0.1 Symmetry: commuting observables

Efficient use of symmetry can be made in order to search and calculate the eigenstates of a given operator. Let us start by stating a useful **theorem**:*There exist a common set of eigenstates for two commuting observables.* Let A and B be two commuting observables. A could be the Hamiltonian, and B a symmetry operation which leaves A invariant, meaning that [A,B]=0. We also say that B leaves the Hamiltonian invariant, or the Hamiltonian is invariant under operation B. Let $|\psi_a >$ be an eigenstate of A with eigenvalue a: $A|\psi_a >= a|\psi_a >$. Since $[A, B] = 0$, we have:

$$AB|\psi_a >= BA|\psi_a >= Ba|\psi_a >= aB|\psi_a >$$

We deduce that $B|\psi_a >$ is also eigenstate of A with the same eigenvalue a.

Therefore, **if a is nondegenerate**, $|\psi_a >$ and $B|\psi_a >$ must be colinear, meaning there exist a number b such that $B|\psi_a >= b|\psi_a >$, implying that **$|\psi_a >$ is also eigenstate of B** .

On the other hand, **if a is degenerate**, we can consider the subspace of the eigenstates of A with the same eigenvalue a: $|\psi_{a,\lambda} >, \lambda = 1, ..., p$, $A|\psi_{a,\lambda} >= a|\psi_{a,\lambda} >$ and diagonalize the matrix of B in that subspace. We can then say that we have found a linear combination of the $|\psi_{a,\lambda} >$ which is also eigenstate of B: $|\psi_{a,b} >= \sum_\lambda c_{\lambda,b}|\psi_{a,\lambda} >$ such that $B|\psi_{a,b} >= b|\psi_{a,b} >$. Therefore, we have found in the degenerate case also a set of kets $|\psi_{a,b} >$ which are eigenstates of both A and B. Note the two *quantum numbers* a, b appear as subscripts of the eigenstates. They serve to label a given state, and tell you how the state transforms under the symmetry operation B, whereas when you had the subscripts a, λ, the index λ was arbitrary and did not have any physical significance, or did not contain any useful information.

This result is useful since we can search for the eigenstates of the Hamiltonian (or any other observable for that matter) among the eigenstates of the symmetry operations which leave the Hamiltonian invariant (i.e. commute with it). As an example, assume we are dealing with an even potential in one dimension: $V(x) = V(-x)$. This symmetry implies that H commutes with the parity operator P defined as: $P\psi(x)\hat{=}\psi(-x)$. We therefore first look for eigenstates of P. They are either even functions with eigenvalue $+1$, or odd functions with eigenvalue -1. We conclude that eigenstates of H can only be either even or odd. We will see another example in the context of Bloch theorem which uses invariance of the Hamiltonian under lattice translations vectors. Yet another example in quantum mechanics is when you have a spin-independent (or rotationally invariant in the spin space) Hamiltonian: $[H, S^2] = 0$; $[H, S_z] = 0$. Its eigenstates must have a definite total spin and spin projection on z-axis: $|\psi_{1/2,\uparrow}>$. Note again the usefulness of quantum numbers to **classify** the eigenstates.

So, in practice, if we have found a set of operators which commute with the Hamiltonian, we can start by diagonalizing them first, and find a common set of eigenstates to these operators in the first place (assuming that they are simple enough to diagonalize, or at least simpler than the Hamiltonian). The eigenstates of the Hamiltonian, are the nondegenerate states obtained. In case a set of found eigenstates are degenerate, we look for eigenstates of the Hamiltonian as a linear combination of the states in that degenerate space. In this found basis, the matrix of the Hamiltonian is block diagonal, and one has unltimately to diagonalize a large number of matrices (blocks) of small dimensions.