Name & Student ID:

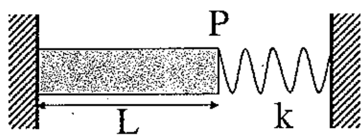
Final Exam, December 11, 2008 Thermodynamics (5D)

Take a deep breath, read all the questions carefully, make sure you understand them well (if not ask me!) and then start working on your problems

1-(*3 points*) A bar of iron of length L, linear thermal expansion coefficient α, and cross section area A is held fixed at one end. The other end is attached to a spring of stiffness constant k, itself fixed to the wall.

* Find the final displacement of the end of the bar (point P), after the temperature is raised by ΔT. (Neglect any change in k or the length of the spring due to temperature change.)

This Pb was actually harder than I thought!

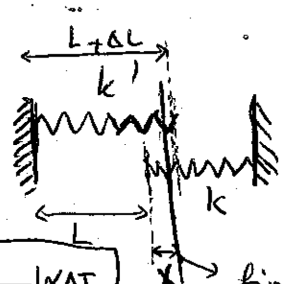


$\frac{\Delta L}{L} = \alpha \Delta T$ due to thermal expansion

Now the equilibrium length of the bar is $L + L\alpha\Delta T$. We can associate to it a spring of $k' = \frac{AE}{L}$ (E = Young Modulus)

(since $F = \sigma A = E \frac{\Delta L}{L} A = k' \Delta L$.)

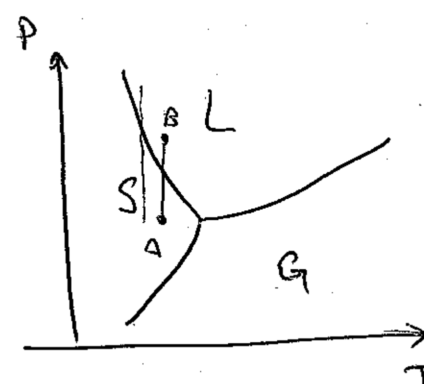
$\Rightarrow$ Now we have the Pb of two springs in series, but now k' has equilibrium length $(L + \Delta L)$



When attached, $kx = k'(\Delta L - x) \Rightarrow \boxed{X_{eq} = \frac{k'}{k+k'}\Delta L = \frac{AE/L}{(k+AE/L)} L\alpha\Delta T}$ final equil. Position

2-(*3 points*) Explain why in the winter, when you pack a ball of snow, it becomes hard (a snow ball). Then explain why this does not happen and the snow stays powder, when the temperature is very cold (like -20° C or -30° C). Hint: refer to the phase diagram of water in $P(T)$ plane.

Near or below zero when snow is squeezed the pressure is increased ($A \rightarrow B$) and the snow melts. Upon releasing the pressure, it freezes again and we get a rigid (icy) snowball.



Way below zero, a much larger pressure is needed to melt the snow & therefore it stays powder & does not form a snow ball.

3-(*2 points*) Find the fractional change $\Delta A/A$ of the area of a disk when T changes by ΔT given that its coefficient of linear expansion is α.

$A = \pi R^2$; After expansion $A + \Delta A = \pi (R+\Delta R)^2$.

where $\frac{\Delta R}{R} = \alpha \Delta T$

$$\frac{A+\Delta A}{A} = 1 + \frac{\Delta A}{A} = \frac{\pi(R+\Delta R)^2}{\pi R^2} = 1 + 2\frac{\Delta R}{R} + \cancel{\theta(\Delta R^2)}^{\,0} \Rightarrow \frac{\Delta A}{A} = 2\alpha\Delta T$$

4-(*2 points*) Find the volume occupied by 1 mol of O_2 at 30^o C and atmospheric pressure. If the atomic mass of O_2 is 32 g, what is the density of oxygen under these conditions?

$$PV = nRT \Rightarrow V = \frac{nRT}{P} = \frac{1 \times 8.31 \times 303}{1.013\ 10^5} = 2.49\ 10^{-2}\ m^3$$
$$= 24.9\ l$$

$$\rho = \frac{M}{V} = \frac{32\ 10^{-3}\ kg/mol}{2.49.10^{-2}\ (\text{volume of 1 mole})}.$$

$$= 1.29\ kg/m^3$$

5-(*4 points*) A piston in a sealed air pump is pushed so that the volume of air inside the pump is halved. Assume this process to be adiabatic.

* Find the new pressure P_2 and the work done on the gas as a function ov P_1 and V_1. Deduce the final temperature as a function of the initial temperature assuming it is an ideal gas.

$$\text{Adiabatic} \Rightarrow P_1V_1^\gamma = P_2V_2^\gamma \Rightarrow P_2 = P_1\left(\frac{V_1}{V_2}\right)^\gamma = 2^\gamma P_1$$
$$\simeq 2.64\ P_1$$

Work done <u>ON</u> the gas $= -\int P\,dV$.

since $PV^\gamma = \text{constant} = C \Rightarrow P = \frac{C}{V^\gamma}$ $\quad W = -\int_{V_1}^{V_2} \frac{C}{V^\gamma}\,dV$.

$$W = \frac{C}{\gamma-1}\left(V_2^{1-\gamma} - V_1^{1-\gamma}\right)$$

Using $C = P_1V_1^\gamma$, we get $W = \frac{P_1V_1}{\gamma-1}\left(\left(\frac{1}{2}\right)^{1-\gamma} - 1\right) > 0$ since we do work on gas.

$$T_2 = \frac{P_2V_2}{nR} = \frac{(2^\gamma P_1)\left(\frac{V_1}{2}\right)}{nR} = 2^{\gamma-1}\frac{P_1V_1}{nR} = 2^{\gamma-1}T_1 \simeq 1.32\,T_1$$

6-(*3 points*) A tank of volume V=0.8 m^3 contains 125 mol of helium at 3 atm. What is the temperature of the gas?

* Find the average speed of the He atoms.

$$T = \frac{PV}{nR} = \frac{3 \times 1.013 \cdot 10^5 \times 0.8}{125 \times 8.31} = 234\ ^\circ K.$$

$$V_{avg} = \sqrt{\frac{8 k_B T}{\pi m}} \simeq 1110\ m/s$$

7-(*4 points*) An igloo, a hemispherical enclosure made of ice (κ =1.67 J/m.s.C) has an inner radius of 2.5m and an outer radius of 2.8m.

* At what rate must thermal energy be generated to maintain inside air at 5°C when the outside temperature is at -20°C? (neglect convection and radiation)

* From a purely thermal standpoint why is it best to make igloos spherical?

$$\left|\frac{dQ}{dt}\right| = K A \frac{\Delta T}{L}$$

$$L = 2.8 - 2.5$$

$$A = \underbrace{2\pi \bar{R}^2}_{\text{hemisphere}} \qquad \bar{R} = \frac{2.8 + 2.5}{2} = 2.65\,m$$

$$= 1.67 \times 2\pi \times (2.65)^2 \times \frac{25}{0.3} = 6.14\ kw.$$

(Note that conduction through the ground has been neglected)

For a given volume, a sphere has the smallest surface area and therefore the largest thermal resistance.

8-(*5 points*) A mass m of water of specific heat c at temperature $T < 100^oC$ receives the amount of heat Q. Discuss and calculate the final state of water (mass and temperature) as a function of the given parameters (Q, m, c, T, L)

$Q_o = mc\,(100 - T)$ is needed to bring water to the boiling point.

so if $Q < Q_o$ $\quad \boxed{m_f = m \;;\; T_f = T + \frac{Q}{mc}}$ all water at T_f

and if $Q > Q_o$ $\quad Q = mc\,(100 - T) + (\Delta m) L$

where Δm = mass of evaporated water.

$$m_f = m - \Delta m = m - \frac{Q - mc\,(100 - T)}{L}$$

$$T_f = 100\,^oC.$$

9-(*5 points*) Consider an ideal gas undergoing a process from volume V_A, temperature T_A to (V_B, T_B).

* Calculate its entropy change per mol in terms of T_A, T_B, V_A, V_B.

* Find ΔS for air if $V_B = 2V_A$ and $T_B = T_A/2$. Is it positive or negative?

$$\Delta S = \int \frac{dQ}{T} \overset{\text{1st law}}{=} \int \frac{dE}{T} + \frac{dW}{T} = C_v \int_A^B \frac{dT}{T} + \int_A^B \frac{P\,dV}{T}$$

ideal gas $\rightarrow$ $P = \frac{nRT}{V} \Rightarrow \frac{P\,dV}{T} = nR\,\frac{dV}{V}$

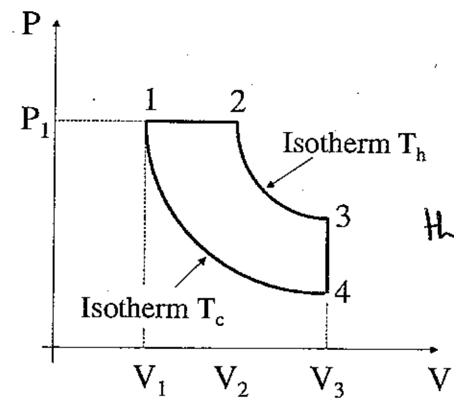
$$\Delta S = C_v \ln \frac{T_B}{T_A} + nR \ln \frac{V_B}{V_A} \qquad C_v = \frac{5}{2} nR \text{ (for air)}$$

$$= \left(\frac{5}{2} \ln \frac{1}{2} + \ln 2\right) \times nR = nR \ln 2 \left(1 - \frac{5}{2}\right) = -\frac{3}{2} nR \ln 2$$

< 0

10-(*11 points*) Calculate the efficiency of the following engine as a function of $T_h, T_c, V_1,$ ~~V_2~~, and V_3. You must first describe how you are going to approach the problem.

* At what fraction of the Carnot efficiency does it function if $V_1 = 1l$, ~~$V_2 = 2l$~~, $V_3 = 4l$, $T_h = 227^oC$, and $T_c = 27^o$ C.



[Sorry, actually V_2 is not needed. If n, T_h, T_c, V_1, V_3 are known, then $P_1 = \frac{nRT_c}{V_1}$, $P_2 = P_1$

$V_2 = \frac{nRT_h}{P_2} = \frac{T_h}{T_c} V_1$

$P_3 = \frac{nRT_h}{V_3}$, $P_4 = \frac{nRT_c}{V_3}$]

$\eta = \frac{W_{cycle}}{Q_{in}}$ → total work done in one cycle; Q_{in} → input heat

$$Q_{in} = Q_{1\to 2} + Q_{2\to 3} = E_2 - E_1 + P_1(V_2 - V_1) + E_3 - E_2 + nRT_h \ln\frac{V_3}{V_2}$$

($Q_{3\to 4} < 0$ $\quad Q_{4\to 1} < 0$ → they don't count)

$$Q_{in} = C_v (T_h - T_c) + P_1 (V_2 - V_1) + nRT_h \ln\frac{V_3}{V_2}$$

$$W = nRT_h \ln\frac{V_3}{V_2} + P_1 (V_2 - V_1) + nRT_c \ln\frac{V_1}{V_3}$$

$P_1 V_1 = nRT_c$; $P_2 V_2 = P_1 V_2 = nRT_h$ $\qquad$ After dividing by nRT_h

$$\eta = \frac{W}{Q_{in}} = \frac{\ln\frac{V_3}{V_2} + \left(1 - \frac{T_c}{T_h}\right) + \frac{T_c}{T_h}\ln\frac{V_1}{V_3}}{\ln\frac{V_3}{V_2} + \left(1 - \frac{T_c}{T_h}\right) + \frac{5}{2}\left(1 - \frac{T_c}{T_h}\right)}$$

$$\ln\frac{V_3}{V_2} = \ln\frac{V_3}{V_1} + \ln\frac{T_c}{T_h} = \ln 4 - \ln\frac{5}{3}$$

$$= \frac{\ln 4 - \ln\frac{5}{3} + 0.4 - 0.6 \ln 4}{\ln 4 - \ln\frac{5}{3} + (0.4)\cdot 3.5} = 0.195$$

$$\eta_c = 1 - \frac{T_c}{T_h} = 0.4 \qquad \frac{\eta}{\eta_c} = \frac{0.195}{0.4} \simeq 49\%$$ of the Carnot efficiency.

11-(*3 points*) Consider two isolated systems at temperatures T_1 and $T_2 (< T_1)$. They are very briefly put in contact so that a heat Q flows from 1 to 2.

* What is the entropy change of each system? Show that the total entropy change is positive.

$$\Delta S_1 = -\frac{Q}{T_1}$$

$$\Delta S_2 = \frac{Q}{T_2}$$

$$\Delta S = \Delta S_1 + \Delta S_2 = Q\left(\frac{1}{T_2} - \frac{1}{T_1}\right) > 0.$$

since $T_2 < T_1$

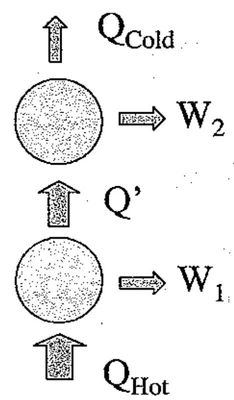
12-(*5 points*) One Carnot engine drives another in series. The heat released from the cold side of the first engine is absorbed by the hot side of the second engine.

* Find the overall efficiency η in terms of η_1 and η_2.

$$\eta_1 = 1 - \frac{Q'}{Q_{hot}}$$

$$1 - \eta_1 = \frac{Q'}{Q_{hot}}$$

Likewise $$1 - \eta_2 = \frac{Q_c}{Q'}$$



$$1 - \eta = \frac{Q_c}{Q_h} = \frac{Q_c}{Q'}\frac{Q'}{Q_h} = (1-\eta_1)(1-\eta_2)$$

$$\Rightarrow \eta = \eta_1 + \eta_2 - \eta_1\eta_2$$

Good luck,
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Eventually needed formulas:

$\frac{dQ}{dt} = -\kappa A \frac{dT}{dx}$	$Q = mL$	$Q = mc\Delta T$	$PV = nRT$
$\Delta E_{int} = Q - W$	$W = \int PdV$	$PV^\gamma = \text{constant}$	$\Delta L/L = \alpha \Delta T$
$N_A = 6.02 \times 10^{23} mol^{-1}$	$R = k_B N_A = 8.31 J/mol.K$	$1 a.m.u = 1.67 \times 10^{-27} kg$	$1 atm = 1.013 \times 10^5 Pa$
$m_{He} = 4\, a.m.u$	$\eta_C = 1 - T_c/T_h$	$\Delta S = \int \delta Q/T$	$\Delta L/L = F/EA$
$v_{rms} = \sqrt{\frac{3k_BT}{m}}$	$v_{avg} = \sqrt{\frac{8k_BT}{\pi m}}$	$v_p = \sqrt{\frac{2k_BT}{m}}$	$\gamma = C_p/C_v$