

EM-110A Final exam - March 18, 2008

Name:

Take a deep breath, read the questions carefully, ponder on them for 5 minutes, and then start writing. Writing down the solution takes only a few lines, so take your time thinking through the solution first. (numbers in parentheses represent the point worth of each question). You will probably not have enough time to answer all the questions, so pick carefully the ones you are more comfortable with.

Good luck,

Keivan Esfarjani

1. (6) Derive the integral expression for the total energy of a charge distribution as a function of the electric field only. Show or explain all your steps rigorously.

2. (20) Consider 3 (infinitely large) metallic plates of charges surface charge densities σ_1 , σ_2 and $-\sigma_1 - \sigma_2$ separated respectively by a dielectric material of constant ϵ_1 and thickness a_1 , and a material of dielectric constant ϵ_2 and thickness a_2 . Find the D, E, P, V fields everywhere, and plot the variations of the electrostatic potential V as a function of distance from the top plate of charge density σ_1 assuming $\sigma_1 = \sigma_2$, $\epsilon_2 = \epsilon_1$ and $a_2 = 2a_1$. (Make this substitution only at the end).

3. (20) Consider a *grounded* ($V = 0$) spherical metallic shell of radius a , and a point charge q at a distance d from its center. What is the electric field inside the sphere? By considering an image charge $q' = -aq/d$ placed at an appropriate location (to be specified) show that the total electrostatic potential has a simple form. Calculate it (draw a picture).

Find the induced charge distribution on the surface of the sphere.

4. (12) Consider a rectangular box of dimensions a, b, c along the $x, y,$ and z directions respectively. The potential on 5 of its facets is set to zero, and that on the facet at $z = c$ is set to be $V(x, y, z = c) = \sin(2\pi x/a) \times \sin(3\pi y/b)$. Find the potential everywhere in the volume of the box.

5. (10) Consider two circular loops of radius a separated by a distance d , ($a \ll d$) such that their axes are both on the z -axis. Calculate their mutual inductance. Specify what are the approximations you are making to get your result.

6. (15) For a loop of self-inductance L , what is the work needed for a current source to have a current I flow in the loop (neglect Joule dissipation effects). This is the energy of the loop.

What is the work needed by the current source to sustain I when another current loop I' is brought from infinity to its proximity? (assume that at the final position the mutual inductance is M , and express your final result as a function of M, I, I').

What is the final energy of the two loops?

7. (20) Consider two coaxial (infinitely long) cylinders of radii R_1 and R_2 . A material of permeability μ_1 is filling the first cylinder, and another material of permeability μ_2 is between the two cylinders. Outside the larger cylinder is vacuum. A solenoidal surface current is flowing on the first shell: $\vec{K}_1 = K_1 \hat{\phi}$, and $\vec{K}_2 = -K_2 \hat{\phi}$ on the second shell. Find the fields H, B, M, A everywhere. Plot the fields B and H as a function of the distance from the axis for $R_2 = 2R_1$; $\mu_1 = \mu_2 = 2\mu_0$ and $K_2 = 2K_1$. (You must do this substitution only at the end for plotting purposes).

8. (10) An alternating current $I(t) = I_0 \cos \omega t$ flows down a long straight wire and returns along a coaxial conducting tube of radius a .
Determine the direction and magnitude of the induced electric field as a function of the distance s from the wire (assume the field goes to zero when $s \rightarrow \infty$.)

Cheat-sheet (a lot of it might not be needed)

$$\int_V \vec{\nabla} \cdot \vec{g} \, d\tau = \oint_{\Sigma} \vec{g} \cdot \vec{d}a ; \int_{\Sigma} (\vec{\nabla} \times \vec{g}) \cdot \vec{d}a = \oint_C \vec{g} \cdot \vec{d}l$$

$$\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3} \right) = -4\pi\delta(\vec{r}) ; \vec{\nabla} f(r) = \hat{r} \frac{df}{dr}$$

$$\vec{\nabla} \cdot (f\vec{v}) = f(\vec{\nabla} \cdot \vec{v}) + \vec{\nabla} f \cdot \vec{v} ; \vec{\nabla} \times (f\vec{v}) = f(\vec{\nabla} \times \vec{v}) + \vec{\nabla} f \times \vec{v}$$

$$\vec{\nabla} \times \vec{V} = \left[\frac{1}{s} \frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z} \right] \hat{s} + \left[\frac{\partial V_s}{\partial z} - \frac{\partial V_z}{\partial s} \right] \hat{\phi} + \frac{1}{s} \left[\frac{\partial(sV_\phi)}{\partial s} - \frac{\partial V_s}{\partial \phi} \right] \hat{z}$$

Electrostatic fields:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d\tau(\vec{r}') ; V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau(\vec{r}')$$

Magnetostatic fields:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d\tau(\vec{r}') ; \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau(\vec{r}')$$

Forces:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) ; d\vec{F} = Id\vec{l} \times \vec{B}$$

Maxwell's equations:

$$\vec{\nabla} \cdot \vec{B} = 0 ; \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} ; (\vec{\nabla} \cdot \vec{D} = \rho_{free})$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} ; \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} ; \left(\vec{\nabla} \times \vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t} \right)$$

Constitutive relations:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon \vec{E} ; \vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) = \mu \vec{H} ; \vec{M} = \chi_m \vec{H}$$

Bound charges/currents:

$$\rho_b = -\vec{\nabla} \cdot \vec{P} ; \sigma_b = \vec{P} \cdot \hat{n} ; \vec{J}_b = \vec{\nabla} \times \vec{M} ; \vec{K}_b = \vec{M} \times \hat{n}$$

Discontinuities:

$$\Delta \vec{E}_{\parallel} = 0 ; \Delta \vec{B}_{\parallel} = \mu_0 \vec{K} \times \hat{n} ; \Delta \vec{H}_{\parallel} = \vec{K}_{free} \times \hat{n}$$

$$\Delta E_{\perp} = \frac{\sigma}{\epsilon_0} ; \Delta D_{\perp} = \sigma_{free} ; \Delta B_{\perp} = 0 ; \Delta V = 0 ; \Delta \vec{A} = 0$$

Flux:

$$\Phi_{11} = L_1 I_1 ; \Phi_{12} = M_{12} I_1 ; \text{emf} = -\frac{d\Phi}{dt}$$

Dipoles:

$$\vec{p} = \alpha \vec{E} ; \vec{p} = \sum_i q_i \vec{r}_i ; \vec{m} = \frac{1}{2} \oint \vec{r} \times Id\vec{l} = I\vec{A}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{r}}{r^3} ; \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} [3(\vec{p} \cdot \hat{r}) \hat{r} - \vec{p}] ; \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\vec{m} \cdot \hat{r}) \hat{r} - \vec{m}]$$

Forces on dipoles:

$$\vec{F}_e = (\vec{p} \cdot \vec{\nabla}) \vec{E} = \vec{\nabla}(\vec{p} \cdot \vec{E}) ; \vec{F}_m = \vec{\nabla}(\vec{m} \cdot \vec{B})$$

Expansions:

$$\text{Spherical } \phi\text{-symmetric : } V(r, \theta) = \sum_{l=0}^{\infty} P_l(\cos \theta) (A_l r^l + \frac{B_l}{r^{l+1}})$$

$$P_0(x) = 1 ; P_1(x) = x ; P_2(x) = (3x^2 - 1)/2 ; P_3(x) = (5x^3 - 3x)/2$$

$$\int_{-1}^1 P_l(x) P_n(x) dx = \frac{2}{2l+1} \delta_{n,l}$$

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx = \frac{a}{2} \delta_{n,m} ; \int_0^a \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi x}{a}\right) dx = \frac{a}{2} \delta_{n,m} \text{ for } (n, m) \neq 0$$

Energy:

$$W_e = \frac{1}{2} CV^2 = \frac{1}{2} \int \rho V d\tau ; W_m = \frac{1}{2} LI^2 = \frac{1}{2} \int \vec{J} \cdot \vec{A} d\tau$$