Newton's Early Computational Method for Dynamics

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Abstract

On December 13, 1679 NEWTON sent a letter to HOOKE on orbital motion for central forces, which contains a drawing showing an orbit for a constant value of the force. This letter is of great importance, because it reveals the state of NEWTON's development of dynamics at that time. Since the first publication of this letter in 1929, NEWTON's method of constructing this orbit has remained a puzzle particularly because he apparently made a considerable error in the angle between successive apogees of this orbit. In fact, it is shown here that NEWTON's implicit *computation* of this orbit is quite good, and that the error in the angle is due mainly to *an error of drawing* in joining two segments of the oribit, which NEWTON related by a *reflection symmetry*. In addition, in the letter NEWTON describes quite correctly the geometrical nature of orbits under the action of central forces (accelerations) which increase with decreasing distance from the center. An iterative computational method to evaluate orbits for central forces is described, which is based on NEWTON's mathematical development of the concept of curvature started in 1664. This method accounts very well for the orbit obtained by NEWTON for a constant central force, and it gives convergent results even for forces which diverge at the center, which are discussed correctly in NEWTON's letter *without* using KEPLER's law of areas. NEWTON found the relation of this law to general central forces only after his correspondence with HOOKE. The curvature method leads to an equation of motion which NEWTON could have solved *analytically* to find that motion on a conic section with a radial force directed towards a focus implies an inverse square force, and that motion on a logarithmic spiral implies an inverse cube force.

I. Introduction

After HOOKE became the secretary of the Royal Society, he initiated a correspondence with NEWTON during the winter of 1679–80, to re-establish contact between him and the Society, and to discuss physical problems of mutual interest. The main subject of this correspondence turned out to be focused on

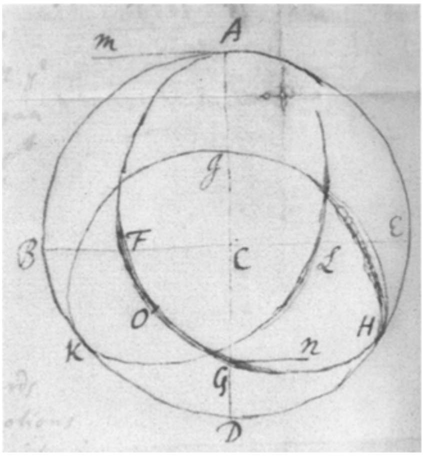

Fig. 1. NEWTON's diagram for the orbit of a body in a constant central field, which is drawn on the right-hand side of the letter of December 13, 1679 to HOOKE. I thank the British Library for permission to reproduce this diagram.

fundamental questions concerning orbital motion under the action of central forces. In particular, on December 13, 1679, NEWTON responded promptly to a letter sent four days earlier by HOOKE [1], with a letter in which he discussed the orbits of a body moving under the assumption of general central attractive forces [2]. On the right-hand side of his letter NEWTON made a drawing for an orbit which is reproduced in Fig. 1, for the special case when the force is a constant. NEWTON did not reveal in his letter any details of the method by which he obtained these orbits, except for a cryptic reference to

> The innumerable & infinitely little motions (for I here consider motion according to the method of indivisibles) continually generated by gravity . . . [3]

and a conclusion

> Your acute Letter having put me upon considering thus far the species of this curve, I might add something about its description by points *quam proximè*. But the thing being of no great moment I rather beg your pardon for having troubled you thus far with this second scribble . . .

It should be of considerable interest to find out what was NEWTON's computational method "by points *quam proximè*", because it could provide some insight on how NEWTON developed orbital dynamics, and discovered eventually the fundamental laws of dynamics. Since the discovery of this letter in 1929 by PELSENEER [4], several conjectures have been proposed for NEWTON's computational method [4–8]. It has also been pointed out that NEWTON's figure does not agree with the results of an exact calculation of an orbit with similar initial conditions, and that the angle between successive apogees in the figure exceeds by a large amount the maximum possible value. For example, KOYRÉ [9] remarks that

> Newton's solution is not quite correct. Which as a matter of fact, is not surprising. The problem he deals with is very difficult, and its solution implies the use of mathematical methods that Newton, probably, did not possess at the time, perhaps not even later. Much more *surprising* [my italics] is the very problem Newton is treating – the problem of a body submitted to a constant centripetal force. In other words, Newton assumes or seems to assume, that gravity is something constant . . .

More recently, ARNOL'D [10] stated that

> Newton, who could not bear the slightest criticism, replied on 13 December with a long letter containing a lengthy discussion and clearly showing that at the time Newton did not know what the trajectory of the ball should look like . . . This letter contains among other mistakes an impossible picture of an orbit . . . the angle between the pericentre and the apocentre is 120° (it should belong to the interval $\pi/2$ and $\pi/\sqrt{3}$) and the orbit is clearly asymmetric.

These mistakes have been attributed generally to some failure in NEWTON's computational method or approximations [5–8]. However, a careful examination of NEWTON's figure reveals that NEWTON made a substantial error only while *drawing* the figure representing the orbit, but not in *calculating* it. This will be demonstrated in detail below. Hence, conjectures made to account for the angular error on the basis of a faulty computational method attributed to NEWTON are incorrect, because NEWTON's computation of the orbit turns out to be quite good. Moreover, the fact has been frequently ignored that in the text of his letter to HOOKE, NEWTON revealed that he had considered also orbital motion for the case that the force increases rapidly towards the center in various degrees, and that he had reached correct conclusions about the geometry of the orbits also for these cases. Referring to Fig. 1, where the trajectory is labeled by $AFOGHAIKL$, and O labels the point on the trajectory closest to the center of force C, NEWTON states that

> . . . Thus I conceive it would be if gravity were the same at all distances from the center. But if it be supposed greater nearer the center the point O may fall in the line CD or in the angle DCE or in other angles that follow, or even nowhere. For the increase of gravity in the descent may be supposed such that the body shall by an infinite number of spiral revolutions descend continually till it cross the center by motion transcendently swift. . .

Indeed, one recognizes that for the case of an inverse-square force the point O would "fall in the line CD", while the case of an "an infinite number of spiral revolutions" implies an inverse-cube force, but these radial dependences of the force were not mentioned in the letter to HOOKE. However, Newton explained them around 1684 when he returned to these problems in a cancelled Scholium to a revision of *De Motu* [11] deposited as his Lucasian lectures that year. Therefore, an important question, which has not been addressed previously, is to verify that a computational method attributed to NEWTON can calculated not only an orbit for the case of a constant central force, after proper account has been taken of the errors of drawing in Fig. 1, but also have led NEWTON to correct conclusions for properties of orbits in the case of forces which diverge toward the center. In addition, such a method must be consistent with the important historical constraint that at the time of the NEWTON-HOOKE correspondence, NEWTON, by his own accounts, had not yet discovered that KEPLER's law of areas or conservation of angular momentum was a general consequence of central forces. For example, in later recollections NEWTON wrote [5] that

> Dr Hook replied . . . and I found *now* [my italics] that whatsoever was the law of the forces which kept the planets in their Orbs the areas described by a Radius drawn from them to the Sun would be proportional to the times in which they were described . . .

In this paper I consider an iterative computational method to calculate orbital motion for central forces, which is based on NEWTON's earlier development of the concept of curvature and of evolutes in 1664 [13]. This method corresponds to one of several conjectures proposed in 1960 by LOHNE [5], and discussed further by WHITESIDE [7], for a method of approximation that NEWTON might have applied to obtain the orbit shown in Fig. 1. It is suggested that NEWTON obtained this orbit by joining a succession of small circular arcs, where each arc has a radius ρ corresponding to the local radius of curvature of the trajectory. The value of ρ is determined by the component of the acceleration which is normal to the trajectory a_n, and by the magnitude of the velocity v along the trajectory according to the relation $\rho = v^2/a_n$. The accuracy of this method depends on the magnitude chosen for the elementary arcs. LOHNE takes small arcs, and his figure [5] agrees very well with the correct orbit, but it does not account for NEWTON's figure. On the other hand, WHITESIDE [7] states that

> The construction . . . which ensues in an obvious manner would, to be sure, exaggerate the central angles between successive 'aphelia' in the same sense and roughly to the same degree as in Newton's manuscript figure.

However, as it will be shown in the next section, WHITESIDE's account for the "exaggerated central angles" in NEWTON's figure cannot be attributed to NEWTON's method of computation.

In section II it is shown that a segment *AFOGH* of NEWTON's orbit, Fig. 1, has an almost perfect reflection symmetry, while the figure as a whole does not satisfy this symmetry. This reveals NEWTON's early understanding of a fundamental dynamical symmetry of orbits in central forces, which he applied in the construction of his diagram. Moreover, it is demonstrated that the large error (approximately 30°) in the angle between successive apogees is due almost entirely to an error of drawing in the application of this symmetry. In section

III, an iterative computational method is discussed to evaluate general non-circular orbits for central forces which is based on NEWTON's own development of the mathematical theory of curvature from 1664 to 1671. This curvature method satisfies a important historical constraint: it does not depend for its implementation on any understanding of KEPLER's law of areas. It is shown that this method leads to an equation of motion which NEWTON could well have established after 1671, when he had developed a fluxional (calculus) approach to curvature. Moreover, he could have solved this equation *analytically* to obtain the relation between elliptical motion and the inverse-square law of force, contrary to prevailing views that this is impossible before understanding of the significance of the law of areas. In section IV, numerical applications of this approximation method are presented in conjunction with the exact symmetry of reflection discussed in section II. For the case of constant central force, good agreement is found between the calculation and NEWTON's diagram, after proper account is taken of NEWTON's error of drawing. When the elementary arc lengths are taken proportional to the curvature, it is shown that the curvature method is remarkably convergent even in the presence of singular forces like $1/r^q$, where $0 < q$, which diverge near the center of force. The relation between this curvature method and a dynamical method based on the action of impulsive forces acting at equal time intervals, which implements physical suggestions first made by HOOKE in 1666, and mentioned in his 1679 correspondence with NEWTON, is discussed in section V. NEWTON introduced this impulse method as the cornerstone of the *Principia* in Theorem I, Proposition I, Book I, where he applied it to demonstrate the general validity of KEPLER's law for central inverse-square forces. A summary and some conclusions are presented in section VI.

II. Solution to the Puzzle of Newton's Angular Error

Since the publication of NEWTON's letter to HOOKE [2, 4] some 64 years ago, the origin of a large error (approximately 30°) in the angle between successive apogees of the orbit, shown in Fig. 1, has been clouded in mystery. This error is paradoxical, because NEWTON shows an orbit which has approximate symmetries and returns repeatedly to the apparent circumscribed circle, as demanded by conservation of *both* energy and angular momentum. How can any approximate method which gives such large errors in the angular position apparently not violate these two fundamental laws of conservation?

Before discussing a possible computational method which NEWTON used to calculate orbital motion it is necessary to understand the source of this error. Fortunately, this can be done by examining carefully certain graphical features of the diagram, Fig. 1. To start, it is apparent that the approximately orthogonal axes AD and BE drawn on this figure do not divide the figure into equal quadrants, although all published *copied drawings* of this figure have ignored this important fact [2, 4–8]. Measurements on the diagram of the distance of the crossing point C of these axes to the circumscribed curve $ABKDHEA$, reveal that this curve is not actually a circle, although it has invariably been drawn as a precise circle in the past, but that only a segment $KDHE$ of this curve is part of a circle centered at C. However, a mirror reflection of this figure suitably rotated, which is an exact dynamical symmetry of the orbit (see section III), maps the segment $AFOGH$ of the orbit nearly perfectly on its mirror image, while the rest of the figure does not satisfy this symmetry. In particular, the crossing point C in the mirror figure, which I will call C_s, does not correspond to C in the original figure, but is displaced by a small amount into the quadrant ACB (see Fig. 5, and the discussion in section IV). Measuring distances to the circumscribed curve $ABKDHEA$ from C_s, one now finds that part of the curve BAE is a segment of a circle centered at C_s, with the *same* radius as the segment $KDHE$ measured relative to C.

These errors of drawing reveal crucial aspects of the graphical construction which NEWTON used to obtain his figure. Assuming that NEWTON had a method to calculate a segment AFO of the orbit, and that for this segment of the orbit the force center is at C_s, he could then have obtained the remaining segment OGH of the curve by a mirror reflection and rotation of the segment AFO. Alternatively NEWTON might have calculated first the segment OGH, now with center of force a C, and then obtained AFO by reflection followed by a rotation. However, he evidently made an error in shifting the center C relative to C_s, and then he incorrectly adjusted the rotation in order to join these two segments of the orbital curve as smoothly as possible. This is also apparent in the drawing of this orbit by NEWTON's patching up the curve with several lines in the segment FOG. In the text NEWTON refers to O as the "nearest approach of the body to the center C . . . ". However, this must be interpreted with some care, because the figure has not one but two centers, C and C_s (see Fig. 4 for a reconstruction of NEWTON's orbit). Indeed, while O is the point on the segment $AFOGH$ nearest to the center C, this center applies only to the segment OGH. Therefore, the appropriate angle subtended between the radial vectors along the maximum (apogee) and minimum (perigee) distances to C is the angle HCO between the radial lines HC and OC. This angle is measured from the diagram to be approximately 107°, which is only about 3° larger than the maximum allowed angle of $108°/\sqrt{3} \cong 103.9°$ for constant central forces [12]. Previously, it had been generally assumed that the angle between apogee and perigee of this segment of the orbit was the angle ACO which is approximately 130°, or half the angle between successive apogees, which is about 120°, without anyone's apparently realizing that the point C does not correspond to the actual center of force for this segment of the orbit. Indeed, measuring the distance of this segment of the curve to the displaced center C_s, one finds that the closest distance lies nearer to F. Therefore, the correct angle between apogee and perigee is AC_sF which is equal to HCO, as expected. In fact, as will be shown in section IV, NEWTON's computation of the segment of the orbit between apogee and perigee is remarkably good.

Due to this error of drawing in NEWTON's construction of this diagram, Fig. 1, it is evident that the additional segments HJK and KL of the orbit which touch or approach the circumscribed curve $ABKDEA$ cannot satisfy the reflection symmetry, because this circumscribed circle is not a true circle Therefore these segments had to be partly sketched in and patched up by NEWTON, as is also quite evident in the segment HJ of the diagram.

III. Computational Method

The recent publication by WHITESIDE of the monumental mathematical papers of NEWTON has enabled one to follow in great detail the development of NEWTON's mathematical ideas, particularly during the years 1664 and 1671 which are most relevant to the period under consideration in this paper [7, 13, 15, 28]. This is important in order to understand his development of orbital dynamics which is not well documented. In 1665 NEWTON applied the method of DESCARTES and HUDDE for the tangent of an algebraic curve to evaluate also its *curvature*, which he called the "crookedness" of a curve [13]. In particular he pointed out that the radius of curvature at a given point on the curve corresponds to the radius of the tangent circle of equal curvature which was named later the osculating circle by LEIBNIZ. He considered several examples including the parabola and the ellipse, and worked out the cartesian coordinates of the evolute. Similar developments on the curvature of a curve and its evolute had been carried out independently by HUYGENS [10, 14], who was interested in constructing an isochronous pendulum. By 1671, NEWTON had obtained the general expression for the curvature and the evolute, now also in polar coordinates, by his method of fluxions, as he called the derivatives of the coordinates describing a curve as function of a parameter [15]. This mathematical theory of curvature turns out to have been extremely important in the development of NEWTON's early ideas about orbital motion, as will be discussed below. The critical role of curvature in NEWTON's dynamics has also been discussed recently by WHITESIDE [7, 16], BRACKENRIDGE [17], and POURCIAU [18].

For a body moving on a circular orbit with radius ρ with a uniform velocity v, NEWTON had shown in 1665 that the acceleration a is uniform and that it is directed towards the center of the orbit, with a magnitude [19]

$$a = \frac{v^2}{\rho} \,. \tag{1}$$

This relation had been obtained also somewhat earlier by HUYGENS [20]. During this time NEWTON evidently had started already to think about the generalization of this result for an elliptical trajectory, as shown by a remark in his manuscript on circular motion [19, 21]:

> If the body b moves in an Ellipsis, then its force in each point (if its motion in that point be given) may be found by a tangent circle of equal crookedness with that point of the Ellipsis.

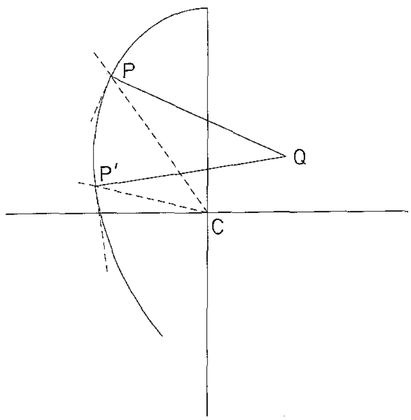

Fig. 2. The segment PP′ of an orbit obtained by rotating the radius of curvature vector PQ into P′Q relative to the fixed center of curvature Q. The center of force is at C, and the dashed lines PC and P′C are the position vectors relative to C.

Referring to Fig. 2, if the force is directed to a fixed center C then the appropriate generalization of Eq. (1) for the acceleration, assumed to be proportional to the force, at a point P on the orbit is [22]

$$a_n = \frac{v^2}{\rho} \tag{2}$$

where $a_n = a\sin(\alpha)$ is the component of the acceleration a normal to the velocity, ρ is the radius of curvature at P, and α is the angle between the radius vector CP and the tangent to the curve at P. During a small interval of time δt the trajectory can then be approximated by rotating the radius of curvature vector through an angle $\delta\phi = v\delta t/\rho$. At the end of this time interval the magnitude of the velocity v changes by an amount

$$\delta v = a_p \delta t \tag{3}$$

where $a_p = -a\cos(\alpha)$ is the component of acceleration along the velocity which is tangential to the orbit at P. Thus at the end of the time interval δt the velocity is $v' \approx v + \delta v$ and the radius of curvature becomes

$$\rho' \approx \frac{v'^2}{a'\sin(\alpha')} \tag{4}$$

where the angle α' can be evaluated purely geometrically, and a' is the acceleration at the new value r' of the radial distance. The dependence of the acceleration a on the radial distance is determined by the central force. The error in evaluating v', a', α' and ρ' depends on the size of the arc $v\delta t$, and for accurate results this should be small compared to ρ, i.e. $\delta\phi \ll 1$.

Given the force (acceleration) as a function of position (and possibly also velocity), iterating this procedure leads to a method for evaluating the orbit of a body which will be called here the *curvature method.* In this method the given acceleration is decomposed at a point P into components perpendicular and parallel to the instantaneous velocity at P. It is important to notice that successive rotations of the radius of curvature vector can be taken either in a clockwise or counter-clockwise sense, depending on the initial direction of the velocity. These are related by reversing the sign of the velocity, or the direction of time (time reversal). For simplicity, it is assumed here that the initial velocity is perpendicular to the initial radial direction. This leads to a reflection or time reversal symmetry of the orbit. As will be explained in further detail below, NEWTON used this symmetry to draw segments of the orbit in Fig. 1, as is demonstrated in section II.

There is an important refinement in this procedure which NEWTON may already have discovered at this time. The first order change δr in the radial distance is given by

$$\delta r = v \delta t \cos(\alpha) \ , \tag{5}$$

and therefore Eqs. (3) and (5) imply that

$$v \delta v = \ - a \delta r \ . \tag{6}$$

Integration of Eq. (6) leads to the law of conservation of energy

$$\frac{v^2}{2} + \int^r dra(r) = E \tag{7}$$

where E is a constant. This conservation law is derived by NEWTON for one dimensional motion in Proposition XXXIX, Problem XXVII in book I of the *Principia,* and it is extended to general motion for central forces in polar coordinates in Proposition XL, Theorem XIII, along similar lines as those presented here. Hence NEWTON could have applied this law to evaluate v^2 in Eq. (2) at different values of r. For example, for constant radial acceleration a,

$$v^2 = 2(E - ar) \ . \tag{8}$$

where E is determined by the initial values of the velocity v_0 and radial position r_0.

While the law of conservation of energy becomes readily evident and can be treated without approximation by the curvature method, this is not the case for the law of conservation of angular momentum. Indeed, for finite step sizes this law is only approximately valid, and in applications one finds, therefore, that areas swept in equal time intervals are only *approximately* equal. Therefore, it is not surprising that if NEWTON had followed this path of discovery, he would not necessarily have found the relation of KEPLER's law of areas to central forces. It was HOOKE's physical ideas on orbital dynamics, which he communicated to NEWTON in the correspondence of 1679/80, that led NEWTON to this discovery, as will be discussed in section V.

Of course, in the limit of vanishingly small steps, the conservation of angular momentum is satisfied implicitly by the curvature method. This can be seen by evaluating the first order change of $\sin(\alpha)$

$$\delta \sin(\alpha) = \delta r\left(\frac{1}{\rho} - \frac{\sin(\alpha)}{r}\right) . \tag{9}$$

Substituting Eq. (2) for ρ and eliminating a by use of Eq. (6), one obtains

$$\frac{\delta \sin(\alpha)}{\sin(\alpha)} + \frac{\delta v}{v} + \frac{\delta r}{r} = 0 . \tag{10}$$

This relation implies the conservation of angular momentum l, where

$$l = rv \sin(\alpha) . \tag{11}$$

However, it will be assumed in this paper that NEWTON had not yet become aware of conservation of angular momentum during the period of his correspondence with HOOKE. Furthermore, it will be shown that NEWTON did not require knowledge of this conservation law to construct the diagram in his letter, Fig. 1, and to evaluate orbital motion in general central force fields.

If the initial velocity v_0 is normal to the initial radial distance r_0, *i.e.* $\alpha = \pi/2$, and it has a magnitude such that the radius of curvature ρ_0 is less than r_0, where $\rho_0 = v_0^2/a_0$, then the trajectory will begin to descend toward the center of force. According to energy conservation, the velocity v increases, but the curvature ρ can either increase or decrease depending on the local magnitude of the acceleration a, the velocity v, and the value of α. For the *simplest* case when a is a constant, which is considered graphically by NEWTON in his diagram, Fig. 1, ρ must increase monotonically until the curvature vector becomes parallel with the radius vector (see Fig. 3). At this position the velocity vector is again normal. to the radius vector, i.e. $\alpha = \pi/2$, as in the initial state. NEWTON could now apply a fundamental symmetry of the curvature method to deduce the subsequent evolution of the orbit. The continuation of this orbit by rotations of the curvature vector gives a curve which is just the reflection across the radial line when $\alpha = \pi/2$ of the orbit obtained by rotations in the opposite sense and of the same step size. Moreover, in the limit of vanishingly small step sizes, the segment of the orbit obtained in this manner is just the same as the segment of the orbit calculated initially, which can be considered to be transversed backwards in time. As discussed in section II, NEWTON was clearly aware of this symmetry principle of the dynamics of central forces, and he applied it to construct graphically segments of the orbit in Fig. 1. On this axis of symmetry, the radius of curvature ρ must be larger than the radial distance r, because if the original orbit descended towards smaller radii, the time reversed trajectory must ascend. Hence, under the action of central forces, the orbit either reaches a minimum value on this symmetry axis, or approaches the origin, as NEWTON pointed out in his letter to HOOKE. Moreover, NEWTON remarked

> And also if its gravity [central force] be supposed uniform it will not descend in a spiral to the very center but circulate with an alternate ascent & descent by it's *vis centrifuga* & gravity alternately overbalancing one another

The term *vis centrifuga* (centrifugal force) was coined by HUYGENS [20] for the radial acceleration a of a body moving with uniform velocity v on a circle of radius r, where $a = v^2/r$, but it is *undefined* for a general orbit. However, if one restricts HUYGENS' definition to the special case when the radial distance is either at a maximum or a minimum, then NEWTON's comment can be understood as comparing the magnitude of the *vis centrifuga* v^2/r with the magnitude of gravity which is equal to the acceleration v^2/ρ at those two points. Then, if the *vis centrifuga* is larger than gravity, $r < \rho$, and the body will undergo "ascent", while if the *vis centrifuga* is smaller than gravity, $r > \rho$, and the body will undergo "descent". In this light, NEWTON's comment has a precise mathematical meaning which cannot be attributed as originating with BORELL's qualitative theory of motion [24], contrary to what has been claimed in the past [25].

It is important to note that, in the limit of small time steps, the curvature method leads to an equation of motion which can be solved *analytically*. Thus, NEWTON could have applied readily his mathematical development of curvature to determine the relation between motion on an ellipse and the dependence of the acceleration (force) on the radial distance r by this curvature method, thus establishing the $1/r^2$ law for this orbital motion *without being aware* of the law of conservation of angular momentum. Writing Eq. (2) in the form

$$\rho \sin(\alpha) = \frac{v^2}{a} \tag{12}$$

where v^2 is given by the energy conservation law, Eq. (7), it is clear that both the left-hand and right-hand side of this equation are functions of the radial distance r. Given a curve, by 1671 NEWTON had obtained an *explicit* expression in polar coordinates r and θ for the *radial component* of the curvature vector [26],

$$\rho \sin(\alpha) = \frac{r(1 + z^2)}{(1 + z^2 - \dot{z})} , \tag{13}$$

where

$$z = \frac{\dot{r}}{r} = \cot(\alpha) . \tag{14}$$

In NEWTON's fluxional notation, the dot means here the derivative with respect to polar angle θ. For example, for conic sections. NEWTON found [27] that in polar coordinates

$$r = L/(1 + \varepsilon \cos(\theta)) , \tag{15}$$

where L is the semi-latus rectum, and ε is a parameter corresponding to the eccentricity in the case of an ellipse, $\varepsilon < 1$, while $\varepsilon = 1$ for a parabola and $\varepsilon > 1$ for a hyperbola. Evaluating Eq. (13) for conic sections, one obtains

$$\rho \sin(\alpha) = 2r - \frac{1}{A} r^2 , \tag{16}$$

where $A = L/(1 - \varepsilon^2)$ corresponds for $\varepsilon < 1$ to the major axis of the ellipse.

Likewise, assuming a power law dependence for the acceleration, $a = c/r^q$, where c is a constant of proportionality, and q is an undetermined exponent, NEWTON could have [28] also readily evaluated the energy integral, Eq. (7), to obtain for the right-hand side of Eq. (12)

$$\frac{v^2}{a} = \frac{2}{(q-1)} r + \frac{2E}{c} r^q, \tag{17}$$

provided $q \neq 1$, while for $q = 1, a = c/r$, and

$$\frac{v^2}{a} = -\frac{2\log(r)}{r} + \frac{2E}{c} r . \tag{18}$$

The total energy E is determined by the initial position r_0 and velocity v_0. Comparing Eqs. (16) and (17) gives the exponent $q = 2$ corresponding to an inverse-square force, and $A = -c/2E$. For bound elliptical motion, $E < 0$, A is the major axis of the ellipse, while for unbounded hyperbolic motion, $E > 0$, A is a negative parameter, and for parabolic motion, $E = 0$, $1/A = 0$. However, the magnitude of the semi-latus rectum L remains undetermined up to this point. This will be discussed further below.

Hence, the curvature method, which is based on the mathematics of fluxions (calculus) which NEWTON had developed by 1671, gives a simple proof of the connection between motion on conic sections and the inverse-square law of force (acceleration) towards the focus of the ellipse, *without* requiring any explicit knowledge of KEPLER's law of areas (conservation of angular momentum). It has been generally assumed in the past that this could not be done, and that NEWTON could not have carried out such a proof before discovering the law of conservation of angular momentum.

There is no direct evidence that NEWTON actually carried out such calculations, although in later years he claimed that

> By the inverse Method of fluxions I found in the year 1677 the demonstration of Kepler's Astronomical Proposition, *viz*[t] that the Planets move in ellipses, which is the eleventh Proposition of the first book of Principles; . . . I wrote the Book of Principles in the years 1684, 1685, & 1686, & in writing it made much use of the method of fluxions direct [derivatives] & inverse [integrals], but did not set down the calculations in the Book it self because the Book was written by the method of composition, as all Geometry ought to be . . . And ever since I wrote that Book I have been forgetting the Methods by which I wrote it [29].

Here, KEPLER's area law is not mentioned, but elsewhere he mentions it explicitly [30].

COHEN [31] concludes that

> Of course, this is bogus history created by Newton in about 1718. If we replace the expression "centrifugal force" by "centripetal force", however, then we have a pretty accurate description of what Newton has done in the tract *De Motu* (1684) and in the *Principia*, rather than in 1676–1677 as he alleged. There is no evidence (nor even a hint of any) that Newton might have achieved this result before his correspondence with Hooke in 1679–1680 . . . It is accordingly, all the more interesting that in this last extract Newton has used the expression "centrifugal force", the concept he had been using in the 1660's, and not "centripetal force", the concept Newton actually uses in all the documents in which he shows that an elliptic orbit is produced by an inverse-square force. This was an obvious slip . . .

Actually, there is more than a "hint" in the letter of December 13, 1679 to HOOKE, which COHEN does not discuss, that "Newton might have achieved this result", at least partially. Evidently, NEWTON had by then developed an accurate method for computing orbits under the action of central forces. Furthermore, the "obvious slip" of referring to "centrifugal forces" may not have been a slip at all if NEWTON's earliest calculations for non-circular orbital motion were based on a direct extension of the concept of *vis centrifuga*, for forces acting continuously in time, which he had developed for circular motion in 1665. Later concepts, based on central forces acting during short pulses equally spaced in time, which constitute the basis of the *Principia*, were developed by NEWTON only after the stimulating suggestions from HOOKE in 1679. NEWTON's diagram and letter are discussed by WHITESIDE [32], who states that

> . . . Nowhere in the present letter, however, is there any indication that he had yet gained the crucial insight that Kepler's area law, in its differential form, $r^2 d\phi/dt = c$ [where c is a constant corresponding to the angular momentum] opens the way, on eliminating the element dt of orbital time, to deriving the polar differential equation . . . of the fall path, without which its precise construction is impossible.

On the contrary, I have shown that without the "crucial insight" of KEPLER's law, a precise construction of orbital motion is indeed possible by the curvature method [33].

It is important to point out that for an *analytic* solution by the curvature method, the simplest curve for orbital motion is the logarithmic spiral, which NEWTON treated in 1665 as an example of his fluxional method [34]. He effectively defined this spiral by the condition that $z = \dot{r}/r$ be a constant, which he chose equal to 2π. In this case Eq. (13) yields immediately

$$\rho \sin(\alpha) = r \ . \tag{19}$$

This implies, according to Eqs. (2) and (17), that the exponent $q = 3$, and the energy $E = 0$ [35]. NEWTON may well have discovered in this way that the inverse cube force implies that a bound orbit reaches the origin "by and infinite number of spiral revolutions . . .", as he described it in his letter of December 13, 1679 to HOOKE [2]. This result cannot be deduced from an approximate solution of orbital motion which can show only the occurrence of a finite number of revolutions in the approach to the origin. It is also noteworthy that later in the *Principia*, NEWTON uses as an example the $1/r^3$ force law, rather than the physically more interesting $1/r^2$ case, to solve explicitly the *inverse problem*, given the force law obtain the orbit (see Theorem XLI, Problem XXVIII, Corollary III, Book I, [36]).

For completeness it should be pointed out that for a conic section

$$\sin\alpha = \sqrt{L}\Big/\left(r\left(2-\frac{r}{A}\right)\right)^{1/2} \tag{20}$$

which together with Eq. (16) leads to a little known geometrical property of conic sections,

$$\rho\sin^3(\alpha) = L\ . \tag{21}$$

Applying the angular momentum conservation law, Eq. (11), to substitute $v = l/r\sin(\alpha)$ in the relation for the normal acceleration, Eq. (2), implies

$$a = \frac{l^2}{\rho\sin^3(\alpha)r^2}\ . \tag{22}$$

Hence [37, 38], the geometrical property Eq. (21) for a conic, section implies immediately that $a = c/r^2$ where c is a constant giving the strength of the inverse square force, and $L = l^2/c$. The problem, which had previously remained unsolved, of determining the semi-latus rectum L from the initial conditions is now clarified by NEWTON's discovery of the law of conservation of angular momentum [39], although, by his own account, he found this law shortly after (or may be even during the period of) his correspondence with HOOKE. WHITESIDE claims that NEWTON did not become aware of this wonderfully succinct derivation of the $1/r^2$ law until the 1690's [21]. However, for the special case $\alpha = \pi/2$, Eq. (22) reduces to

$$a = \frac{l^2}{\rho r^2}\ , \tag{23}$$

and it did not escape NEWTON's attention that for the ellipse, the equality of the curvature at apogee and perigee, or ". . . alike crookedness at both ends . . ." [40], implies that the ratio of the acceleration at these turning points vary inversely as the square of the distance from the focus. This is the simplest proof that elliptic motion with a central force towards a focus together with KEPLER's law of areas implies an inverse-square law, and it appeared in Proposition 2 of a manuscript of NEWTON [40], similar to the one received by JOHN LOCKE in 1690. The original date of this manuscript is controversial [41], but it seems likely that it would have been the first proof NEWTON would construct and remember to reconstruct, after the fateful visit of HALLEY in 1684 [42]. As David Gregory commented in 1705, "the best way of overcoming a difficult

Problem is to solve it in some particularly easy cases. This gives much light into the general solution. By this way Sir Isaac Newton says he overcame the most difficult things" [41].

To show the equivalence of the equation of motion Eq. (12), where $\rho\sin(\alpha)$ is given by Eq. (13) and v^2 is given by the law of energy conservation Eq. (7), to the more familiar form for the second-order differential equation of an orbit in polar coordinates, we write it in the form,

$$\ddot{u}+u=-\frac{a}{v^2u^2}(\dot{u}^2+u^2)\ . \tag{24}$$

where $u=1/r$. This equation can also be written in the form

$$\frac{d}{d\theta}\log(\dot{u}^2+u^2)=\frac{d}{d\theta}\log(v^2)\ , \tag{25}$$

which implies that

$$l=\frac{v}{(\dot{u}^2+u^2)^{1/2}}\ . \tag{26}$$

is a constant. This invariance of the equation of motion, Eq. (24), is precisely the angular momentum, Eq. (11). Solving this equation for $\dot{u}$ gives then the solutions of Eq. (24) in the familiar integral form

$$\theta=l\int du(v^2-l^2u^2)^{-1/2}. \tag{27}$$

A corresponding integral is also obtained for the time t by recognizing that $du/dt=lu^2\dot{u}$. Both of these results where given by NEWTON in similar form in Proposition XLI, Problem XXVIII of the *Principia*. Moreover, applying Eq. (26) to the right-hand side of Eq. (24) reduces it to the modern form of the radial equation of motion

$$\ddot{u}+u=\frac{a}{l^2u^2}\ . \tag{28}$$

IV. Numerical Application of the Curvature Method

It is straightforward to apply the curvature method described in section III to evaluate numerically and/or graphically the trajectory for a central force which is given as a function of the radial distance r. For this method the case of constant central force is *simplest*, because then the value of the acceleration a in Eq. (2) does not have to be recalculated after each iteration. As NEWTON explained later in a letter to HALLEY [43] during the priority quarrel with HOOKE

> ... I then took the simplest case for computation, which was that of gravity uniform in a medium not resisting ...

This is an important remark which has mystified historians of science (for example, see the quotation by KOYRÉ in the Introduction), because for constant central acceleration the conventional integration to obtain the orbit in polar

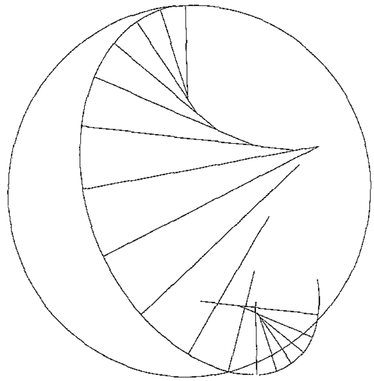

Fig. 3. The case of constant central force showing the curvature vectors drawn as lines normal to the orbit. This orbit is obtained by counterclockwise rotations of the curvature vector.

coordinates cannot be carried out analytically in terms of functions known to NEWTON. However, NEWTON's remark makes perfect sense if he had applied a computational method along the lines described here.

The results of numerical calculations for central forces with various dependences on the radial distance are shown in Figs. 3 to 9. To obtain convergent results in cases where the curvature decreases rapidly with distance, the angular step size $\delta\phi = v\delta t/\rho$ was kept fixed during the iterations. Initial conditions were chosen to obtain a trajectory similar to NEWTON's for a constant central force, as shown in Fig. 3. It is important to realize that for a fixed step size the iteration gives only an approximate trajectory which becomes less accurate with increasing number of steps. This is apparent in Fig. 3, were the iteration is continued until the trajectory crosses the circumscribed circle and re-enters the circle. This crossing occurs because with the curvature method angular momentum is conserved only approximately. Clearly, Fig. 3 does not approximate well the corresponding segment *AFOGH* of the orbit in NEWTON's diagram shown in Fig. 1 which, starting very near A emerges tangentially to the apparent circumscribed circle at H. However, the segment *AF* of the orbit matches the corresponding segment in NEWTON's figure, Fig. 1, provided the center of force in this figure is shifted somewhat relative to the crossing point *C* of NEWTON's axis. This will be discussed further below.

As NEWTON indicated in his letter to HOOKE, he had found that orbits for central force approach a minimum distance from the center of force, or may even approach this center (see quotation in the Introduction). The curvature method implies that when r is a minimum or a maximum, the radius vector is perpendicular to the oribit, *i.e.* $\alpha = \pi/2$. In this case the radius of curvature vector becomes parallel to the radial vector. It is then clear from the reflection symmetry argument discussed in section III that the iteration of the orbit past this minimum distance is the same as the original iteration, but in *reverse order, provided that the limit is taken of vanishingly small steps.* It is evident, from

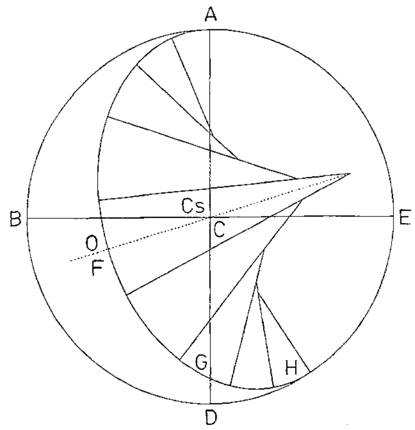

Fig. 4. The segment OGH of the orbit is obtained by reflection symmetry about the axis OC of the segment AO, which is evaluated by the curvature method.

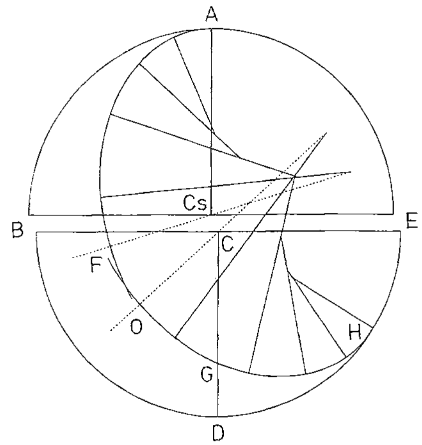

Fig. 5. Simulation of NEWTON's mistake in drawing, by shifting the center of force C in Fig. 4 for the segment OGH of the orbit relative to C_s by the amount found in NEWTON's diagram (see section II) and rotating the segment by 30°.

NEWTON's diagram, Fig. 1, that he made use of this symmetry, although it is only approximate for finite steps, to draw successive branches of the orbit. This is shown explicitly in Fig. 4, where the segment *OGH* of the orbit is obtained as the mirror reflection of the segment *AF* with the minimum distance *OC* as the axis of symmetry. This orbit is in good agreement with the exact orbit. If now the centers C_s and *C* are dsplaced by an amount corresponding to that mistakenly introduced in NEWTON's figure (see section II), and the lower segment *OGH* of the orbit is rotated by approximately 30°, Fig. 5 is obtained. This figure gives a good approximation to NEWTON's diagram, as can be verified by superimposing Fig. 5 and Fig. 1.

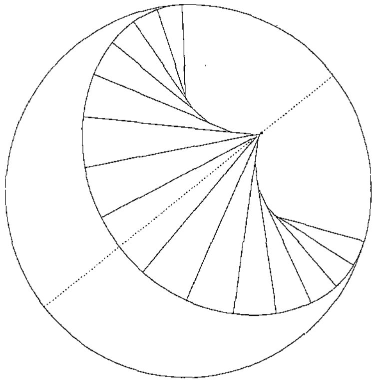

Fig. 6. Orbit for a force varying inversely with the distance obtained by the curvature method with reflection symmetry.

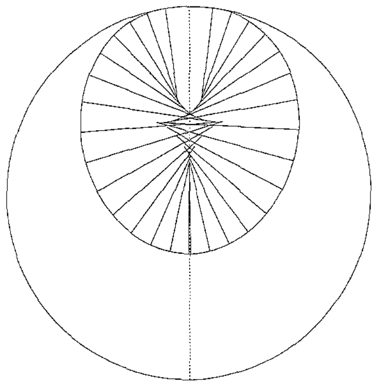

Fig. 7. Orbit for the inverse square force obtained by the curvature method.

It is clear from NEWTON's letter to HOOKE that he understood also the geometrical properties of orbits in the case that the force increases with decreasing distance from the center. The result by the curvature method for a force varying inversely with distance, which he considered later in the revised version of the *De Motu* [11] and in the *Principia*, is shown in Fig. 6. The angle from perigee to apogee is approximately 127°, in good agreement with the exact upper bound $180/\sqrt{2}$ which NEWTON gave in the *Principia* [12], but only roughly in agreement with the lower of NEWTON's two estimates in 1684, namely 136°. For the case of an inverse-square force the corresponding results are shown in Fig. 7. For the chosen step size, the point nearest to the center does not quite lie on the line *CD*, and consequently the curve does not close. For this step size the orbit does close for a force proportional to r^q, where $q \approx 2.2$, and it is approximately an ellipse. The significance of such a result would not have escaped NEWTON's attention, because he had deduced some time earlier the

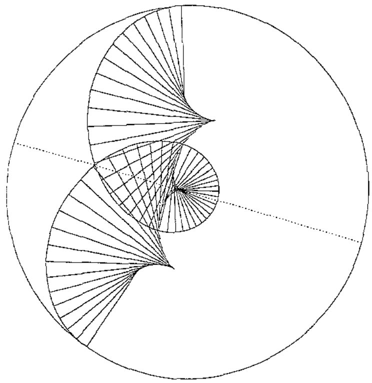

Fig. 8. Orbit for the force $f = 1/r^{2.5}$.

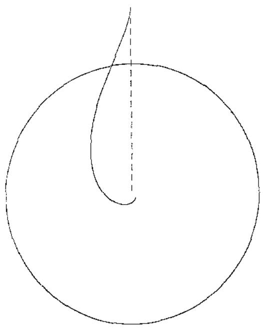

Fig. 9. Orbit for the inverse-cube force as seen in the rotating frame.

inverse-square dependence of the gravitational force from KEPLER's third law applied to circular motion. Figure 8 shows the corresponding result for a force proportional to $r^{-2.5}$ in which case the point O of closest approach lies "... in the angle DCE ...". Finally, Fig. 9 shows the case of an inverse cube force, where the orbit is approaching the center with "an infinite number of spiral revolutions ...", as NEWTON remarked in his letter to HOOKE. However, it should be emphasized that this conclusion can be reached only by an analytic argument, and not by a numerical approximation. This orbit is shown as it appears to an observer who is rotating uniformly in a frame with the initial angular rotation of the body, because it bears a remarkable resemblance to an orbit which NEWTON had sketched in an earlier letter to HOOKE [44], representing an orbit spiraling to the center of the earth. It has often been remarked that NEWTON made another error in drawing this orbit, but that is not correct, because he did not specify in that letter the force law near the center. Nevertheless one cannot jump to the conclusion from NEWTON's rough diagram that he had indeed drawn the orbit for an inverse cube force.

V. Relation between the Curvature and the Impulse method

In the autumn of 1684 NEWTON sent a manuscript entitled *De Motu Corporum in Gyrum* [45, 46] which was registered in the Royal Society. This is the earliest known draft of the *Principia,* first published in 1687. In *De Motu,* NEWTON's analysis of orbital motion starts with an approach which is quite different from the curvature method discussed in sections III and IV. The action of the central force is now represented by instantaneous pulses which occur at small equal intervals of time, and consequently the orbital motion is decomposed into straight line segments rather than into arcs, as discussed in the previous sections. In between pulses the motion is inertial and the velocity is constant, while the effect of the pulse is to change instantaneously both the magnitude and the direction of the velocity. This method is the mathematical realization of physical ideas proposed by HOOKE since the middle 1660's, which he had outlined to NEWTON in the 1679–80 correspondence. HOOKE suggested that the orbital motion of planets around the sun could be understood by compounding the tangential velocity with an impressed radial velocity due to the gravitational attraction of the sun. In 1682, he had also proposed that the gravitational attraction occurs in pulses [47]. Although in his analysis of circular motion in 1664 NEWTON had developed concepts similar to HOOKE's, there is no evidence that he generalized these to non-circular orbits before his correspondence with HOOKE. In the first letter in his correspondence with NEWTON, HOOKE had asked [48]

> ... if you will let me know your thoughts of that of compounding the celestial motions of the planets of a direct motion by the tangent & and an attractive motion towards the central body, ...

NEWTON replied four days later [44] that

> ... And perhaps you will incline the more to believe me when I tell you that I did not before the receipt of your last letter so much as hear (that I remember) of your hypotheses of compounding the celestial motions of the Planets, of a direct motion by the tangent to the curve... If I were not so unhappy as to be unacquainted with your Hypothesis above mentioned....

These remarks support the belief that at the time of the correspondence NEWTON had developed ideas different from HOOKE's about "compounding the celestial motions" most likely, as we have argued, along the lines indicated by the curvature approach. NEWTON could not yet see, or at least would not tell HOOKE, the connection between his approach and the physical ideas HOOKE proposed. If the force acts as very short pulses during fixed equal time intervals, the curvature approach, which assumes forces acting continuously in time, is not applicable at all. Assuming that the time interval between pulses is δt, $\vec{r}$ is the position and $\vec{v}$ is the velocity of the body at time t, the impulse approach can be expressed in the analytic form

$$\vec{r}' = \vec{r} + \vec{v}\delta t \;, \tag{29}$$

$$\vec{v}' = \vec{v} + \vec{a}'\delta t \;, \tag{30}$$

where $\vec{r}'$ and $\vec{v}'$ are the velocity and position of the body at the end of the time interval δt, and $\vec{a}'$ is the instantaneous acceleration or force which, for central forces, is directed along $\vec{r}'$. It is then easily verified that these equations lead to the law of conservation of angular momentum

$$\vec{r}' \times \vec{v}' = \vec{r} \times \vec{v} . \tag{31}$$

These equation represent in analytic form the geometrical construction which is the basis for the fundamental theorem 1 in *De Motu*, "All bodies circulating about a centre [of force] sweep out areas proportional to the times", which appeared later as the cornerstone of the *Principia* in theorem I, proposition I, book I. This geometrical construction was also obtained by HOOKE in an unpublished manuscript, dated September 1685, which has recently come to light [47, 49]. HOOKE applied it graphically to solve for the orbital path under the action of a force which vary linearly with distance. In reference [47] it is shown that HOOKE was able to demonstrate that this orbit is an ellipse satisfying KEPLER's law of areas, almost two years before the publication of the *Principia*.

The curvature method, which was discussed previously, can be readily compared with this impulse method to first order in δt, assuming that equivalent effective forces (accelerations) act continuously rather than impulsively in time. A vector $\vec{q}$ which is fixed during the time interval δt, is defined by

$$\vec{q} = \vec{r} - \vec{\rho} \tag{32}$$

where $\vec{\rho}$ is the radius of curvature vector. Mathematically, the vector $\vec{q}$ gives the coordinates of a curve which is the evolute of the curve given by $\vec{r}$ as a function of t. Then rotating the vector $\vec{\rho}$ while keeping the vector $\vec{q}$ fixed, one obtains at the end of the time interval δt

$$\vec{r}' = \vec{q} + \vec{\rho}' \tag{33}$$

where to first order in δt

$$\vec{\rho}' = \vec{\rho} + \vec{\delta\phi} \times \vec{\rho} \tag{34}$$

is the rotated curvature vector, and the angular rotation vector is

$$\vec{\delta\phi} = \frac{v\delta t}{\rho} \vec{z} , \tag{35}$$

where $\vec{z}$ is a unit vector normal to the plane of the orbit. Likewise

$$\vec{v}' = (v + a_p \delta t)(\vec{u}_p + \vec{\delta\phi} \times \vec{u}_p) \tag{36}$$

where $a_p = a\cos(\alpha)$ is the component of the acceleration along the velocity, and $\vec{u}_p = \vec{v}/v$ is a unit vector along $\vec{v}$ and is therefore tangent to the curve.

Since $\vec{z} \times \vec{\rho} = \rho \vec{u}_p$, and $\vec{z} \times \vec{u}_p = \vec{u}_n$, where $\vec{u}_n = -\vec{\rho}/\rho$ is a unit vector normal to the curve, we obtain to first order in δt

$$\vec{r}' = \vec{r} + \vec{v}\delta t , \tag{37}$$

and

$$\vec{v}' = \vec{v} + \vec{a}\delta t \tag{38}$$

where $\vec{a} = a_p\vec{u}_p + (v^2/\rho)\vec{u}_n$.

These equations differ from the corresponding equations obtained for the impulse method, Eqs. (29) and (30), only in that the acceleration vector $\vec{a}$ is evaluated at time t instead of at time t'. This difference should be neglected in the first order approximation, but it is important in computations with a finite time interval, because then the impulse equations preserve angular momentum exactly [51], while this is not the case for the curvature method, or for its linear approximation.

WESTFALL [6] interprets NEWTON's letter of December 13, 1679 as indicating that "he had accepted Hooke's conceptualization, ignoring his own ideas . . ." to obtain the diagram, Fig. 1. However, WESTFALL does not work out the orbit resulting from this assumption. This was done recently by ERLICHSON [8] who claims to have shown that NEWTON applied the impulse method to obtain his diagram. There are at least three reasons why ERLICHSON's explanations, and by implication, WESTFALL's suggestions, are not correct:

1. In his paper, ERLICHSON shows two figures for orbits obtained with the impulse method applied to the case of a constant central force. One of his orbits, Fig. 3 in ref. [8], is similar to NEWTON's in the quadrant ACB, but deviates from it in the quadrant BCD. I have extended it graphically until it reaches the circumscribed circle, and find an angle between successive apogees of about 200°, in good agreement with the exact orbit, but not with NEWTON's diagram. In the calculation of the second orbit, Fig. 6 in ref. [8], the time steps are about a factor four smaller, but, surprisingly, ERLICHSON obtained a large angle of 130° between apogee and perigee. However, this sizable angular error is not a consequence of the impulse method, but to large errors in ERLICHSON's calculation. Moreover, he does not explain why such calculational errors would still give NEWTON an orbit which becomes tangent with the circumscribed circle. I find that a calculation with the same time step as in ERLICHSON's Fig. 6 gives results in excellent agreement with the exact orbit (see Fig. 10 in this paper).

2. The impulse method with equal time steps similar to those applied by ERLICHSON, does not give convergent results when the central force increases rapidly with distance towards the center, *e.g.* $1/r^2$ forces, as has been shown in [47]. Thus, this method does not explain how NEWTON would have reached any conclusions about forces which become singular at the origin which he, nevertheless, described correctly in his letter to HOOKE. This important point is not considered in ERLICHSON's paper [50].

3. The impulse technique implies KEPLER's law of areas or conservation of angular momentum even for finite time steps. However, it is precisely this law which NEWTON repeatedly admitted to have discovered *during or after* his correspondence with ROBERT HOOKE [5], but certainly not before.

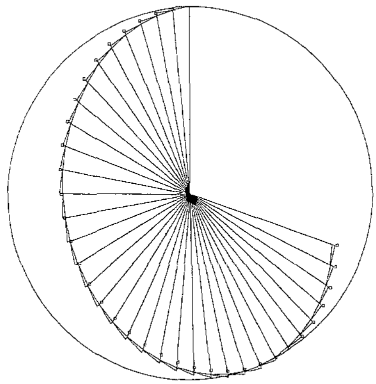

Fig. 10. Orbit for a constant central force obtained by the impulse method. The squares indicate the radial position of the exact orbit.

VI. Summary and Conclusions

It has been argued in this paper that NEWTON's diagram, included in his letter of December 13, 1679 to HOOKE, shows that he had developed a quantitative and remarkably accurate method to calculate an orbit under the action of constant central forces. This has not been appreciated in the past, particularly because of a large error in the diagram for the angle between successive apogees, and the generally held belief that he had an insufficient understanding of the dynamics of non-circular motion. The angular error has been invariably attributed to NEWTON's faulty method of computation. Instead, it has been shown in section II that the error occurred when NEWTON drew the diagram improperly, either unwittingly, or because he did not want HOOKE to know too many details of his method of calculation. Indeed, NEWTON succeeded in misleading also 20th century scholars, who have invariably reproduced his figure incorrectly, by drawing an exact circle circumscribing the orbit with perfectly orthogonal axes, sometimes even showing incorrectly a closed orbit, although NEWTON had carefully avoid doing this. The letter and diagram reveal that NEWTON's understanding of orbital dynamics at the time of his correspondence with HOOKE was deeper than has been previously realized, in spite of the fact that he had not yet understood the significance of KEPLER's law of areas. How NEWTON might have carried out his orbital calculations without any understanding of this law has been discussed in section III.

By 1671 NEWTON had fashioned the necessary mathematical and physical tools, the calculus of curvature in polar coordinates and the dynamics of circular motion, which would have enabled him to attack the general problem of non-circular orbital motion for central forces. Accordingly, a dynamical orbit can be viewed as composed of motion during small intervals of time on an arc of the osculating circle associated with the orbit, while the center of this circle is also in motion and its radius varies in time. The curve traced by the center of curvature is the evolute of the orbital curve, and NEWTON had obtained expressions for it as functions of both the cartesian and polar coordinates of this curve. In the limit of "innumerable & infinitely little motions" one obtains an equation of motion in polar coordinates which NEWTON could have solved with ease analytically for such curves as the conic section and the logarithmic spiral, and found the inverse square law and the inverse cube law for these two orbits, respectively, without yet knowing the significance of KEPLER's law of areas. Indeed, the conservation of angular momentum is more or less hidden in the curvature method for orbital dynamics. NEWTON mentioned briefly his central idea for a curvature approach already back in 1664 indicating that the "tangent circle with equal crookedness . . . " would allow the calculation of the force ". . . if the body moves in an Ellipsis . . . " [19]. A new feature of non-circular motion is the change of the magnitude of the velocity or tangential acceleration, which is determined by the component of the force tangential to the orbit. This leads directly to the law of energy conservation, known also to other 17$^{\text{th}}$ century scientists like HUYGENS and HOOKE [47]. The diagram in NEWTON's letter of December 13, 1679 to HOOKE reveals not only his ability to compute a non-circular orbital motion, but also his understanding of an exact reflection symmetry of this orbit. This letter also indicates that NEWTON understood the geometrical characteristic of orbits for central forces which increased with decreasing distance to the center. NEWTON's important remark that for constant central force he ". . . took the *simplest* case for *computation* [italics are mine] . . . " makes sense if he computed his approximate orbit by the curvature method. However, this remark is unintelligible if one believes, as KOYRÉ and other historians of science do, that he computed the orbit by the standard integral for the polar angle as a function of radial distance. NEWTON could have done this only after discovering the law of conservation of angular momentum, which evidently he did not know before his correspondence with HOOKE in 1679.

The missing ingredient for a complete solution, which must include the temporal [52] as well as the spatial dependence of the motion, was provided by the fundamental idea of HOOKE to view orbital motion as compounded by a tangential inertial velocity and a change of velocity impressed by the central force. This idea can be expressed in simple mathematical form for forces which act in short pulses, for which the curvature method is not applicable (the curvature method assumes central forces which are constant in time), and leads immediately to the conservation of angular momentum which is hidden from view in the curvature approach to dynamics. After the correspondence with HOOKE, NEWTON evidently understood the full equivalence, for vanishingly small time steps, of these two distinct physical approaches to orbital motion, but never credited HOOKE for his seminal contribution.

From the perspective outlined in this paper, it is not surprising that NEWTON chose the impulse approach, as the starting point for his exposition and mathematical proofs of the laws of motion in the *Principia*. If the curvature method was NEWTON's earlier approach to orbital dynamics, that is, before his encounter with HOOKE in 1679, it would have been less straightforward for him to explain this approach to his contemporaries. The curvature method, which is based on an advanced development of the calculus, would imply that his audience would have consisted of only a few people, *e.g.* LEIBNIZ and HUYGENS [53]. However, the fingerprints of curvature concepts appear in the later editions of the *Principia* more or less hidden and tucked away under various propositions, lemmas, and corollaries [17, 18]. NEWTON's claim that "By the inverse Method of fluxions I found in the year 1677 the demonstration . . . that the Planets move in ellipses . . ." can simply not be dismissed as bogus history [29] as is the prevalent view of many NEWTONIAN scholars today. It has been shown here that by 1671 NEWTON had developed the mathematical techniques and physical concepts which would have enabled him to compute orbital motion under the action of central forces, and to obtain analytic solutions for special force laws like $1/r^2$ and $1/r^3$, without explicitly knowing KEPLER's area law, *i.e.* conservation of angular momentum.

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3. The term *indivisibles* was used by BONAVENTURA CAVALIERE in the title of his book *Geometria Indivisibilibus Continuorum* published in 1635, and later by JOHN WALLIS in his *Arithmetica Infinitorum* (Oxon, 1656) which is devoted to the integration of curves by *the method of indivisibles.* NEWTON's early mathematical studies were also influenced by the work of WALLIS, see *The Mathematical Work of John Wallis* by J. F. SCOTT, (London, 1938)
4. J. PELSENEER, *Une lettre inédite de Newton,* Isis **12**, 237–254 (1929).
5. J. LOHNE, *Hooke versus Newton,* Centaurus **7**, 6–52 (1960). NEWTON's diagram, Fig. 1, was first published in this paper.
6. R. S. WESTFALL, *Force in Newton's Physics,* (American Elsevier, NY, 1974) pp. 427–429.
7. *The Mathematical Papers of Isaac Newton* vol. VI, 1684–1691, ed. D. T. WHITESIDE (Cambridge University Press, 1974) pp. 12, footnote (36).
8. H. ERLICHSON, *Newton's 1679/80 solution of the constant gravity problem,* American Journal of Physics **59**, 728–733 (1990).
9. A. KOYRÉ, *An Unpublished Letter of Robert Hooke to Isaac Newton,* Isis **43**, 312–337 (1952).
10. V. I. ARNOL'D, *Huygens & Barrow, Newton & Hooke* (Birkhaüser, 1990) p. 19.
11. Ref. [7], pp. 149–153; D. T. WHITESIDE, *The Preliminary Manuscripts for Newton's 1687 Principia, 1684–1685* (Cambridge University Press, 1989), pp. 89–91.

   In a revised treatise of *De Motu,* written probably in the winter or early spring of 1684–1685, NEWTON discussed again the problem of orbital motion "should the centripetal force act uniformly at all distances . . ." in the Scholium (which is crossed out and did not appear in this form in the *Principia*) following proposition XII,

problem VII. He now gives explicitly the magnitude of the angle between perigee and apogee to be about 110°, in close agreement with the angle 107° which I obtained (see section III) from his diagram of 1679, before knowing the existence of this manuscript. NEWTON also gives the corresponding angle for the case of a force which varies "reciprocally proportional to the distance from the center" as "something like 136° or 140°". This is a rough numerical approximation, since in this case the maximum angle is $180°/\sqrt{2} \approx 127.3°$. For discrete steps, the curvature method will tend to give a somewhat larger value of this angle, because the radius of curvature increases monotonically until the minimum distance to the center of force is reached. He continues with the statement

> And universally, if the centripetal force were to decrease in less than the doubled ratio of the distance from the centre, the body would return to its auge [apogee] before it could complete a circle, while if that force were to decrease in a greater than doubled but less than tripled ratio, the body would complete a circle before it could return to its auge. If, however, the same force were to decrease in the tripled or more than tripled ratio of the distance from the centre, and the body should begin to move in a curve which at the start of motion intersected the radius AS at right angles, this, were it once to begin to descend, would continue to descent right to the centre, while if it were once to begin to ascend, it would go off to infinity.

In this Scholium, NEWTON describes the radial dependence of the forces which he had not revealed to HOOKE in 1679. The only other significant change is that he now refers to force as "centripetal" rather than "centrifugal". His computational results for the case of constant force give almost the same result for the angle between perigee and apogee as in the diagram of 1679. Again NEWTON does not reveal in this treatise what computational method he applied to obtain these results, nor has a diagram been found for the orbit in a constant central force field, but there is no reason to suppose that these are not the same as the calculations he sent to HOOKE in 1679, applying, as we suggest, the curvature method.

12. In Corollary II of Proposition XLIV, Theorem XIV, Book I of the *Principia*, NEWTON shows that the central force f acting on a body moving on the "revolving ellipse"

$$r = \frac{L}{1 + \varepsilon\cos(\nu\theta)} \tag{39}$$

is given by

$$f = \frac{F^2}{r^2} + \frac{L(G^2 - F^2)}{r^3} \tag{40}$$

where $\nu = F/G$. In this case the angle between perigee and apogee is $180°/\nu$.

This result gives a powerful approach to obtain approximate orbits for general attractive central forces when the orbit is nearly circular, *i.e.* $\varepsilon \ll 1$. For example, the force $f' = c/r^n$ can be approximated by Eq. (40) by setting $f' = f$ and $df'/dr = df/dr$ ar $r = L$. This gives $n = 3 - \nu^2$, a result which NEWTON derives in Proposition XLV, Problem XXXI. For a constant force $n = 0$, $\nu = \sqrt{3}$, and the angle between perigee and apogee in this case is $180°/\sqrt{3} \approx 103°55'23''$. For a force proportional to the inverse of the distance $n = 1$, $\nu = \sqrt{2}$, and the corresponding angle is $180°/\sqrt{2} \approx 127°16'45''$. The approximate numerical values for the angles are those given by NEWTON.

13. *The Mathematical Papers of Isaac Newton* vol. I, 1664–1666, ed. D. T. WHITESIDE (Cambridge University Press, 1967) pp. 252–255.
14. J. G. YODER, Unrolling Time (Cambridge University Press, 1988) pp. 71–115.
15. *The Mathematical Papers of Isaac Newton* vol. III, 1670–1673, ed. D. T. WHITESIDE (Cambridge, 1969) pp. 151–159.
16. D. T. WHITESIDE, *How forceful has a Force Proof to be? Newton's Principia, Book 1: Corollary 1 to Propositions 11–13* Physis (Rivista Internazionale di storia della scienza) vol. XXVIII 727–749 (1991).
17. J. B. BRACKENRIDGE, *The critical role of curvature in Newton's developing dynamics* in *The Investigation of difficult things: Essays on Newton and the History of the Exact Sciences in honour of D. T. Whiteside* edited by P. M. HARMAN & ALAN E. SHAPIRO (Cambridge University Press, 1992) pp. 231–260.
18. B. POURCIAU, *Radical Principia*, Archive for History of Exact Sciences (1992) **44**, pp. 331–363.
19. J. HERIVEL, *The Background to Newton's Principia, A Study of Newton's Dynamical Researches in the Years 1664–84*, p. 130 (Oxford, 1965).
20. CHRISTIAN HUYGENS, *De Vi Centrifuga*, in Oeuvres Complètes de Christiaan Huygens XVI, 253–301 (The Hauge, 1929).
21. D. T. WHITESIDE, *The Prehistory of the Principia from 1664 to 1668*, Notes Rec. R. Soc. Lond. **45**, pg. 15 (1991).
22. Corollary III of Proposition VI, Theorem V, Book I of the third edition of the *Principia* ends with the statement that

For PV is QP^2/QR.

In the diagram in the *Principia* corresponding to this proposition, $PV = 2\rho\sin(\alpha)$ $QP = v\delta t$ and $QR = (1/2)a\delta t^2$, where $\delta t \propto SP \times QT$ in accordance with KEPLER's law of areas. NEWTON does not provide any proof for this result, which corresponds to Eq. (2) in the form

$$\rho\sin(\alpha) = \frac{v^2}{a} \tag{14}$$

It should be pointed out that this expression does not depend on an application of KEPLER's law of areas, because the time interval δt cancels. Corollary III is absent from the first edition of the *Principia.*

23. In Corollary III of Proposition VI, Theorem V, book I of the third edition of the *Principia*, NEWTON introduced a second measure for force (acceleration), using explicitly the conservation of angular momentum l, stating that "... the centripetal force will be inversely as the solid SY^2PV ... ". In our notation. $SY = r\sin(\alpha) = l/v$ is the component of the position vector along the curvature vector, and $PV = 2\rho\sin(\alpha)$ is the component of the curvature vector along the position vector, *i.e.* the chord of the osculating circle which passes through the center of force. Hence, NEWTON's second measure of force corresponds (apart from a factor $2l^2$) to $a = l^2/(r^2\rho\sin^3(\alpha))$, which is the same as Eq. (22).

24. A. ARMITAGE, *Borell's Hypothesis and the Rise of Celestial Mechanics*, Annals of Science, **5**, 342–351 (1950).
25. D. T. WHITESIDE, *Newton's early thoughts on planetary motion: a fresh look*, The British Journal for the History of Science, **2**, 118 (1964).
26. *The Mathematical Papers of Isaac Newton*, vol. III, 1670–1673, ed. D. T. WHITESIDE (Cambridge University Press, 1969), pp. 169–173.

27. Reference [25], p. 123, footnote 23.
28. *The Mathematical Papers of Isaac Newton*, vol. II, 1667–1670, ed. by D. T. WHITESIDE (Cambridge University Press, 1968) p. 207. NEWTON starts his treatise entitled *Analysis by Equations unlimited in the number of terms* with his Rule 1:

    If $ax^{m/n} = y$, then will $[na/(m+n)]x^{(m+n)/n}$ equal the area ...
29. I. B. COHEN, *Introduction to Newton's Principia* (Cambridge 1971) pg. 295. This quotation is taken from a draft of a letter to DES MAIZEAUX, written in 1720 or somewhat earlier. Other statements of NEWTON make it plausible that NEWTON had antedated his discovery, possibly because of his dispute with LEIBNIZ on the development of the calculus. I believe it is significant that in this letter NEWTON does not refer to KEPLER's law of areas, which he demonstrated only after his correspondence with HOOKE. Historians of science have dismissed NEWTON's claim that his first proofs were analytic, based on his fluxions (calculus), primarily because there is no direct documentary evidence, but also because they have not been able to figure out what technique he might have used. However, the analysis presented here, based on the mathematical and physical ideas which NEWTON had developed by 1671, make NEWTON's claim entirely plausible.
30. Reference [29] p. 291. Here NEWTON states that

    ... At length in the winter between the years 1676 & 1677 I found the Proposition that by a centrifugal force reciprocally as the square of the distance a Planet must revolve in an Ellipsis about the center of the force placed in the lower umbilicus of the Ellipsis & with a radius drawn to that center describe areas proportional to the times.

    However, the dates 1676 & 1677 contradict NEWTON's statement in several letters that he had discovered KEPLER's law at the time of his correspondence with HOOKE, which was two years later.
31. I. B. COHEN, *The Newtonian Revolution* (Cambridge University Press, 1980), pp. 248–249.
32. *The Mathematical Papers of Isaac Newton*, vol. VI, 1684–1691, ed. by D. T. WHITESIDE (Cambridge University Press, 1974), p. 12, footnote 36.
33. The *direct problem* in dynamics, given an orbit in polar coordinates where the origin is the center of force to obtain the acceleration (force) a towards this origin, can be readily solved by quadratures using the curvature method. One finds that

$$\frac{a}{a_0} = \frac{w_0}{w} \exp\left[-2 \int_{r_0}^{r} dr'/w(r') \right], \tag{42}$$

    where $w = \rho \sin(\alpha)$, and w_0, a_0 and r_0 are the initial values of the corresponding variables. For example, for a logarithmic spiral, $w = r$, Eq. (19), one obtains $a/a_0 = (r_0/r)^3$, while for a conic section, $w = r(2 - r/A)$, Eq. (16), $a/a_0 = (r_0/r)^2$.
34. Reference [13] pp. 370–378.
35. For general bound motion, $E < 0$, in an inverse cube force c/r^3, the general solution is $r = r_0/\cosh(b\theta)$, with $E = -cb^2/2(1+b^2)r_0^2$, and angular momentum $l = c^{1/2}/(1+b^2)^{1/2}$.
36. In the original tract *De Motu* NEWTON added a comment in a *Scholium* to Problem 1, WHITESIDE, (1684–1691: VI, p. 45), stating that

    'In a spiral which cuts all its radii at a given angle [logarithmic spiral], the centripetal force tending to the spiral' a pole is reciprocal in the tripled ratio of the distance'.

I think that it is significant that at that time NEWTON did not given any proof or discuss the method by which he had obtained this result. A geometrical proof based on an elegant self-similarity argument appeared later in the revised *De Motu* as Proposition VIII, Problem III, with this *Scholium* deleted, WHITESIDE (1684–1691: VI, p. 137), and in the same form in Proposition IX, Problem VI, book I of the *Principia*.

37. D. T. WHITESIDE, *The Prehistory of the Principia from 1664 to 1686* Notes Rec. R. Soc. Lond. **45**, 11–61 (1991).
38. J. FAULKNER, *Curvature of the ellipse and dynamical consequences* (to be published). I thank JOHN FAULKNER for calling this result to my attention.
39. The problem of determining the orbital parameter from initial conditions was included by NEWTON in an extension to Corollary I of Proposition XIII Problem VIII in section III, book I, added in the second edition of the *Principia*. However, NEWTON does not explain how to obtain these parameters which depends on knowing his unpublished mathematical results of 1671 on curvature.

> For the focus, the point of contact, and the position of the tangent being given, a conic section may be described, which at that point shall have a given curvature.

The "position of the tangent" is given by $\sin(\alpha)$ for a conic section in Eq. (20), and the curvature is obtained from this equation and Eq. (16)

$$\rho = \frac{1}{\sqrt{L}}\left[2r - \frac{r^2}{A}\right]^{3/2}. \tag{43}$$

These equations can be solved to determine the parameters L and A in terms of given or initial values r_0, $\sin(\alpha_0)$ and ρ_0.

$$L = \rho_0 \sin^3(\alpha_0) \tag{44}$$

and

$$A = \frac{r_0^2}{2r_0 - \rho_0 \sin(\alpha_0)}. \tag{45}$$

NEWTON then continues

> But the curvature is given from the centripetal force and velocity of the body being given;

This statement refers to the initial value of ρ_0 given by the fundamental dynamical curvature equation, Eq. (12), in the form

$$\rho_0 = \frac{v_0^2}{a_0 \sin(\alpha_0)}, \tag{46}$$

where the magnitude a_0 of the initial acceleration is taken to be equal to the initial value of the "centripetal force". It is clear NEWTON means that not only the magnitude, but also the direction of the initial velocity at the position vector $\vec{r}_0$ must be given, since only then can one evaluate $\sin(\alpha_0)$ from

$$\cos(\alpha_0) = \frac{\vec{r}_0 \cdot \vec{v}_0}{r_0 v_0}. \tag{47}$$

Hence, substituting the value of ρ_0 obtained from the basic equation of the curvature method, Eq. (46), in Eq. (44) and Eq. (45) one can obtain, explicitly, the orbital parameters L and A in terms of the initial values of $r_0, v_0, \sin(\alpha_0)$ and a_0,

$$\frac{1}{A} = \frac{2}{r_0} - \frac{v_0^2}{c} = \frac{-2E}{c}, \tag{48}$$

and

$$L = \frac{v_0^2 r_0^2 \sin^2(\alpha_0)}{c} = \frac{l^2}{c} \tag{49}$$

The geometrical construction in Proposition 1 justifies the uniqueness theorem at the end of this Corollary:

> and two [different] orbits, touching one the other, cannot be described by the same centripetal force and the same velocity.

This completes NEWTON's proof that conic sections are the only possible orbits for an inverse square force, as he announced at the start of this Corollary. The uniqueness theorem is also demonstrated by Proposition XLI, book I of the *Principia.* However, the solution of the orbital integral in Proposition XLI for $1/r^2$ was not mentioned by NEWTON in any of his three editions of the *Principia.* A direct solution was first published by JOHANN BERNOULLI in the *Memoire de l'Academie Royale des Sciences* 1710, pp 519–533. Recently there has been a lively debate in the literature, initiated by R. WEINSTOCK, concerning the question whether NEWTON actually proved in the *Principia* that given a central force $1/r^2$, the only orbits are conic sections. R. WEINSTOCK, *Dismantling a Centuries-Old Myth: Newton's Principia and Inverse Square Orbits*, American Journal of Physics **50**, 610–617 (1982). For further references, and a careful study of this question see G. H. POURCIAU, *On Newton's Proof that Inverse-Square Orbits Must be Conics*, Annals of Science **48**, 159–172 (1991); *Newton's Solution of the One-Body Problem*, Archive for History of Exact Sciences, vol. **44**, 125–146 (1992). M. NAUENBERG, *The Mathematical Principles Underlying the Principia Revisited* (to be published in the College Mathematics Journal).

40. Reference [19] pp. 246–256.
41. R. S. WESTFALL, *A note on Newton's demonstrations of motion in ellipses*, Archives Internationales d'Histories des Sciences, **22**, 52–60 (1969). For a recent discussion see, J. B. BRACKENRIDGE, *The Locke/Newton Manuscripts revisited: conjugates, curvatures, & conjectures* (to be published in Archives Internationales d'Histories des Sciences).
42. D. T. WHITESIDE, *The Preliminary Manuscripts for Newton's 1687 Principia, 1684–1685* (Cambridge University Press, 1989) p. xv. According to a memorandum which CONDUITT had from DEMOIVRE in November 1727,

> . . . in 1684 Dr. Halley made Sir Isaac a visit at Cambridge and there in a conversation the Dr. asked him what he thought the Curve would be that would be described by the Planets supposing the force of attraction towards the Sun to be reciprocal to the square of their distances from it. Sir Isaac replied immediately that it would be an Ellipsis. The Doctor struck with joy and amazement asked him how he knew it. Why saith he I have calculated it. Whereupon Dr. Halley asked him for his calculation without any further delay. Sir Isaac looked among his papers but could not find it, but he promised him to renew it, and then to send it to him. . .

A couple of months later NEWTON sent HALLEY the manuscript *De Motu.* Partly HALLEY's urging that he publish his results, NEWTON then spent the next two years in further work which culminated in his monumental *Principia.*

43. Reference [1] pp. 433–434.
44. Reference [1] pp. 300–303.
45. Reference [19] pp. 304–326.
46. D. T. WHITESIDE, *The Preliminary Manuscripts for Isaac Newton's 1687 Principia, 1684–1685* (Cambridge University Press, 1989) pp. 3–11.
47. M. NAUENBERG, *Hooke, Orbital Motion and the Principia* (to be published in the American Journal of Physics).
48. Reference [1] pp. 297–298.
49. P. J. PUGLIESE, *Robert Hooke and the dynamics of motion in a curved path*, in *Robert Hooke, New Studies*, edited by M. HUNTER and S. SCHAFFER, (Boydell Press, 1989) pp. 181–205
50. To avoid this lack of convergence, NEWTON could have applied an adaptive algorithm to iterations in the impulse method. For example, the time step δt could be taken proportional to the radial distance from the center.
51. The impulse equations correspond to a canonical or simplectic mapping of the coordinates $\vec{r}$ and $\vec{v}$. This is the consequence of the fact that these equations are the exact solutions of a Hamiltonian with impulsive forces. For a discussion of simplectic transformations in Hamiltonian mechanics, see, for example, V. I. ARNOL'D, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, 1984).
52. In Theorem 4 of *De Motu* NEWTON proves that

    Supposing that the centripetal force be reciprocally proportional to the square of the distance from the centre, the squares of the periodic times in ellipses are as the cubes of their transverse axes.

53. The first treatise on calculus was published by l'HOSPITAL, based on lectures by his tutor, JOHANN BERNOULLI, about a decade after the publication of the *Principia*, entitled *Analyse des Infiniment Petits, Pour l'intelligence des lignes courbes* (A Paris, de l'Imprimerie Royale, MD-CXCVI). An unpublished translation of this book into English (which I found among HOOKE's manuscripts in the Royal Society) was provided by CHARLES HAYES who subsequently incorporated it into a book entitled *A treatise on fluxions* (London 1704) (D. T. WHITESIDE, private communication). An exact translation of l'HOSPITAL's book was published in 1730 by E. STONE under the title *The Method of Fluxions both Direct and Inverse* (The Former being a translation from . . . , and the later supply'd by the Translator). NEWTON's original treatise of 1671 was first published by JOHN COLSON in English in 1736, in a book entitled *The Method of Fluxions and Infinite Series* with its Application to the Geometry of Curve-Lines, By the Inventor Sir Isaac Newton, Kt., Late president of the Royal Society. Translated from the AUTHOR's Latin Original not yet made publick. London, Printed by Henry Woodfall; M.DCC.XXXVI.

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There is evidence that two years after his correspondence with HOOKE, NEWTON continued to apply his early ideas on orbital dynamics. For example, in a letter of 1681 to CROMPTON (H. W. TURNBULL, The correspondence of Isaac Newton, vol. II, Cambridge University Press, 1960, p. 361) following some arguments with FLAMSTEED, NEWTON explains the correct theory of the orbit of a comet around the sun by "... the *vis centrifuga* at C [perihelion] overpow'ring the attraction & forcing the Comet there notwithstanding the attraction, to begin to recede from ye Sun" (compare with the quotation from letter to HOOKE on top of page 231). In an extract omitted from a letter sent on April 16, 1681 to FLAMSTEED (TURNBULL, vol. II, p. 366) he added that "... I think I have a way of determining ye line of a Comets motion (what ever that line be) almost to as great exactness as the orbit of ye Planets ...".