{I}{\sqrt{v_0^2 + 2gh}} = \frac{I}{\sqrt{v_0^2 + 2gh}} = \frac{I}{\sqrt{(\frac{I}{\pi r_0^2})^2 + 2gh}} \tag{13.18}$$

but $A = \pi r^2$, so finally

$$r == \frac{1/\sqrt{\pi}}{((\frac{1}{\pi r_0^2})^2 + 2gh/I^2)^{\frac{1}{4}}} \tag{13.19}$$

This looks kind of yucky. But you can see that for large h the radius shrinks proportional to $h^{-1/4}$.

13.4.2 Bernoulli's equation

When we derived the equation of continuity, in Sec. 13.4.1 by looking at the flow in a pipe of varying cross section, I mentioned that the pipe wasn't a real pipe but another one of these physics constructions that us physicists are found of making. To understand this better, look at Fig. 13.11. This figure represents

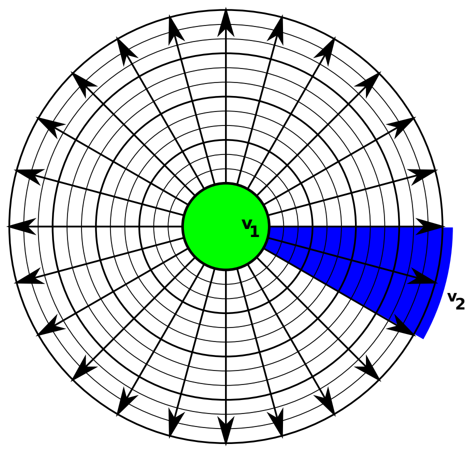


Figure 13.11:

the flow of water out of a hose pipe in the middle of a swimming pool (assuming the flow is laminar). It's circularly symmetric, and in steady state. The green region is the mouth of the pipe itself. The arrows represent the direction of flow, and are often used to denote the "velocity field". The velocity exiting the pipe is v_1 and the velocity will diminish as we move away from the center. The blue region is some region between the arrowed lines. We can think of the boundaries of this blue region as forming a pipe.

So we can apply the equation of continuity, eqn. 13.15 to this situation. Remember that since we don't have a real pipe, it's perfectly reasonable that the velocity on the sides of the pipe is the same as in the middle.

But aside from the equation of continuity, we have something else that we can use, assuming that energy in the flow is conserved. Now that's not always a good assumption. Fluids dissipate energy creating heat all the time, but there are some situations, where that dissipation can be ignored. In that case we can use conservation of energy, or more easily, by the work-energy theorem, which says that the work $W = \Delta K$, the right hand side being the change in kinetic energy.

If we, using the same variables as in Sec. 13.4.1, consider a short time interval Δt, fluid will pass into the blue pipe and move a distance $\Delta x = v_1 \Delta t$. The work done is $W_1 = F\Delta x$. But the force $F = p_1 A_1$, so $W_1 = p_1 A_1 \Delta x = p_1 \Delta V$. This is the work done on the system in injecting fluid into the swimming pool. At the other end of the blue region, fluid comes out, and likewise, work is done by the system, giving a contribution of $W_2 = -p_2\Delta V$. So altogether

$$W = (p_1 - p_2)\Delta V \tag{13.20}$$

The kinetic energy of a region of fluid of volume ΔV, so the kinetic energy of it is $K = \frac{1}{2}mv^2 = \frac{1}{2}\rho\Delta V v^2$. Using the work energy theorem yields

$$\Delta K = \frac{1}{2}\rho\Delta V(v_1^2 - v_2^2) = W = (p_1 - p_2)\Delta V \tag{13.21}$$

But we can easily add the effects of gravity while we're at it since the fluid may change its vertical height giving $W_G = mg(z_2 - z_1) = \Delta V\rho g(z_2 - z_1)$. Adding this to the work energy theorem, gives

$$p_1 + \rho g z_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho g z_2 + \frac{1}{2}\rho v_2^2 \tag{13.22}$$

So you can think of $p + \rho g z + \frac{1}{2}\rho v^2$ as a conserved quantity like the total mechanical energy. It doesn't change as the fluid flows. Going back to Fig. 13.11, we can now see how the pressure will vary will position. Forgetting again about gravity, we see that because the velocity is decreasing as we go away from the center, the pressure must increase to compensate. The circles in that figure represent lines of constant pressure.

13.5 Viscosity

Bernoulli's equation applies to situations where there is no dissipation, so-called "inviscid" flow. But in reality there are many situations where there is a lot of dissipation. Dissipation in liquids is what gives rise to drag on objects moving in them. As you drag an object through a liquid, it resists motion: the larger the velocity, the larger the resistive force. This resistance is a property of the particular fluid being considered, and is characterized by a number called the "viscosity".

To define viscosity, consider the experimental situation shown in Fig. 13.12. A liquid (blue) is in between two plates (orange), separated by a distance of

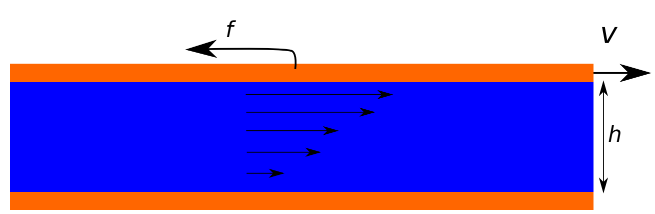


Figure 13.12:

h. The upper plate is moved *sufficiently slowly*, parallel to the lower plate at a speed v. There is a force exerted on the upper plate by the liquid of f that will be proportional to v, i.e. $f \propto v$. Why is there such a force?

The velocity of the liquid right next to a plate will be the velocity of the plate. So velocity next to the bottom plate is zero, and the velocity next to the top plate is v. In between, the velocity will interpolate between those two values as shown by the arrows in the liquid. The force f on the upper plate will be zero if both plates are stationary because nothing is moving. But the force will also be zero if both plates are moving with the same velocity. Why? Because if you change to a moving reference frame, the plates look stationary again. So if the velocity in the fluid is constant, independent of position, then $f = 0$. It's the variation of f with position that gives you the force. The faster the variation, the larger the force. So this suggests that $f \propto 1/h$. Also the total force should be proportional to the area of the plates A. Together with $f \propto v$, we should be able to write:

$$\frac{f}{A} = \eta \frac{v}{h} \tag{13.23}$$

where this proportionality constant η is called the *viscosity*.

The viscosity depends greatly on the liquid we're considering. For water at atmospheric pressure and at $20°C$ the viscosity is about $10^{-3} Pa\ s$. For a gas like helium, or air, at the same pressure, it's about $2 \times 10^{-5} Pa\ s$. On the other hand, molasses is around $5 Pa\ s$. Viscosity plays a big role in how a fluid behaves. High viscosity fluids are well ... viscous. Not exactly the most profound statement ever made.

13.5.1 Turbulence

We talked about laminar flow when discussing Bernoulli's equation, but in reality a lot of the time flow is not laminar, but turbulent. It used to be easy to see turbulence around you, because people smoked like chimneys. The smoke would swirl around all over the place, revealing that the disgusting smelly air in the room was actually highly turbulent. If it was laminar, you would be able to position yourself to safely sit at arm's length and the smoke would flow smoothly flow under the door. But no, the smoke swirls around, forming eddies, and if you look at those eddies, they frequently break up forming even smaller eddies. Before you know it, the whole room is smoky.

The same is true for the ocean. When you watch waves crashing on the shore, this is also highly turbulent flow. On the other hand, for small enough scales, say inside a biological cell, the flow *is* laminar. At small scales, viscous effects dominate. These tend to suppress quick changes in velocity as a function of position. At large scales however, these won't be as important and inertial effects will dominate. These allow for a fluid to swirl around a lot, creating turbulence.

Figuring out exactly when the flow will be laminar and when it will be turbulent is very hard, but you can get a back of the envelope idea of when this will occur based on the "Reynolds number" (invented by Stokes). This is a dimensionless number that tells us of the relative importance of viscous and inertial effects. It depends on a velocity scale, and a distance scale. Think again about the experiment in Fig. 13.12. If v and h are both large, you'd expect turbulence. When they're both small, you'd expect laminar flow as pictured. Call the distance scale L, and the velocity scale v.

Dimensionally, we can get an idea of inertial effects by thinking about the relative strengths of inertial and viscous forces. For inertial forces, think of the fluid as rotating with a radius L at a speed v. Forget about factors of 2, this is just giving you the right quantity to consider. In that case we know that the acceleration of this fluid will scale like v^2/L. Its mass $M \propto \rho L^3$. So the force per area $A = L^2$ is

$$\frac{F_{inertial}}{A} = \frac{Mv^2/L}{L^2} = \frac{\rho L^3 v^2/L}{L^2} = \rho v^2 \tag{13.24}$$

This makes sense because the Bernoulli equation eqn. 13.22 involved a pressure, which has units of force per area, and also had a term ρv^2.

To get the magnitude of the contribution from viscous forces, we use the definition of the viscosity eqn. 13.23

$$\frac{F_{viscous}}{A} = \eta \frac{v}{L} \tag{13.25}$$

The Reynolds number is the relative importance of these two forces $R = F_{inertial}/F_{viscous}$ which from the above two equations is

$$\text{Re} = \frac{\rho v L}{\eta} \tag{13.26}$$

which is dimensionless.

As an example, think about the flow past a sphere. Far away from the sphere, the we'll call the velocity v. The radius of the sphere will play the role of L. Then if $\text{Re} > 10$, the flow is no longer laminar. Swirly flows will start appearing around the sphere.

Let's look at what the Reynolds number is typically in cell biology. The size of a cell is of order $10\mu = 10^{-5}m$. The velocity of motion varies, but a fast moving cell would be sperm, which goes at about $10^{-4}m/s$. The viscosity is seldom less than that of water $\eta \approx 10^{-3} Pa\ s$, and the density is pretty similar to that of water $\rho = 1gm/cm^3 = 10^3 kg/m^3$. Plugging this into eqn.

$$\text{Re} = \frac{\rho v L}{\eta} = \frac{10^3 \times 10^{-4} \times 10^{-5}}{10^{-3}} = 10^{-3} \tag{13.27}$$

This says that inertial forces are unimportant, and by far the largest factor contributing to the flow are the effects of viscosity. You don't see turbulent or non-laminar flow at the scale of the cell.

13.5.2 Drag on a sphere

Now consider a small sphere (we need low Reynolds number for this to work) in a liquid moving past it with at some velocity. Let's say that the velocity far from the sphere is uniform and has a value of v. What's the force on the sphere? Stokes figured this out and the math is a bit complicated. But he found that the drag force F_d is

$$\mathbf{F} = 6\pi\eta r\mathbf{v} \tag{13.28}$$

Here r is the radius of the sphere. You could change reference frames, and have the fluid stationary far away, and have the sphere moving at some velocity $\mathbf{v}$. You'd get the same result, except now the force would be opposite the direction of motion of the sphere. In this case the formula says that the force is proportional to the velocity, which is good, because this agrees with what we learned before, that in a liquid, at small enough velocities, $\mathbf{F} = -b\mathbf{v}$ where the drag coefficient b is

$$b = 6\pi\eta r. \tag{13.29}$$

This tells us that the drag coefficient is proportional to the radius of the sphere. You might think this is a bit odd. Sure the drag should increase with the radius, but you'd think it should be the surface area that would come into this equation, which would give a factor of r^2, not r. But no, at small Reynolds numbers the linear dimension is what comes in.

The fact that it depends linearly may make you wonder what the drag on other shapes would look like, say a long cylinder. If it's not the surface are that comes in, what is it? To a first approximation it's the same answer as you get for the sphere, where the length of the cylinder replaces the diameter of the sphere! There are some additional factors of the cylinder's radius that come in, but they do inside a logarithm meaning that the answer depends only weakly on the cylinder's radius. This counterintuitive fact has many consequences for understanding biology at small scales.

Simple viscometer

Suppose you want to measure the viscosity η of a liquid but don't have tons of money to do this. How can you get η easily? Take a sphere of known density and drop it in the liquid. It can be lighter or denser than the liquid. It works both ways. We found out earlier in the chapter, how to calculate the force including buoyancy, eqn. 13.10. As the sphere drops (or rises), it will come to steady state, moving a constant velocity. You can get this velocity by equating the drag, eqn. 13.28, with this gravitational force:

$$F_{net} = (\rho_W - \rho_O)Vg = F_d = 6\pi\eta r v \tag{13.30}$$

But the volume $V = \frac{4}{3}\pi r^3$. So we can solve for η

$$\eta = \frac{2}{9}\frac{|\rho_W - \rho_0|r^2 g}{v} \tag{13.31}$$

where ρ_W is the density of water and ρ_0 is the density of the sphere.

13.5.3 Example

Kinesin is a microscopic biological motor that walks along a filament called a *microtubule*. It is often used to transport payloads inside of a cell that are attached to it. The average velocity of kinesin without a load is about $0.78\mu m/s$. Kinesin stops being able to move forward when the force on it is greater than the *stall force*. For kinesin with reasonable conditions, that force is about $5pN$ (pico-Newtons). How large of a spherical payload can the kinesin carry without stalling assuming it goes at its average speed? Take the cytoplasm to have 8 times the viscosity of water.

13.5.4 Solution

The drag force from eqn. 13.28 can be equated to the stall force F_s:

$$6\pi\eta r v = F_s \tag{13.32}$$

giving

$$r = \frac{F_s}{6\pi\eta v} = \frac{5 \times 10^{-12} pN}{6\pi 8 \times 10^{-3} Pas \times 7.8 \times 10^{-7} m/s} = 4.25 \times 10^{-5} m \tag{13.33}$$

This is $42.5\mu m$. This is larger than the typical size of many cells (about $10\mu m$). So kinesin motor molecules have no difficulty carrying any size of cargo in a cell.

Chapter 14

Temperature and Heat

In this chapter we'll try to understand the properties of systems in the world around us in a different way that we've done previously. Just as with liquids, we'll be concerned with a collection of a large number of molecules and what we can say about them, despite the fact that at a microscopic level, they're shooting about and jiggling in an incredibly complicated way. But now we'll try to understand specifically, the properties of this complicated jiggling, often called "thermal motion". We'll try to understand the concept of *temperature* and how this is closely related to another mysterious but very important quantity, called the *entropy*

14.1 Conservation of energy

If you have a bunch of molecules in a liquid or a gas in a well insulated container, you'll see a picture of them shooting around like you showed before (Fig. 14.1). With such complicated motion, how can you say anything at all? The answer

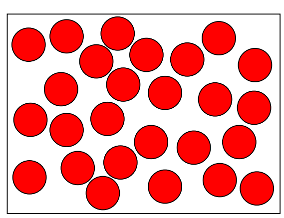


Figure 14.1:

is two fold.

1. First you can use conservation of energy to help out: no matter how complicated the configuration, the energy does not change.

2. The motion is so complicated that eventually the system will get into pretty much every configuration with the correct total energy

Let me stress here, that we're talking about an isolated system, so that it can't exchange energy with anything else. If that's not the case, all is not lost, but it's easier to understand my argument when you make sure the system is isolated. The second point, that the system can get into pretty much any configuration is a very important idea. This is a non-precise description of "ergodicity". We're accepting that the motion is so fiendishly complicated that we have no hope of solving for the trajectories of the particles, so much so that the system becomes "ergodic", that is, it can get into any configuration that has the right total energy.

These two statements might not appear to help you much. Suppose you want to know how the particles all move in time, as we did for projectile motion. Well then you're out of luck. But why would you want to know that anyway, since you couldn't even write down the coordinates of 10^{23} molecules to begin with? What kind of questions could you sensibly ask. Here are a few:

1. What is the average of the speed of a single molecule in a liquid over time?

2. What is the distance between the ends of a DNA molecule, averaged over time?

3. What is the pressure on the walls of a container of water, again averaged over time.

These are all questions about the *time average* of quantities of interest.

We won't be able to explain the theory behind how you can get all these things, but it's worth giving you some idea of how you'd get answers to these questions. Because when you average over time, and here I'm being a theoretical and saying you can average over a very very long time, you don't have to know how the different molecules move in time. You just have to know what configurations they will get into, and *average over those configurations instead*. So you don't need to solve Newton's laws, just use the fact that you know all of the configurations that you need to average over. These are the ones with the right total energy. This is a huge simplification. Let's investigate a classic example of this.

14.1.1 The ideal gas

Understanding and "ideal gas" is actually pretty subtle. We have a container at constant energy, containing N molecules. The gas is so dilute that we can effectively ignore any interactions between the molecules. Actually you can't really do that or you get total nonsense. What you need to say is that the interactions are there but very infrequent, so as far as the energy is concerned, you can ignore configurations where the molecules get close to each other. But in order for the system to be ergodic, you need them to infrequently exchange energy. Without that, the energy of every molecule is separately conserved and you don't want that because of course, that's not physically sensible, a helium atom's kinetic energy isn't the same for all time. So we'll ignore any potential energy and just consider the kinetic energy of these molecules. For the moment, we'll ignore any rotation kinetic energy. Then the total energy E is

$$E = K_1 + K_2 + \cdots + K_N = \frac{m}{2}(v_1^2 + v_2^2 + \cdots + v_N^2) \tag{14.1}$$

Ergodicity means that all configurations with constant energy are equally likely. So to find the time average of v_1^2, we just need to average over all allowed configurations, those defined by the above equation that can be rewritten

$$v_1^2 + v_2^2 + \cdots + v_N^2 = \frac{2E}{m} \equiv R^2 \tag{14.2}$$

Here I defined $R^2 = 2E/m$. The reason I've introduced the symbol R is to make you think of a radius. You know that $x^2 + y^2 = R^2$ describes points that make up a circle. $x^2 + y^2 + z^2 = R^2$ describe the points that make up a sphere. So eqn. 14.2 describe points that make up a hypersphere in N dimensions!

Wow, this does sound a bit esoteric. The point I'm making is that intuitively you can understand the set of allowed points for the molecules the same way you understand a sphere, except now you're not in 3 dimensions by 10^{23}. But a lot of things are the same as for a 3d sphere. We can talk about averages the same way. The average value of x by symmetry is zero. But suppose you want to find the average value of x^2, call it $\langle x^2 \rangle$ averaging x over all points on the sphere, you can get this pretty easily by symmetry, without needing to do any multiple integration. You know that

$$\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle. \tag{14.3}$$

You also know that

$$\langle x^2 + y^2 + z^2 \rangle = \langle R^2 \rangle = R^2 \tag{14.4}$$

Which means that

$$\langle x^2 + y^2 + z^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle = 3\langle x^2 \rangle = R^2. \tag{14.5}$$

So this says that $\langle x^2 \rangle = R^2/3$.

Now going back to the N dimensional problem, you can see that exactly the same method will work for that case and that for any of the molecules so that for any one of them

$$\langle v^2 \rangle = \frac{R^2}{N} = \frac{2E}{m}. \tag{14.6}$$

You could also write this as

$$\langle \frac{m}{2} v^2 \rangle = \frac{E}{N}. \tag{14.7}$$

This says that the energy is evenly distributed among all N particles. Seems pretty sensible if you think about it.

14.1.2 Stat Mech vs Thermo

The last section was an example of a statistical mechanics calculation. If you continue to study physics, you'll see that you can study quite complicated systems this way, and understand for example, how it is that liquids can become gasses, how permanent magnets can form, why rubber is elastic. These are all examples of things you can do with *statistical mechanics*. But because you don't have two years of physics under your belt, we won't be able to do that here.

On the other hand, we can think about systems in a "top-down" way. Instead of thinking of them being made out of atoms all interacting in well defined ways, we can instead try to understand what can be said about system knowing empirical properties of it, such as what happens when you compress it, add energy to it, etc. We can come up with a few very simple postulates (such as conservation of energy) to allow us to figure out a lot of things that aren't at all obvious. This is known as *thermodynamics*. Stat mech and thermo used to be taught as two separate subjects but really thermo in principle (and largely in practice) is derivable from the underlying microscopic laws. Thermodynamics is a very powerful way looking at systems and we'll see that you can derive some pretty amazing results with it.

14.2 Thermodynamics

One of the most important concept in thermodynamics is that of *temperature*. Let me stress, that you don't need to postulate the existence of it, it can be derived from statistical mechanics, but we're not going to do that here.

We're all familiar with hot, and cold, and the concept of temperature. It's built into our senses. One thing that you'll notice is that if two objects are at different temperatures, like chocolate chip cookies just taken out of the oven, the two objects, cookies and table, will eventually come to the same temperature. This might take some time, but this will happen, so make sure not to wait too long! If you have three objects, A, B, and C, then if A and C are in thermal equilibrium and B and C are as well, then A and B will also be in thermal equilibrium.

14.2.1 Temperature Scales

You probably already learned this but it's pretty easy to convert between Fahrenheit T_F and Centigrade T_C. $T_F = 32 + 1.8T_C$. Kind of arbitrary, and not really worth getting all worked up about this. I know a lot of people that think people in the US are idiots because they use Fahrenheit to measure temperature.

However us scientists use the Kelvin temperature scale. An object at zero degrees Kelvin can't get any cooler. As it turns out the temperature for ice melting at atmospheric pressure, is about $273°K$. So instead of saying "It's a great day out! It's $20°C$ or $68°F$, us scientists say, "It's $293°K$". No wonder no one understands us. Of course we know that we're simply The Best, because we use the most sensible temperature scale. Actually, there really isn't any best scale and depending on the problem you study, you may want to use a completely different scale.

14.2.2 Volume expansion

If you heat up a solid, it will change its size. Most of the time it'll get bigger (but not always). If the change in temperature is small, we expect that the change in volume ΔV, will be proportional to both the total initial volume V, and the change in temperature ΔT

$$\Delta V = \beta V \Delta T \tag{14.8}$$

where β is the coefficient of volume expansion for the material.

There are many other properties of a material that depend on temperature, for example, the pressure. From the point of view of thermodynamics we can take many of these as empirical facts, but now we'll talk about the laws of thermodynamics that allow you to deduce interesting relationships between them.

14.3 Thermodynamic States

How do you characterize the state of a glass of water? On one hand we can think of it microscopically, and then we'd need to specify all of the positions and velocities of the atoms that make up the water. Not only is that impossible to do, but it misses the point that most properties of the water, its taste, its temperature, its viscosity, you don't need all that information. What information do you need? Basically two things: its temperature and pressure. If you know that, you'll have an extremely good idea about all of its other properties. But you could add in the magnetic field of the earth as well. It'll make a small difference to the properties you measure, but for some substances, it'll make a big difference. That's fine, we'd need three parameters, not 10^{23} and these parameters in a practical sense define the state of the system.

If you take your glass of water and you heat it up, then cool it back down to its original temperature, it'll be indistinguishable from its original state. If you put it in a plastic bag, and squeeze it, then release it, it'll also return to its original state. In all these cases the history of the water is unimportant. There are systems, like magnets, where history is important (unless you wait for an extremely long time), but in the following, we're not considering such materials.

The classic example that is the easiest to understand is a gas in a cylinder with a piston on top. Unlike the glass of water, you don't have to worry about evaporation. We can easily control the volume by moving the piston. We can raise it's temperature. There are three parameters that appear to characterize the state, the pressure, p, volume V, and temperature T. And you can think of others. An important one is the total internal energy E of the gas. But if you take a gas and cool it, the pressure will also decrease. So these four parameters are not independent. You only need two, say P, V. Or V, T will do.

So will P, E, etc. Graphically we can think about what happens if you slowly change a system from one state, A, to another, B, and back again to the original state. This is shown in Fig. 14.2. The initial state is characterized by point A,

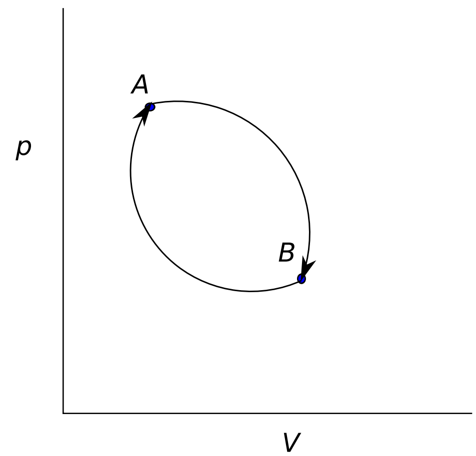


Figure 14.2:

which corresponds to a pressure value of P_A and volume value of V_A. Then by changing external conditions, like the volume, and the temperature, the system will slowly change, following the path as shown until it ends up at state B. You can now follow a *different* path, and end up at state A. These types of diagrams are crucial in understanding thermodynamics.

14.4 The First Law of Thermodynamics

When you take a cookie out of the oven and put it on the counter, the cookie cools down. We can say that *heat* is flowing between the cookie and the rest of the environment (the counter and the air). We often denote heat by the letter Q. Heat has the units of energy and is transfered between two objects if there is a temperature difference between them. Just as with temperature, people often use a different unit to measure heat, the calorie.

$$1\ cal = 4.184J \tag{14.9}$$

This shouldn't be confused with the calories we worry about when we eat too many twinkies which are "Calories". The ominous capitalization is because $1Calorie = 1000cal$, which quite a lot of energy!

But nevertheless, heat is a form of energy. It's microscopic thermal energy: all those molecules crashing into each other and jiggling around like crazy. Without understanding the exact motion that they execute, we can use conservation of energy to say something useful.

Imagine you have a gas, similar to the one we considered Sec. 14.1.1. The gas is in a container with a piston at the top maintaining some pressure, see Fig. 14.3. The side of the container can be places next to another object at a slightly different temperature, shown on the left in light blue.

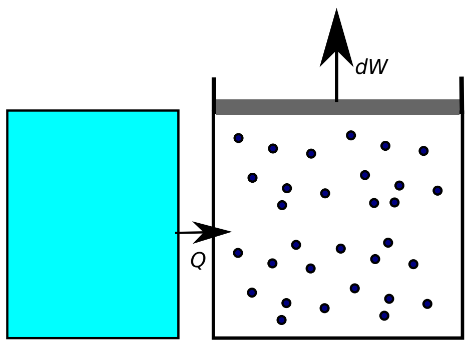


Figure 14.3:

If the gas is brought in contact with the blue container, suppose a small amount of heat δQ flows into the gas from the blue container. This means that the total energy *internal energy*, E, of the gas must increase by an amount dQ.

There's another thing you can do. You can lift up the piston by some small amount meaning that the gas is doing work δW. Putting this together, we have that

$$dE = \delta Q - \delta W \tag{14.10}$$

This is just a restatement of conservation of energy in terms of heat and work.

To understand this better, lets consider the two terms on the right hand side.

14.4.1 Heat

If you add a small amount of heat Q to a system, say a gas, then the temperature of the system will increase by some amount ΔT. There's a relationship between these two quantities

$$Q = C\Delta T. \tag{14.11}$$

C is called the "heat capacity" and that depends completely on the details of the system. Even for gas, you get a different result depending on whether it's being held in a fixed volume, or under a constant pressure. The heat capacity also changes a huge amount with temperature.

It tells you how much the temperature rises when you start off with a system with total energy E, and add in a bit of heat to it dE, that is

$$C = \frac{dE}{dT} \tag{14.12}$$

One thing about almost all heat capacities is that they're *extensive*. That means that you expect that the heat capacity will be proportional to how many particles there are in the system. You can therefore write

$$C = cM \tag{14.13}$$

where M is the mass of the system and c is known as the "specific heat". For example, for water at room temperature $c = 1cal/(g\ K)$.

For substances undergoing phase changes, say from solid to liquid, things are different. Say you have a bunch of ice that you're trying to melt. You can add in a substantial amount of heat and all you do is melt some of the ice, but the temperature doesn't change. It'll stay at $0°C$. At this special melting temperature, we don't have a linear relationship between heat and temperature. If you have a gram of ice, you need to add a certain amount of heat to it to completely melt it. It's only after you've added that heat in, that the temperature will then increase when more heat is added. The same behavior exists between other phases as well, such as from liquid to gas phases. So we can define a "heat of transformation", L, to transform from one phase to another. Because again we expect this quantity to be extensive (i.e. proportional to mass), we have that amount of heat Q that must be added is

$$Q = Lm \tag{14.14}$$

For example, for melting ice, $L = 3.33 \times 10^5 J/kg$. The heat of transformation from liquid water to vapor (i.e. when you boil water) is $L = 2.256 \times 10^6 J/kg$.

14.4.2 Work

Consider a container of gas like we did above, and now consider in more detail, what's the work that you do when you lift the piston up and let it do work against a force F, as shown in Fig. 14.4. The work $\delta W = Fdz$. But the force is related to the pressure p and the area of the piston A, $F = pA$. And the change in volume of the container $dV = Adz$. Putting this together, we see that the work done by our system is

$$\delta W = pdV \tag{14.15}$$

Now we ask what happens if we move the piston very slowly, by a sizeable amount, so the volume goes from V_1 to V_2. But as we lift the piston and change the volume, the pressure in general, will also change, because the pressure is a function of volume, $p(V)$. In that case we can write the work as

$$W = \int dW = \int_{V_1}^{V_2} p(V)dV. \tag{14.16}$$

So the work is the integral of the pressure vs volume curve for the initial to the final volume. This is illustrated in Fig. 14.5. The pressure is some function of the volume. The exact shape will depend on the path that the experimenters choose to take, assuming they have the knobs to do it. There are only two

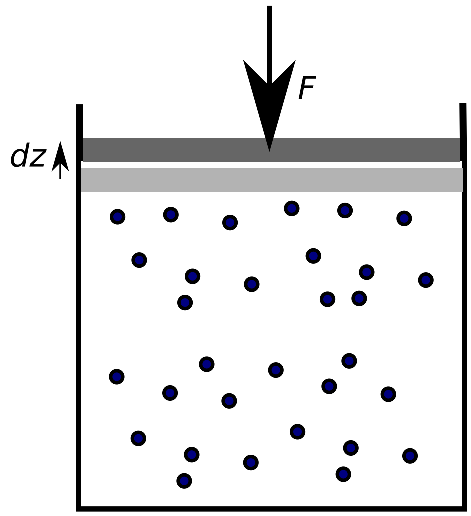


Figure 14.4:

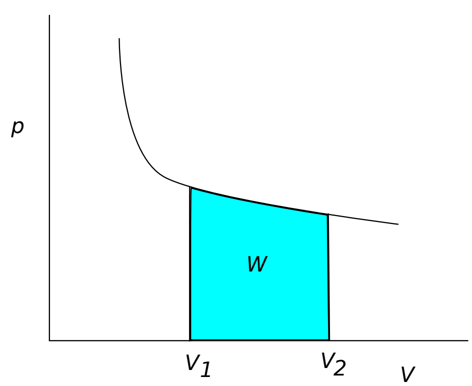


Figure 14.5:

knobs you really need, one that adjusts volume, and the other that adjusts temperature. As discusses earlier in Sec. 14.3, we can take multiple paths to get between any two states. So suppose we cycle the system by going from a state A to state B through one path, and B to A through another path, as shown in Fig. 14.6. How much work in total is done by the system after one cycle? The work going from A to B is the area under the upper curve (blue plus green regions). The work going from B to A is the *negative* of the area under the lower curve (the green region). It's negative because we're going backwards, from larger to smaller volumes. Subtracting these two area gives the blue area in the middle. That area is the total work done in one cycle.

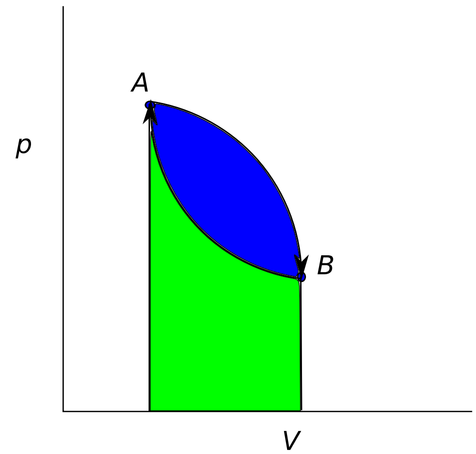


Figure 14.6:

How is it that we can get work after one cycle? Isn't that getting something for nothing? The answer is that to take these two paths, we've got to add and subtract heat from the system. Some of this net heat is being turned into work. Is all heat turned into work? As we'll see later in this chapter, the answer is no. Thermodynamics puts a rigorous bound on how much heat energy can be turned into work.

14.4.3 Is heat a parameter?

You can change a systems volume, or temperature, or pressure. You can cycle around the $P - V$ plane and come back to a system with the same P and V, and therefore T. These parameters all characterize the thermodynamic state of the system. But we've also talked about heat. Does the system contain a certain amount of heat? In other words, if you boil an egg, does that egg now contain a heat value of $30,000J$?

Initial attempts at thermodynamics the "caloric theory of heat" did ascribe a heat content to objects. There was suppose to be some special fluid that would run from cold to hot bodies causing them to heat up. This would mean that heat was conserved. But then Count Rumford started thinking about all the heat that was generated from boring cannons. He saw that you could change mechanical energy into heat, so heat was no conserved. So a glass of water on the table does not have a well defined heat that characterizes its state. It has a well defined temperature, volume, and internal energy. There doesn't seem to be a way of dividing up the internal energy into a heat and non-heat part.

But it might seem to you as if there should be something similar to heat that tells you about the state of a system. There is, and it's called the *entropy*.

14.5 Entropy

Let's start off by revisiting the work that you do in a cyclic process as shown in Fig. 14.6. Work is clearly not a state of the system either. Because after going around a loop in the P-V plane the work done is not zero. But suppose we wanted to make something related to work into a state of the system. We know that $\delta W = PdV$, so $dV = \delta W/P$. What happens to $\delta W/P$ when we go around a loop? Well the integral of dV around a loop is zero. First the volume gets bigger, then it gets smaller, returning to the same size. So we've figured out a way to make δW into something that involves a change in the system state: you just divide it by the pressure.

Similarly if we think about going around this loop and monitoring δQ, as Count Rumford showed, the change in Q is not zero. The mathematical way to fix this up is not to consider δQ but $\delta Q/T$, where T is some function that has been designed to make sure that we get zero when we go around the loop. As it turns out this is also a *definition of the temperature*! Something that we haven't considered until now, despite having talked about it at some length. We'll call this quantity the infinitessimal change in entropy, dS

$$dS = \delta Q/T \tag{14.17}$$

When we fo between two states, A and B, as shown in Fig. 14.6, we can compute the change in the entropy S by adding up the infinitessimal changes dS as we go around:

$$S_B - S_A = \int_A^B dS = \int_A^B \delta Q/T \tag{14.18}$$

We can now rewrite the first law of thermodynamics eqn. 14.10

$$dE = TdS - pdV \tag{14.19}$$

14.5.1 How do you calculate the entropy?

Ludwig Boltzmann first figure out how to calculate the entropy from the microscopic dynamics of atoms. A lot of people thought he was talking nonsense, but a lot of them, for example Ernst Mach, didn't believe that atoms existed. But Boltzmann's formula has stood the test of time though it's a little hard to explain in an introductory course.

Let's go back to the ideal gas of Sec. 14.1.1 Remember we said that allowed configurations were ones of constant energy. This constraint meant that the velocities must be on the surface of a N-dimensional sphere as shown in eqn. 14.2. We also have the constraint that all molecules must be in the box. Phase space is this N dimensional world where we describe the particles by their positions and velocities: $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$. We want to calculate the volume allowed in this high dimensional space. For the positional part we have the volume is V^N. For the velocities it's proportional roughly to R^N

(remeber that.$R^2 = 2E/m$). The total volume Ω multiplies these two volumes together and the this volume depends on energy E, N, and V as

$$\Omega \propto V^N E^{3N/2} \tag{14.20}$$

We don't care about multiplicative constants like m. There is also a prefactor dependence on N but we'll see that this won't come into our discussion.

The entropy, according to Boltzmann is

$$S = k_B \ln \Omega \tag{14.21}$$

Where k_B is called "Boltzmann's constant" with a value of $k_B = 1.38 \times 10^{-23} m^2 kg s^{-2} K^{-1}$. This is a general formula, if Ω is the volume of accessible phase space, but mercifully, here we'll only consider the ideal gas phase space volume.

For an ideal gas, this

$$S = constant + N k_B (\ln V + \frac{3}{2} \ln E)] \tag{14.22}$$

It might seem strange that this weird formula is the entropy, and even though this wasn't easy to follow, we'll see that it does actually work!

Going back to the first law of thermodynamics, eqn. 14.19, let's first of all consider cases where we keep the volume constant, so $dV = 0$. Then we have $dE = TdS$. This says that

$$\frac{1}{T} = \frac{dS}{dE} \tag{14.23}$$

Differentiating eqn. 14.22, realizing that V isn't changing, we only have to deal with the second term in parentheses, giving

$$\frac{1}{T} = N k_B \frac{3}{2E} \tag{14.24}$$

or

$$E = \frac{3}{2} N k_B T \tag{14.25}$$

This gives the relationship between internal energy and temperature. The heat capacity at constant volume C_V is calculated by differentiating this according to eqn. 14.12:

$$C_V = \frac{dE}{dT} = \frac{3}{2} N k_B \tag{14.26}$$

This says that the heat capacity doesn't depend on the temperature. This is a well known fact for dilute gasses, that aren't too low in temperature where quantum effects play a role.

Now let's keep the energy E constant and vary the volume V. Again using the first law of thermodynamics, eqn. 14.19, we have $dE = 0 = TdS - pdV$ so

$$TdS = pdV \tag{14.27}$$

This says that with the energy kept constant:

$$\frac{p}{T} = \frac{dS}{dV} \tag{14.28}$$

Differentiating eqn. 14.22, realizing that E isn't changing, we have

$$\frac{p}{T} = k_B \frac{N}{V} \tag{14.29}$$

which is the ideal gas law $pV = Nk_BT$. This also is experimentally well verified for dilute gasses.

We see that if we know this mysterious quantity, the entropy we can calculate a lot of properties of the system. We can deduce how the total internal energy depends on temperature, and how the pressure depends on the volume and the temperature.

Unlike heat the entropy is a function only of the state of the system. When you slowly move a piston or add heat to an ideal gas, as shown in going from A to B in Fig. 14.6 the entropy will change. If you go back to your original state, along any path *sufficiently slowly*, your entropy will return to its intial value.

14.6 Reversible versus Irreversible processes

In the above figures, such Fig. 14.6, we thought about taking a system very slowly from one configuration to another. If for example, you take an ideal gas with a piston shown in Fig. 14.4, and slowly increase the height of the piston, what's the change in entropy? Here the gas and cylinder are isolated, so there's no flow of heat in or out of the gas. So $\delta Q = 0$, therefore $dS = \delta Q/T = 0$. So the entropy doesn't change $\Delta S = 0$. This is called an *adiabatic* process. However work is performed by the gas as it expands so what's its final temperature?

Since we know the form of the entropy from eqn. 14.22 then

$$S_1 = const. + Nk_B[\ln V_1 + \frac{3}{2}\ln E_1] = S_2 = const. + Nk_B[\ln V_2 + \frac{3}{2}\ln E_2] \tag{14.30}$$

Rearranging this we have

$$\ln(V_1/V_2) = \frac{3}{2}\ln(E_2/E_1) \tag{14.31}$$

or

$$E_2 = E_1(V_1/V_2)^{\frac{2}{3}} \tag{14.32}$$

And since from eqn. 14.25 we know that $E \propto T$, so

$$T_2 = T_1(V_1/V_2)^{\frac{2}{3}} \tag{14.33}$$

So if $V_2 > V_1$, $T_2 < T_1$: the system cools down. This makes sense since it's taking some of its internal energy and doing work as it expands out. This is called adiabatic expansion, and during this process, we see that the system cools down.

Now instead of doing this, suddenly move the piston out from V_1 to V_2. You don't need to do that if you're not keen on lightning fast movements of heavy and expensive experimental equipment. You could install a valve between two chambers, one filled with gas and one empty, and suddenly open the valve. The gas will flow rapidly into the second chamber. The physics of these two problems is much the same. So in this case, let's compute the entropy change.

Because no work is being done, the internal energy stays the same $E_1 = E_2$. You can also see from eqn. 14.25 that means the temperature stays the same. The system no longer cools down. We can again use the formula for the entropy, eqn. 14.22 to see that the change in entropy is

$$\Delta S = S_2 - S_2 = N k_B \ln V_2/V_1) \tag{14.34}$$

For this experiment to work, $V_2 > V_1$, which means that $\Delta S > 0$.

These two processes differ in a fundamental way because the first one was done slowly and the second quickly. The first one is a reversible process, in that you can reverse all the steps and end up in your initial state. The second process is not reversible. If you try to push the piston back quickly, molecules will be in your way, ad you certainly will not end up in your original state.

The fact that the entropy change of the gas was zero when we expanded slowly was convenient but not necessary. If we had connected the cylinder with gas to a "heat bath", that is a big system whose temperature was very well fixed, we could still slowly expand the gas. The gas would do work, but would not cool down. The overall change in entropy of the gas would be non-zero. But heat would flow in from the heat bath. If you consider this composite system, it is still isolated from the outside world so $\delta Q = 0$ for the entire system so that dS for the whole system is zero. The entropy will not change. You could easily reverse the process and end up in your initial state.

So what is important here is that your process is reversible when you change it slowly enough.

14.7 Second Law of Thermodynamics

The example with reversible versus irreversible processes can be generalized to lead to the mysterious and profound *second law of thermodynamics*. There are many equivalent ways of writing the second law, but we can think of as follows. If you have an isolated system, completely insulated from the outside world, and it is evovling from some initial to a final state. The change in entropy of this sytem $\Delta S \geq 0$. $\Delta S = 0$ only if the process is reversible.

What do you do if the system isn't isolated from the rest of the universe? Well then you'd say that the entropy of the universe will always increase (or remain the same). So if you smash a cup, that's an irreversible process. You've increased the entropy of the universe by smashing it. If you noticed the shards of glass spontaneously coming back together and reforming your grandma's precious china in a pristine state, then without an external agent of some kind helping to do this, you've witnessed a violation of the second law. Sorry, that'll never happen.

Many physicists believe that the arrow of time that we see in the microscopic world, is a consequence of the second law, because the microscopic equations (e.g. Newton's laws) are time symmetric. The increase in entropy that we see is related to the initial conditions of the universe. This is a fascinating topic, but one that probably should be postponed until you've taken more physics classes. Hopefully that'll happen.

14.7.1 Heat engines

Let's discuss how this second law helps us undestand engines better. Specifically engines powered by heating and cooling. Engines are cyclical, after one cycle, the engine should return to the state it started from. For a heat engine, it should take heat from a high temperature source, and use that to extract usable work. We'll make this high temperature source a heat bath. As we discussed above, a heat bath is so big that its temperature stays the same. Heat that flows into or out of it, does not affect its temperature.

The only thing extra thing that we need to consider is waste heat. As we'll see, not all the heat can be turned into work, and the remaining waste heat will be deposited in another heat bath, at some lower temperature. Call the higher heat bath temperature T_H, and the lower one T_L. This engine is shown schematically in Fig. 14.7.

The engine shown as the yellow circle in the middle, is periodically in contact with the upper heat bath, the lower one, or none at all. The engine itself can be very simple, for example just the simple gas with a piston shown in Fig. 14.3 that is sometimes in contact with a heat bath, like the one shown in light blue in that figure. The engine operates very slowly so that it is reversible. These are the four stages of the engine:

1. The engine at temperature T_H is put in contact with the upper bath at the same temperature, and expands out at constant temperature (i.e. *isothermally*) doing some work.

2. It is then disconnected from the bath T_H and continues to expand, thus doing work, as it cools. Eventually the gas lowers its temperature to T_L.

3. The engine is now put in contact with the lower bath at temperature T_L, and work is done on the engine (not by the engine) to contract it down in volume.

4. It is then disconnected from the bath T_L and continues to be compressed, now raising its temperature until finally it ends up back at a temperature of T_H.

This is called the "Carnot" cycle. The engine is doing work in its first two stages, and then having work done on it in the last two. Altogether however,

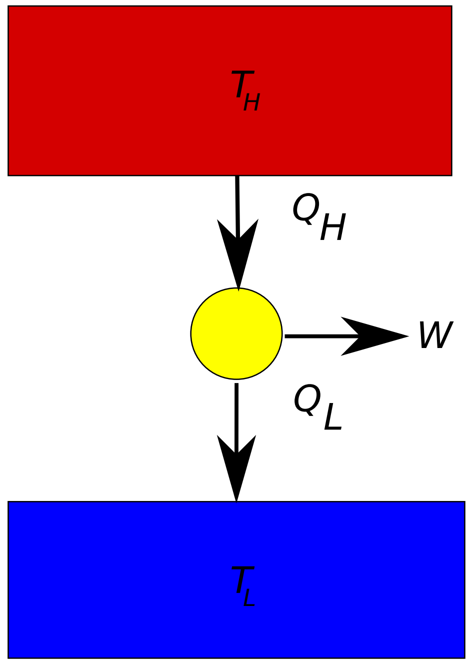


Figure 14.7:

it'll do a net amount of work. How much work will it do? Because its reversible, we can analyze the entropy change in all of these stages. The total change in entropy of the engine by itself (not including the heat baths) is zero because its initial and final states are the same. But what is the change in entropy of the heat baths?

In the first stage, the temperature is constant, and so since $dS = dQ/T$, the change in entropy for the upper part of the cycle will be $\Delta S_H = -Q_H/T_H$, Where Q_H is the heat extracted from the upper reservoir. Likewise in the third stage, the $\Delta S_L = Q_L/T_L$, where Q_L is the heat added to the lower reservoir. But conservation of energy tells us that the heat coming out of the top reservior must either become work, or waste heat, so

$$Q_H = W + Q_L \tag{14.35}$$

We can define the efficiency η as the ratio of the work done to the heat extracted from the upper heat bath

$$\eta = \frac{W}{Q_H} = 1 - \frac{Q_L}{Q_H} \tag{14.36}$$

The last equality comes from rearranging eqn. 14.35.

Seeing that this process is reversible, $\Delta S_L + \Delta S_H = 0$, which says that

$$\frac{Q_H}{T_H} = \frac{Q_L}{T_L} \tag{14.37}$$

implying that

$$\frac{Q_L}{Q_H} = \frac{T_L}{T_H} \tag{14.38}$$

Using this in our expression for η gives that the efficiency is

$$\eta = 1 - \frac{T_L}{T_H} \tag{14.39}$$

If we were to include irreversibility in the derivation, in that case, the total entropy change needs to be positive, and one can show that the efficiency must be less than this maximum Carnot efficiency.

Because the Carnot cycle is reversible, it can be run backwards. It can be shown that there is no way of beating this efficiency. Therefore we have a thermodynamic bound on how efficient you can make an engine! The bigger the ratio of the temperatures T_H and T_L, the more efficient the engine. But if these two temperatures are similar, the efficiency will be low.

Using these kinds of arguments, you can reformulate the second law multiple ways. One is that you can't take heat from a single heat bath, say the ground, and turn this into work. If you could do this, you'd have a great source of energy. It doesn't violate conservation of energy, but it does violate the second law of thermodynamics. So don't believe any snake oil salesmen that want you to invest your college fund this kind of invention.

14.8 Dynamics at finite temperature

14.8.1 Speed of molecules a gas

Earlier when we talked about the escape velocity from the Earth's gravitational field, we mentioned a formula for the speed of molecules as a function of temperature and their mass. Now we'll be able to derive how fast they actually move.

We have all the elements necessary to do this. For a gas with N molecules, we know from eqn. 14.7 that the center of mass kinetic energy of a single moelcule is the total energy E/N. And we also can relate E to T using eqn. 14.25. So we get

$$\langle \frac{m}{2} v^2 \rangle = \frac{E}{N} = \frac{3}{2} k_B T. \tag{14.40}$$

We're interested in the root mean square (RMS) velocity of a molecule. Why not just its mean velocity? Because that's zero. What is normally used to get an idea of the magnitude of the speed of a single particle is $\sqrt{\langle v^2 \rangle}$. We can get that be rearranging the above equation:

$$\sqrt{\langle v^2 \rangle} = \frac{E}{N} = \sqrt{\frac{3}{m} k_B T}. \tag{14.41}$$

This says that the RMS velocity is $\propto 1/\sqrt{m}$ and also increases with temperature. For nitrogen or oxygen molecules at room temperature, this is about $500m/s$. Pretty darned fast! But the bigger the molecule, the smaller this RMS velocity.

But this is just a measure of the speed you'd expect to find if you measured the velocity of a lot o molecules. Suppose you want more than that. Some of the molecules are going left, some are right, some fast, some slow. Sounds like Dr. Suess. The way to quantify what you see is to do a histogram: how likely is it for you to find that a molecule has its x component of the velocity v_x, between two values v_x and $v_x + dv_x$. This distribution was worked out a long time ago and is known as the Maxwell-Boltzmann distribution:

$$P(v_x) \propto e^{-\frac{v_x^2}{2mk_BT}} \tag{14.42}$$

A plot of this is shown in Fig. 14.8. This is a *Gaussian* distribution with

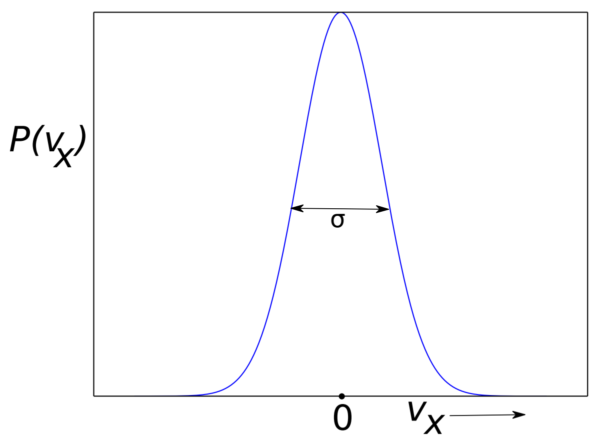


Figure 14.8:

standard deviation $\sigma = \sqrt{k_BT/m}$.

This gives you a histogram of how likely it is for v_x, but v_y and v_z look the same. They're also independent, that is, if you found a certain value of v_x it wouldn't change the odds for v_y and v_z.

In a gas, the velocity remains pretty much constant until it collides with another molecule, and in normal conditions, this distance, commonly characterized by the "mean free path", can be rather large. In air under normal conditions, it's around $700A$,

The velocity of molecules in a liquid follows the same Maxwell-Boltzmann distribution. But because of frequent collisions, the mean free path is typically of order of Angstroms.

Now we can ask what happens on a large timescale to a molecule as it bounces around colliding with other molecules.

14.8.2 Diffusion

We'll first think about how individual molecules move in a liquid or a gas, and then ask how large collections of molecules move. For example, if you put of drop of dye in water, what will it do after some time?

Random walks

If we tag one molecule, and what it move through a liquid or a gas, what will we see? Probably not much unless you're wearing really thick glasses, but suppose you could see a small molecule, or how about more realistically, something quite a bit larger, like a protein that is fluorescing. The molecule collides with others. Every time it does, it shifts its direction. Because the motion these small scales is so incredibly complicated, you can think about it as being pretty much random. Let's come up with a simple model of this random motion.

Let's say the the molecule undergoes a collision at regular time intervals of Δt. If the molecule's position at time t is $\mathbf{r}_t$, then its position after the next collision is

$$\mathbf{r}_{t+\Delta t} = \mathbf{r}_t + \Delta\mathbf{r}_t \tag{14.43}$$

We're saying that the molecule steps in a random directon $\Delta\mathbf{r}_t$. We choose this vector to in a random direction that changes every time it collides. We can use the same root mean square trick to characterize the behavior of $\Delta\mathbf{r}_t$. Say we want typically for it to go a mean free path length l.

$$\langle|\Delta\mathbf{r}_t|^2\rangle = l^2 \tag{14.44}$$

What we've just done, is describe a random walk. The molecule will move in a random direction every time it takes a "step". The outcome will look random but a typical random walk will have a characteristic look to it, as shown in Fig. 14.9.

If a random walk starts at some origin and then goes for N steps, where will it be? It could be going in any direction so on average, it hasn't gone anywhere. But from looking at a random walk, you see that is seems to get further and further away from its starting point, the longer you let it walk. The way to characterize this is by looking at squares as we've been doing of late.

$$\langle|\mathbf{r}_t - \mathbf{r}_0|^2\rangle = Nl^2 \tag{14.45}$$

So the RMS distance it goes is propotional to $\sqrt{N}$. The longer the walk, the further, on average, that the particle goes. But it's not a linear function of the number of steps, it is proportional to $\sqrt{N}$. Now the time that has elapsed after N steps is $t = \Delta tN$, so

$$\langle|\mathbf{r}_t - \mathbf{r}_0|^2\rangle = \frac{l^2}{\Delta t}t \tag{14.46}$$

We can define a *diffusion coefficient* $D = \frac{l^2}{6\Delta t}$ so that

$$\langle|\mathbf{r}_t - \mathbf{r}_0|^2\rangle = 6Dt \tag{14.47}$$

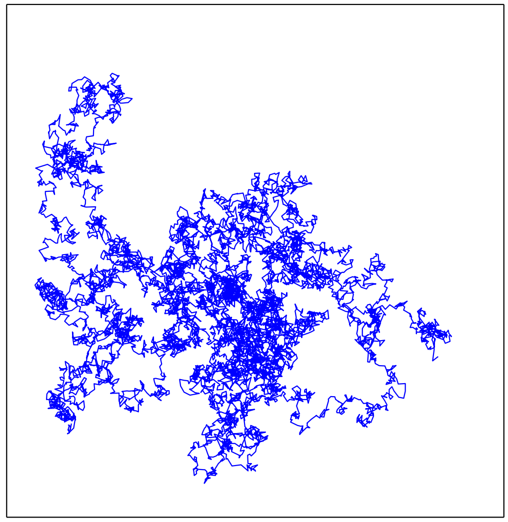


Figure 14.9:

The diffusion coefficient of a molecule varies greatly depending on its size, and the environment that it's in. It's is very important in many contexts, including many situations in biology.

Many particles

Now consider many particles diffusing at the same time. To illustrate this, consider a tiny drop of dye inserted into water. Each molecule of dye will follow a random walk, and with many you'll see the average behavior: how the density of the dye spreads out over time. Fig. 14.10 shows snapshots of the density of molecules that are initially added in a tiny drop. As time progresses, the molecules spread out. The colors represent density with red being the highest, yellow intermediate, and blue being the lowest. The way the spreading occurs can be modelled very well mathematically, but we won't discuss diffusion in this level of detail here.

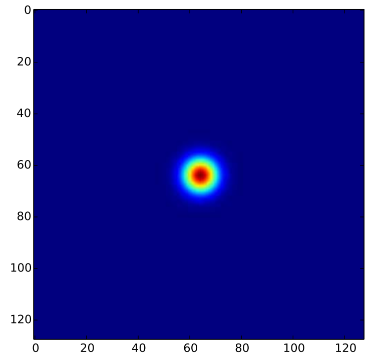
If you looked at a slice of the density of dye particles, you'd see they'd be clustered around their starting point. As you waited, the cluster would get more and more spread out. Fig. 14.11 shows the distribution of dye particles $P(x)$ at three separate times. As you progress to longer times, the particles will continue to spread indefinitely, but they spread *slower and slower*. This is because the width of the distribution obeys diffusive scaling, which we said above was propotional to $\sqrt{t}$ and not t. If you plot $\sqrt{t}$ you see it flattens out.

Diffusion and Drag

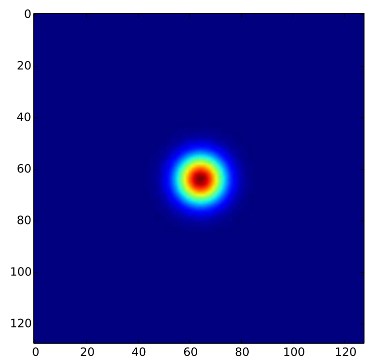
Einstein found a relationship between the diffusion coefficient of a particle in a fluid, and its drag coefficient. Remember that the drag coefficient b is the coeffient in the formula relating force $\mathbf{F}$ you get when you drag an object with velocity $\mathbf{v}$ in a liquid: $\mathbf{F} = -b\mathbf{v}$. He should through quite a profound argument that

$$D = \frac{k_B T}{b} \tag{14.48}$$

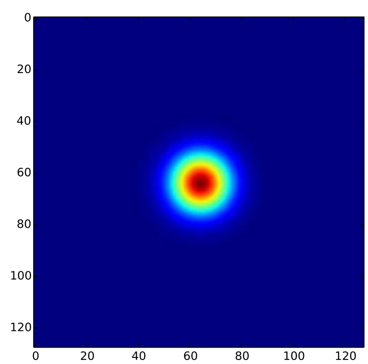
It might seem a bit surprising that diffusion and drag are related. You might think that it'd be possible to have some situations where a particle could could have little drag and also not diffuse quickly. The only way to do that is to lower the temperature.



t=1



t=2



t=3

Figure 14.10:

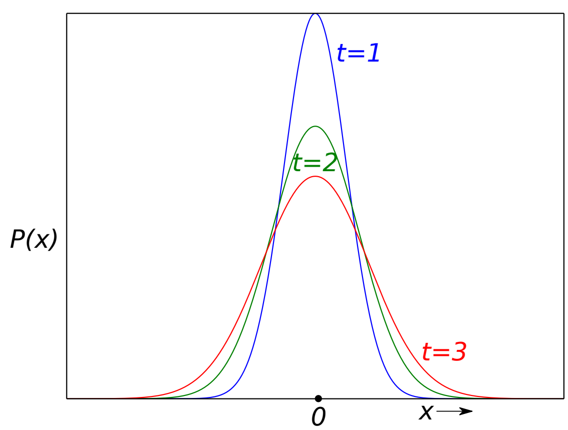


Figure 14.11: