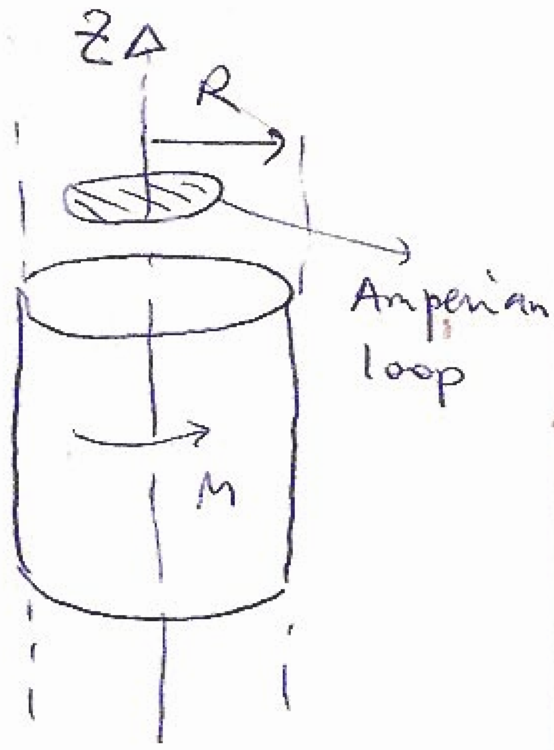
EM-110A Quiz #4 - March 11, 2008

Name:

1. A long circular cylinder of radius R carries a magnetizattion $\vec{M} = ks^2\hat{\phi}$ where k is a constant, s is the distance from the axis and $\hat{\phi}$ is the azimuthal unit vector. Find the bound currents and the magnetic field due to M for points inside and outside the cylinder. (If you can not do the problem for some reason, maybe you can explain how it can done).



Magnetization distribution, and thus $\vec{B}$ & $\vec{H}$ are azimuthal as if they were created by a current flowing in the $\hat{z}$ direction

$$\begin{cases} \vec{J}_b = \vec{\nabla}\times\vec{M} = \frac{1}{s}\frac{\partial}{\partial s}(sM_\phi)\hat{z} = 3ks\,\hat{z} \\ \vec{K}_b = \vec{M}\times\hat{n} = ks^2\,\hat{\phi}\times\hat{s} = -ks^2\,\hat{z} \text{ (for } s=R) \end{cases}$$

$\vec{B}$ can either be calculated from $\vec{J}_b, \vec{K}_b$ by using Ampère's law, or through $\vec{H}$: $\vec{B} = \mu_0(\vec{H}+\vec{M})$; but $\nabla\times H = 0$ (no free currents), but $\nabla\cdot H = -\nabla\cdot M = \frac{1}{s}\frac{\partial}{\partial\varphi}M_\phi = 0 \Rightarrow \begin{cases}\nabla\times H = 0 \\ \nabla\cdot H = 0\end{cases}$ everywhere $\Rightarrow \vec{H} = 0$

1 $\Rightarrow \vec{B}_{in} = \mu_0\vec{M} = \mu_0 ks^2\hat{\phi}$; $\vec{B}_{out} = \mu_0\vec{H}_{out} = 0$

2 Ampère on $\vec{B}$ using $\vec{J}_b, \vec{K}_b$

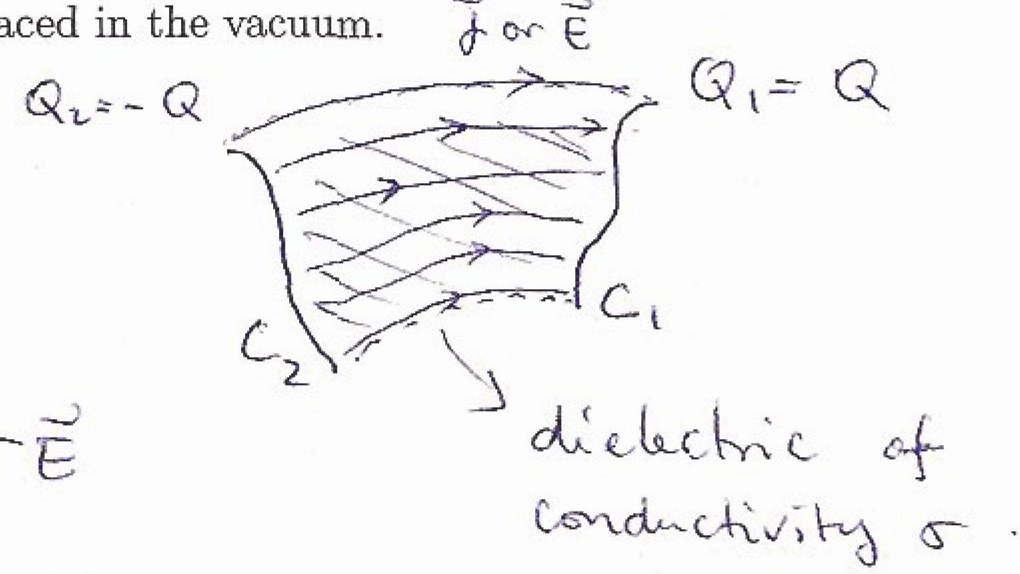
$\underline{s<R}$ $\quad 2\pi s\,B(s) = \mu_0\int\vec{J}_b\cdot d\vec{a} = \mu_0\int_0^s 2\pi s'\,ds'\;3ks' = \mu_0\,6\pi k\frac{s^3}{3}$

$\underline{s>R}$ $\quad 2\pi s\,B(s) = \mu_0\left(6\pi k\frac{R^3}{3} - kR^2\times 2\pi R\right) = \mu_0 kR^3(2\pi - 2\pi) = 0.$

$$\begin{cases}\vec{B}_{in} = \mu_0 ks^2\,\hat{\phi} \\ \vec{B}_{out} = 0\end{cases}$$ in agreement with the previous calculation

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0, \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t};\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J};\ \vec{\nabla}\cdot\vec{B} = 0;\ \vec{B} = \mu_0(\vec{H}+\vec{M})$$
$$\vec{\nabla}\times\vec{M} = \vec{J}_b;\ \vec{M}\times\hat{n} = \vec{K}_b; \vec{J} = \sigma\vec{E};\ I = \int\vec{J}\cdot d\vec{a}; V = -\int\vec{E}\cdot d\vec{l}$$

2. A weakly conducting material of conductivity σ is placed between two metallic coaxial cylinders of radii a and b. Show that the product of the resistance per unit length times the capacitance per unit length satisfies $RC = \epsilon_0/\sigma$. (If you can prove this also in a general case for an arbitrary geometry of two conductors, it is fine too, otherwise you can try to do it in this specific case.) To prove it, you can assume the two conductors are subject to a voltage and get R by relating the current flowing between the two, to the applied voltage. Likewise, to get C try to relate the voltage to the accumulated charge on the two conductors if they were placed in the vacuum.



General proof:

In the volume between the two conductors $\vec{J} = \sigma \vec{E}$

If they are charged (subject to voltage V)

$Q_2 = \int_2 \sigma_2 \, da$ (σ_2 surface charge density on conductor C_2 not to be confused with conductivity σ!

$Q_2 = C(V_2 - V_1)$

$V_2 - V_1 = R I.$ $\quad I = \int \vec{J} \cdot d\vec{a}$ can be taken on any surface, bu we will do the surface integral

$$= \int_{C_2} \sigma \vec{E} \cdot d\vec{a} = \sigma \int_{C_2} \frac{\sigma_2}{\varepsilon_0} da = \frac{\sigma Q_2}{\varepsilon_0}$$

$$V_2 - V_1 = RI = R \frac{\sigma}{\varepsilon_0} Q_2 = R \frac{\sigma}{\varepsilon_0} C (V_2 - V_1) \Rightarrow R \frac{\sigma}{\varepsilon_0} C = 1$$

or $\boxed{RC = \frac{\varepsilon_0}{\sigma}}$ QED.