Solutions to Quiz 1

1) $h(x,y) = 3\,(4xy - 6x^2 - 2y^2 - 7x + 5y + 1)$

$$\frac{\partial h}{\partial x} = 3(4y - 12x - 7)$$

$$\frac{\partial h}{\partial y} = 3(4x - 4y + 5).$$

Extremum is obtained by solving $\begin{cases} \partial h / \partial x = 0 \\ \partial h / \partial y = 0 \end{cases}$

$\Rightarrow \begin{cases} 4y - 12x - 7 = 0 \\ 4x - 4y + 5 = 0 \end{cases}$ the sum gives $-8x = 2 \Rightarrow$ $\boxed{x = -\frac{1}{4}}$
giving $-1 - 4y + 5 = 0 \Rightarrow$ $\boxed{y = 1}$

because $\frac{\partial^2 h}{\partial x^2} = -12 \times 3$ and $\frac{\partial^2 h}{\partial y^2} = -4 \times 3$ are both < 0

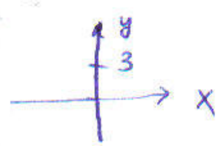
we are dealing with a maximum here.

$$h_{max} = h\left(-\tfrac{1}{4}, 1\right) = 3\left(4\left(-\tfrac{1}{4}\right) - 6\left(\tfrac{1}{4}\right)^2 - 2(1) - 7\left(-\tfrac{1}{4}\right) + 5 + 1\right)$$

$$= 3\left(-1 - \frac{3}{8} - 2 + \frac{7}{4} + 6\right) = 13 + \frac{1}{8} = 13.125$$

Slope at $(2,3)$ $\begin{cases} \frac{\partial h}{\partial x}(2,3) = 3(4 \times 3 - 12 \times 2 - 7) = -57 \\ \frac{\partial h}{\partial y}(2,3) = 3(4 \times 2 - 4 \times 3 + 5) = 3 \end{cases}$

Slope $= \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} \simeq 57.08$



2) Vector with zero div and zero curl.

$\vec{V} = (x, -y, 0)$ or $x\hat{x} - y\hat{y}$

$\vec{\nabla}\cdot\vec{V} = 1 - 1 = 0$

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix}\partial_x \\ \partial_y \\ \partial_z\end{vmatrix} \times \begin{vmatrix} x \\ -y \\ 0\end{vmatrix} = \begin{vmatrix}\partial_y(0) + \partial_z(y) \\ \partial_z(x) - \partial_x(0) \\ \partial_x(-y) - \partial_y(x)\end{vmatrix} = \begin{vmatrix}0 \\ 0 \\ 0\end{vmatrix}$$

3) Derive $\int \vec{\nabla} g \, d\tau = \oint_S g \, da$

use divergence theorem for $\vec{A} = g\vec{C}$ with $\vec{C}$ = constant

$$\int \vec{\nabla}\cdot\vec{A} \, d\tau = \oint_\Sigma \vec{A}\cdot\vec{da}$$

$$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot(g\vec{C}) = \vec{\nabla}g\cdot\vec{C} + g\times(\vec{\nabla}\cdot\vec{C}) = \vec{\nabla}g\cdot\vec{C}$$

$$\Rightarrow \int \vec{\nabla}\cdot\vec{A} \, d\tau = \int \vec{\nabla}g\cdot\vec{C} \, d\tau = \vec{C}\cdot\int\vec{\nabla}g \, d\tau$$
$$= \oint_\Sigma \vec{A}\cdot\vec{da} = \oint_\Sigma \vec{C}\cdot g\,\vec{da} = \vec{C}\cdot\oint_\Sigma g\,\vec{da}$$

true for all $\vec{C}$ directions

$$\Rightarrow \int \vec{\nabla}g \, d\tau = \oint_\Sigma g \, \vec{da}$$

Derive $\int_\Sigma \vec{\nabla}g \times \vec{da} = -\oint_P g \, \vec{dl}$.

use Stokes theorem for $\vec{A} = g\vec{C}$ with $\vec{C}$ = constant

$$\int_\Sigma (\vec{\nabla}\times\vec{A})\cdot\vec{da} = \oint_P \vec{A}\cdot\vec{dl} = \oint_P g\,\vec{C}\cdot\vec{dl} = \vec{C}\cdot\oint_P g\,\vec{dl}$$

$$\hookrightarrow \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(g\vec{C}) = \vec{\nabla}g\times\vec{C} + g(\vec{\nabla}\times\vec{C}) = \vec{\nabla}g\times\vec{C}$$

$$\int_\Sigma (\vec{\nabla}\times\vec{A})\cdot\vec{da} = \int_\Sigma (\vec{\nabla}g\times\vec{C})\cdot\vec{da} = \int\vec{C}\cdot(\vec{da}\times\vec{\nabla}g) = -\vec{C}\cdot\int\vec{\nabla}g\times\vec{da}$$

True for all $\vec{C}$ $\Rightarrow$ $-\oint_P g\,\vec{dl} = \int\vec{\nabla}g\times\vec{da}$