Newton's Principia and Inverse-Square Orbits

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For over a decade now, Robert Weinstock has been claiming "that no known proof [that an inverse-square force implies conic section orbits] comes from the mind or pen of Isaac Newton" [**1**]–[**4**]. In his original 1982 article, provocatively titled "Dismantling a centuries-old myth: Newton's *Principia* and inverse-square orbits" [**1**], and again in reference [**4**], he states explicitly that in Corollary 1 of Propositions 11–13 in Book I of the *Principia* [**5**], "Newton has [not] even outlined a proof" of this theorem. In this note I first demonstrate that several statements which Weinstock offers to support his claim are incorrect [**6**]. I then show that, contrary to Weinstock's assertion, this Corollary 1, *in conjunction with* other Propositions in the *Principia*, does give a proof that inverse-square forces lead to conic section orbits [**7**]–[**10**].

The discussion by Weinstock in his article "Credit where credit won't do" [**4**] repeats arguments presented in earlier papers, so I quote here from these works where appropriate.

Claim 1. In reference to Propositions 11–13, Weinstock claims that

> The conic section here described is, on the other hand, merely a *geometrical construct*; there is no evidence as yet that any particle actually traverses it [a conic section] as an orbit. This *must* be understood. [**1**]

Actually, what "must be understood" is that Props. 11–13 are special examples of Prop. 6, where Newton describes, in precise mathematical fashion, the construction of the *orbit* of a particle (its position as a function of time) on a given *planar* geometrical curve. I shall call these *Kepler orbits*. In Props. 11–13 the curve is a conic section (ellipse, hyperbola, and parabola, respectively) and the construction of Kepler orbits on these curves is worked out in complete detail in Props. 30 and 31. The principle for the construction is to apply Kepler's second law [**11**]: that in equal time intervals equal areas are swept by the position vector SP relative to a given fixed point S; see Figure 1. Thus, as a measure of the time elapsed for a particle traversing an arc AP of the curve, Newton uses the area ASP of a sector made up of this arc and the radial lines from the fixed point S to the ends of the endpoints A and P of the arc. The constant of proportionality between area and time is determined by the initial velocity and position of the particle. This constant

SECT. VI.

De inventione motuum in Orbibus datis.

Prop. XXX. Prob. XXII.

Corporis in data Trajectoria Parabolica moventis, invenire locum ad tempus aſſignatum.

Sit S umbilicus & A vertex principalis Parabolæ, ſitq; 4 AS x *M* area Parabolica APS, quæ radio SP, vel poſt exceſſum corporis de vertice deſcripta fuit, vel ante appulſum ejus ad verticem deſcribenda eſt. Innoteſcit area illa ex tempore ipſi proportionali. Biſeca AS in G, erigeq; perpendiculum GH æquale 3 *M*, & circulus centro H, intervallo HS deſcriptus ſecabit Parabolam in loco quæſito P. Nam demiſſa ad axem perpendiculari P*O*, eſt HG*q*. + GS*q*. (= HS*q* = G*Oq*. + $\overline{HG - POq.}$) = G*Oq*. + HG*q* − 2 HG x P*O* + P*Oq*. Et deleto utrinq; *HGq*. fiet GS*q*. = G*Oq*. − 2 HG x P*O* + P*Oq*. ſeu 2 *H*G x P*O* (= G*Oq*. + P*Oq*. − GS*q*. = A*Oq*. − 2 GA*O* + P*Oq*.) = A*Oq*. + ¾ P*Oq*. Pro A*Oq*. ſcribe A*O* x $\frac{POq}{4AS}$, & applicatis terminis omnibus ad 3 P*O*, ductiſq; in 2 AS, fiet ⁴⁄₃ G*H* x AS (= ⅙ AO x PO + ½ AS x PO = $\frac{AO + 3AS}{6}$ x PO = $\frac{4AO - 3SO}{6}$ x P*O* = areæ APO − SPO) = areæ APS. Sed *GH* erat 3 M, & inde ⁴⁄₃ *HG*

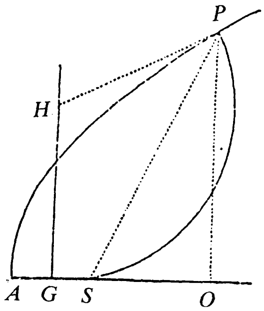

Figure 1

Discussion and diagram for parabolic motion, Proposition 30, Book I, *Principia* (1687 edition).

corresponds to twice the rate at which the area is swept, $l = vSY$ (l is also the angular momentum for unit mass), where v is the velocity of the particle when it is at P, and SY is the perpendicular distance from S to the tangent line to the curve at P. [For a short time interval δt the arc AP is approximated by a straight line segment with length $v\,\delta t$ and the sector ASP by a corresponding triangle with vertices at A, S, and P. Therefore the area of this sector is approximately $\frac{1}{2}(vSY)\,\delta t$.]

I discuss here the simplest case (Prop. 30), which is the construction of a Kepler orbit on a parabola with the focus S as the fixed point; see Figure 1. In Cartesian coordinates x and y, with the vertex A as the origin, P a point on the parabola, and O its projection on the x axis,

$$y^2 = Lx, \tag{1}$$

where $x = AO$, $y = PO$, and $L = 4AS$ is the latus rectum. The area of the sector APS swept by the radius vector SP is given by Newton in analytic form in the

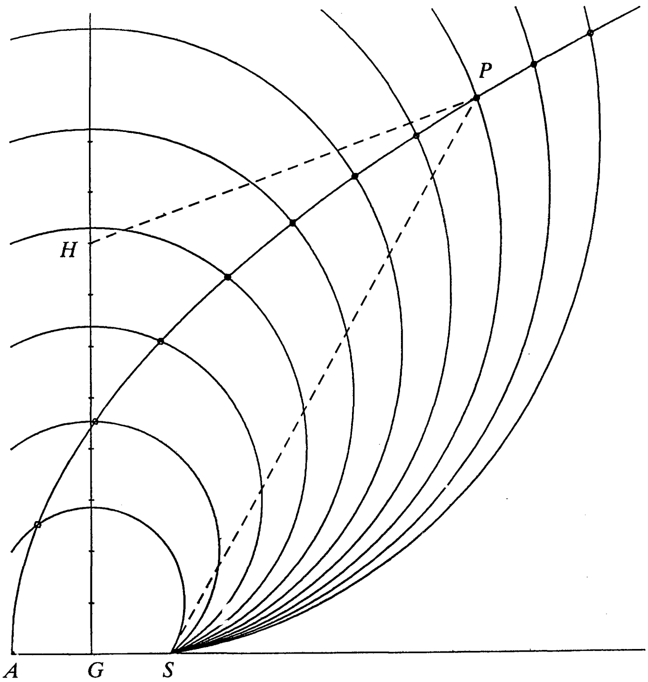

Figure 2
Graphical solution, according to Proposition 30, of the position (dots) of a particle on a parabolic orbit at 10 equal time intervals.

second line from the bottom of the text in Figure 1:

$$APS = \tfrac{1}{6}y\left(x + \tfrac{3}{4}L\right). \tag{2}$$

Newton also gives a construction to evaluate this area geometrically. Setting a perpendicular to the x-axis at the midpoint G between A and S, Newton shows that the point H on this axis which is equidistant from S and P is at a height GH proportional to the area of the sector APS,

$$GH = \frac{3}{L}APS = \frac{y}{2L}\left(x + \frac{3}{4}L\right) \tag{3}$$

Thus the location of the point P on the parabola at any "time" GH is readily obtained by drawing a circle centered at H with radius HS, and finding its intersection P with the parabola, solving (3) geometrically. Applying this construction, Figure 2 illustrates a Kepler orbit on a parabola, showing the location of a particle (marked by dots) at ten equal time intervals (indicated by tick marks on the time-axis GH). This refutes graphically Weinstock's claim #1 that "there is no evidence as yet that any particle actually traverses it [the parabola in this case] as an orbit" [**12**].

Claim 2. According to Weinstock,

> Nowhere in the *Principia* is it proved that there actually exists a central force law requiring a particle to move in a conic section with a focus as a center. [**1**]

On the contrary, in Prop. 2 Newton proves that

> Every body, that moves in *any* [italics mine] curved line described in a plane and by a radius drawn to a point either immovable, or moving forwards with an uniform rectilinear motion, describes about that point areas proportional to the times [Kepler orbit], is urged by a centripetal force directed to that point. [**13**]

Then in Prop. 6 Newton describes, in precise mathematical fashion, a geometrical construction to evaluate the *magnitude* of this centripetal (central) force (proportional to the acceleration) which is required to constrain a particle to move along the prescribed Kepler orbit [**14**]. Thus, Weinstock's claim #2 is also refuted.

The only question remaining at this point is whether there might be other curves besides conic sections, along which Kepler orbits under an inverse-square force could occur for certain initial conditions. In Prop. 17 Newton gives the initial conditions (position and velocity) for which the orbit is either an ellipse, a parabola, or a hyperbola. For example, for the case of a parabola, equation (1), he shows that the latus rectum L is given by

$$L = 2SP + 2KP, \tag{4}$$

where KP is a length defined in terms of the initial position and velocity vector. It can be readily shown that

$$KP = 2\frac{SY^2}{SP} - SP \tag{5}$$

where SY is the perpendicular distance from the focus to the tangent line at P. Since the (constant) rate at which the radius vector sweeps out area is proportional to $l = vSY$, where v is the speed at P, it follows from (4) and (5) that

$$\frac{1}{2}v^2 = \frac{l^2}{2SY^2} = \frac{cL}{4SY^2} = \frac{c}{SP}, \tag{6}$$

where $c = l^2/L$ is a constant (Prop. 16). This says the "total energy" $E = v^2/2 - c/SP = 0$, corresponding to parabolic motion for an inverse-square force. Similarly, Newton shows that for elliptic orbits $L < 2SP + 2KP$, corresponding to $E < 0$, and for hyperbolic orbits $L > 2SP + 2KP$, corresponding to $E > 0$.

In the first edition of the *Principia* (1687) Newton asserted without proof, in Corollary 1 of Props. 11–13, that

> From these three last Propositions it follows that if any body... is urged by the action of a centripetal force that is inversely proportional to the square of the distance of the places from the center, the body will move in one of the conic sections, having its focus in the center of force: and conversely.

Already in 1710 Johann Bernoulli publicly criticized this corollary, stating that [**15**] "M. Newton l'a supposé [une Section Conique], pg. 55, Corol. 1, sans le démontrer." In later recollections [**16**], Newton stated that

> The Demonstration of the first Corollary of the 11^{th}, 12^{th} & 13^{th} Propositions being *very obvious* [my italics], I omitted it in the first edition & instead contented my self with adding the 17^{th} proposition where it is proved that a body in going from any place with any velocity will in all cases describe a conic Section: which is that very Corollary... But at the desire of M^r Cotes I... .

As we have seen in the case of a parabola, Newton shows in Prop. 17 how to evaluate the latus rectum L [the single parameter that determines a parabola apart from orientation, equation (1)] in terms of initial conditions, "any place with any velocity," equation (4). Evidently at the urging of Roger Cotes, the editor of the second edition of the *Principia* (1713), Newton became aware that this corollary required clarification. Since the orbits in Props. 11–13 are conic sections, these three propositions cannot be invoked, *without further justification*, to rule out the existence of other possible orbits under the action of an inverse-square force corresponding to curves that are *not* conic sections. In other words, conic section orbits could turn out to be only a subclass of all the possible orbits that exist under the action of inverse-square forces [**10**].

In a letter dated October 11, 1709, Newton instructed Cotes [**17**] to add an addendum to Corollary 1, which in the third edition became (γ):

> For the focus, the point of contact, and the position of the tangent, being given, a conic section may be described, which at that point shall have a given curvature. But the curvature is given from the centripetal force and velocity of the body being given; and two orbits, touching one the other, cannot be described by the same centripetal force and the same velocity.

Claim 3. In Weinstock's view,

> What (γ) says in the context of its appearance is that *if* the inverse-square orbit is a conic section, then that conic is uniquely determined by initial conditions. [**3**] [**4**]

Weinstock fails to note that in the second edition of the *Principia* Newton introduced a second measure of force in Prop. 6, and added Corollary 4:

> The same things being supposed, the centripetal force is as the square of the velocity directly and that chord inversely.

Here the word "chord" corresponds to twice the component of the radius of curvature vector along the radial direction. In modern notation this corresponds to the relation (apart from a factor 2) for the component a_n of the acceleration normal to the orbit [**18**],

$$a_n = \frac{v^2}{\rho} \tag{7}$$

where v is the velocity and ρ is the radius of curvature [**19**]. Thus Newton provided an explicit expression to evaluate "the curvature [ρ]...from the centripetal force [a] and the velocity [v] of the body." This provided a new argument, independent of Prop. 17, that for arbitrary initial conditions, there is a Kepler orbit along a conic section of suitable curvature, meeting those conditions. The final clause of (γ) is, as Weinstock suggests, a uniqueness theorem. A proof of this theorem is readily demonstrated for polygonal orbits [**20**] by the geometrical constructions in Prop. 1, and it was assumed by Newton to be also valid in the limit of continuous orbits [**21**]. It is demonstrated independently in Prop. 41, which will be discussed below.

In conclusion, Newton proved that (a) the class of Kepler orbits along conic sections all have an inverse-square acceleration (Props. 11–13); (b) there is a member of the class satisfying each pair (position and velocity) of possible initial conditions [Prop. 17; addendum (γ) together with Corollary 4 of Proposition 6]; and (c) there is a unique orbit for each pair of initial conditions (Prop. 1; Prop. 41). This constitutes a valid proof that under an inverse-square force every orbit is a conic section. Weinstock's claim #3 that addendum (γ) *assumes* that an inverse-square orbit is a conic section is clearly incorrect.

Proposition 41. In Prop. 41, Newton develops a second, more straightforward method to obtain a general solution in integral form to the *inverse problem*, as it was called in the eighteenth century,

> Supposing a centripetal [central] force of any kind, and granting the quadratures [integrals] of curvilinear figures; it is required to find as well the curves in which bodies will move, as the times of their motion in the curves found.

Here Newton gives explicitly the standard integral formulas for the polar angle of the orbit and for the time along the orbit as a function of the radial distance. This proposition gives also a proof of the uniqueness theorem announced in Corollary 1 to Props. 11–13. Of course, Newton's notation is not that found in modern textbooks, because the *Principia* is written in a geometrical language (familiar to his contemporaries). Moreover, calculus textbooks [**22**] were not yet available in 1687. In a few lines in Prop. 41 he obtains the relation

$$A \times KN = \frac{Q \times IN}{\sqrt{ABFD - ZZ}}\,. \tag{8}$$

From the diagram accompanying Prop. 41 and the text, one finds that $A = r$, $KN = r\,\delta\theta$, $Q = r^2 \times \delta\theta/dt$, $IN = dr$, $Z = Q/A$, and $ABFD = 2(E - V(r))$, where r is the radial distance and θ is the polar angle. Hence (8) corresponds to the standard form found in modern textbooks of mechanics,

$$\frac{d\theta}{dr} = \frac{l}{r^2\sqrt{2(E - V(r) - l^2/2r^2)}} \tag{9}$$

where $l = Q$ is the conserved angular momentum, E is the conserved total energy, and $V(r)$ is the potential energy, $V(r) = -\int f(r) r\,dr$. For an inverse-square central force, $V(r) = -c/r$ [**23**]. Newton also gives an expression corresponding to $dt/dr = (r^2/l)\,d\theta/dr$ which by integration determines the time t as a function of the radius r.

Twenty-three years after the publication of the *Principia*, Johann Bernoulli obtained the solution for $V(r) = -c/r$ (inverse-square forces) by integrating (9) directly, with a change of variables [**15**]. He obtained this equation by effectively repeating Newton's derivation in Prop. 41, admitting that

> La démonstration de ce Lemme [Prop. 40] se trouve dans le Livre de M. Newton *De Princ. Math. Nat.* pag. 125,

but stating that

> les quadratures des espaces curvilignes étant donne & même plus commodément que M. Newton ne l'a trouvée dans la pag. 127. & c. de ses *Princ. Math.*

However, Newton had already discussed the solution of the inverse problem in Prop. 17 of the first edition [**6**] of the *Principia* (1687), and in the second edition (1713) he provided an addendum to Corollary 1 to Props. 11–13 which demonstrated again that conic sections are the unique orbits for inverse-square forces. Evidently, he did not think it was necessary to re-state his results a third time as a corollary [**24**] to Prop. 41. After all, he had developed (although not yet published) the fundamental theorem of calculus and he could have easily shown [**23**] [**25**] that conic sections were the solution of the integral given by equation (8). Instead, he chose to discuss in Corollary 3 of Prop. 41 the solution of (8) for the inverse-cube force, which he had not yet treated in complete generality earlier (in Prop. 9).

Conclusion. To summarize, in Prop. 1 Newton shows that the orbit under the action of a centripetal force obeys Kepler's area law, and in Prop. 2 he proves the converse, that the force on a particle that moves according to the area law is centripetal. The proofs of these two propositions are based on a model of forces corresponding to instantaneous impulses acting on a particle at equal time intervals. Such forces lead to polygonal orbits. It is clear that each orbit is then determined *uniquely* by the initial position and velocity vectors [**10**]. Newton describes a limiting process from discrete impulses and polygonal orbits to a continuous force and smooth curved orbits, as the time interval between impulses becomes vanishingly small [**21**]. He asserts that in this limit a discrete orbit (uniquely determined by initial position and velocity vectors) gives rise to a curved orbit corresponding to the action of a continuous force. Prop. 6 shows how to calculate the forces for a given orbit. It starts with a chosen *planar* curve and a fixed point in the plane, which in Props. 11–13 is one of the conic sections (ellipse, parabola, or hyperbola) with one focus as the fixed point. The motion of a particle on a given curve, the dynamical orbit, is *defined mathematically* by giving its position on the curve as a function of time, where time is measured by the area law. Hence the position of a point particle is determined purely *geometrically* by requiring that its motion on the curve satisfy the area law with respect to the chosen fixed point. This definition requires that there be no retrograde motion [**14**] along the given curve relative to this fixed point, a restriction that Newton did not discuss in the *Principia*. The initial conditions, position and velocity, determine the constant of proportionality between area change and time change (the angular momentum per unit mass) which completes the mathematical construction of a *unique* orbit. In particular, the *acceleration* vector associated with the motion of the particle is now completely determined by this geometrical construction. Prop. 2 demonstrates that the acceleration always points toward the chosen focus. In Props. 30 and 31, Newton explicitly solves analytically and geometrically for the dependence of time (area) on position, and vice versa, for the case of conic section curves (see Figures 1 and 2 for the parabolic orbit).

This is pure mathematics. Physics enters with the application of the second law of motion. Newton sets the resulting acceleration vector *proportional* to a given centripetal force vector (directed toward the chosen fixed point), which can therefore be assumed to act on the particle. The acceleration is then uniquely specified by the radial dependence of the given force. It is in this sense that the force is said to be the *cause* for the particle to move on its orbit. Prop. 6 gives a geometrical expression to calculate the magnitude of the acceleration of the body on its given orbit (apart from a missing constant of proportionality l^2 and a factor 2), and consequently the centripetal force acting on it. For the special case of a conic section with the force center at a focus, this force is shown, in Props. 11–13, to be inversely proportional to the square of the distance of the particle from the chosen focus.

In an addendum (γ) to Corollary 1 to Props. 11–13, Newton stated the geometrical proposition that given (a) a focus, (b) a point (initial position), (c) a tangent line at that point (initial velocity), and (d) the curvature at that point, a conic section is uniquely specified. In Corollary 4 of Prop. 6 the curvature is determined by the velocity and the component of the force along the radius of curvature vector, equation (7). Hence, for inverse-square forces each initial condition is satisfied by a conic section orbit, as is demonstrated also in Prop. 17. Finally the uniqueness theorem, that the pair of initial conditions determine a single orbit, is implied by Prop. 1 and explicitly demonstrated in Prop. 41. This completes

Newton's proof that conic sections are the only orbits for inverse-square forces. Moreover, in Prop. 41 Newton gave a general integral (quadrature) for the solution to the *inverse problem* for arbitrary central forces, which can be applied readily to the case of inverse-square forces. However, he chose not to include this case as a corollary to Prop. 41 in the *Principia* [**24**]. It should be noted that he had already worked out geometrically the time (area) dependence of conic section orbits in Props. 30–31. The analytic solution of Newton's equation (9), showing by direct integration that it corresponds to a conic section, was first published in 1710 by Johann Bernoulli [**15**]. It was historically important [**26**], but added little beyond simplifying the proof of the inverse problem.

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References

1. R. Weinstock, Dismantling a centuries-old myth: Newton's *Principia* and inverse-square orbits, *American Journal of Physics* 50 (1982) 610–617.
2. R. Weinstock, Long-buried dismantling of a centuries-old myth: Newton's *Principia* and inverse-square orbits, *American Journal of Physics* 57 (1989) 846–849.
3. R. Weinstock, Newton's *Principia* and inverse-square orbits: The flaw reexamined, *Historia Mathematica* 19 (1992) 60–70.
4. R. Weinstock, Issac Newton: Credit where credit won't do (in this issue).
5. I. Newton, *Philosophiae Naturalis Principia Mathematica*. The first (1687) edition has been reproduced in facsimile by William Dawson & Sons Ltd. The third edition (1726) has been translated into English by Andrew Motte and revised by Florian Cajori (University of California Press, 1934). For a historical background on the composition of the *Principia* see I. B. Cohen, *Introduction to Newton's Principia*, Cambridge University Press, 1971; and for comparisons between the different editions see A. Koyre and I. B. Cohen, *Isaac Newton's Philosophiae Naturalis Principia Mathematica*, Vols. 1 and 2, Harvard University Press, 1972.
6. Already in the earliest known draft of the *Principia*, known as *De Motu*, written in 1684, Newton discussed "Problem 4":

   > Given that the centripetal force is inversely proportional to the square of the distance, and knowing the magnitude of the force [it is] requireid to find the ellipse which a body describes when projected from a given point with given velocity in a straight line.

   In addition he considered also the initial conditions for which the orbit becomes a parabola, or a hyperbola. A revised version of Problem 4 became Prop. 17 in the *Principia*. See John Herivel, *The Background to Newton's Principia; A Study of Newton's Dynamical Researches in the Years 1664–1684*, Oxford, 1965, pp. 284–285.
7. B. Pourciau, On Newton's proof that inverse-square orbits must be conic, *Annals of Science* 48 (1990) 159–172; Newton's solution of the one-body problem, *Archive for History of Exact Sciences* (1992) 126–146. My arguments are similar to those presented in these papers.
8. H. Erlichson, Comment on "Long-buried dismantling of a centuries-old myth: Newton's *Principia* and inverse square orbits," by Robert Weinstock, *American Journal of Physics* 58 (1990) 882–884. Some of the arguments in this paper are similar to those in [**7**], and herein. However, Erlichson fails to notice that his statement that "any plane curve is an orbit for some central force, relative to an arbitrary force center..." is *not correct* if the curve and force center leads to the occurrence of retrograde motion; see [**14**]. Moreover, he states that "Newton needed Proposition 17 to show that the conic sections did indeed exhaust the possibilities for an inverse square force." However, after the addendum to Corollary 1 was included in latter editions of the *Principia* Newton no longer needed Prop. 17 for this purpose.
9. E. J. Aiton, The solution of the inverse-problem of central forces in Newton's *Principia*, *Archives Internationales d'Histoires des Sciences* 38 (1988) 271-276. Aiton makes the same mistake as Weinstock, stating that "before Newton can treat it as a physical orbit... he must first show that it

is possible for a body to move in a conic. . . . Newton does not in fact offer such a demonstration . . . . Newton regarded the result as obvious." This is not the case, as I discuss. However, Aiton concludes that Newton's "outline of a proof is nevertheless valid."

10. M. Nauenberg, The mathematical principles underlying Newton's *Principia* revisited. This forthcoming paper contains considerably more mathematical detail than could be included in the present survey article.
11. Johannes Kepler, *Astronomia Nova, 1609*; tr. William H. Donahue, *New Astronomy*, Cambridge University Press, 1992, pp. 577–579.
12. In Corollary 3 of Prop. 30 Newton also describes a geometrical construction to obtain the time, given the position P on the parabola. Since $HP = HA$, it follows that

    > the time may be found in which the body has described any assigned arc AP. Join AP, and on its middle point erect a perpendicular meeting the right line GH in H.

    In Prop. 31 Newton works out the more difficult problem of obtaining the dependence of time (area) on position for the case of the ellipse, and includes a geometrical and analytic method to solve the Kepler problem **[11]**.
13. Although in this proposition Newton considers only polygonal orbits with triangles of equal areas, in Prop. 1 he discusses the limit leading to a continuous curve when "those triangles be augmented and their breadth be diminished *in infinitum*."
14. In Props. 2 and 6 (or elsewhere in the *Principia*) Newton did not discuss the implication of motion on a general curve with *retrograde motion* with respect to a fixed point S. This occurs when the radial position vector becomes tangent to the curve at a *turning point* P, and then the angle QSP of a point Q moving past P changes sign. In this case an orbit on this curve cannot satisfy Kepler's area law past point P (the area must increase monotonically in time), and correspondingly there does not exist a central force that can account for this orbit. This may be regarded as a "small gap" in Newton's demonstrations in the *Principia*.
15. Extrait de la réponse de M. Bernoulli à M. Herman, dateé de Basle le 7. Octobre 1710, *Memoire de l'Academie Royale des Sciences*, pp. 519–533.
16. I. B. Cohen, *Introduction to Newton's Principia*, Cambridge University Press, 1971, p. 294.
17. H. W. Turnbull, *The Correspondence of Isaac Newton, 1709–1713*, Vol. 5, Cambridge University Press, 1975.
18. M. Nauenberg, Newton's early computational method for dynamics, *Archive for History of Exact Sciences* 46:3 (in press).
19. It is a curious fact that nowhere in the *Principia* can one find a definition of the concept of curvature which Newton developed from 1664 to 1671, but did not publish. This work first appeared in D. T. Whiteside, ed., *The Mathematical Papers of Isaac Newton 1670–1673*, Vol. 3, Cambridge University Press, 1969, pp. 169–173. The calculus of curvature was essential to Newton's early development of orbital dynamics. For further details see **[18]**, and J. B. Brackenridge, The critical role of curvature in Newton's developing dynamics, in P. M. Harman and A. E. Shapiro, eds., *The Investigation of Difficult Things: Essays on Newton and The History of the Exact Sciences in Honour of D. T. Whiteside*, Cambridge University Press, 1992, pp. 231–260.
20. M. Nauenberg, Hooke, orbital motion and Newton's *Principia*, *American Journal of Physics* 62:4 (1994). For the case of a centripetal force that varies linearly with the distance from the center, Hooke found graphically a polygonal approximation to an ellipse. Hooke evidently understood (as his diagram and corresponding text demonstrate) that the geometrical construction in Prop. 1 leads to a *unique* orbit, given the force law and the initial coordinate and velocity of the body. In this respect Hooke was ahead not only of his contemporaries, but of many modern scholars who do not seem to realize the uniqueness theorem implicit in the geometrical construction in Prop. 1.
21. A proof that the limit exists had to await a rigorous formulation of the calculus, which occurred only 150 to 200 years later. See C. B. Boyer, *The History of the Calculus and Its Conceptual Developments*, Dover, 1949. Arnol'd points out that Newton had shown that the solutions depend smoothly on the initial condition, which is sufficient to insure uniqueness in the continuum limit; see V. I. Arnol'd, *Huygens & Barrow, Newton & Hooke*, Birkhauser, 1990, pp. 32–33.
22. The first treatise on differential calculus, *Analyse des Infiniment Petits, Pour l'intelligence des lignes courbes*, was published by the Marquis de l'Hôpital in 1696. It was based on lectures by his tutor, Johann Bernoulli.

23. It is straightforward to *verify*, by taking a first-order derivative of the radius r with respect to the polar angle θ for the equation of a conic section in polar coordinates,

$$r = \frac{L_s}{1 + \epsilon \cos(\theta)}, \tag{10}$$

that the conic section is the *unique* solution of equation (9), with $L_s = l^2/c$ and $\epsilon = \sqrt{1 + 2El^2/c^2}$, apart from an arbitrary constant of integration determining the orientation of the axis of the conic section. The parameter L_s can also be determined by the "focus, point of contact... tangent... [and] curvature [ρ]" according to the relation [**18**]

$$L_s = \rho \sin^3(\alpha) \tag{11}$$

where α is the angle between the radius vector and the velocity vector.

24. Had Newton stated in a corollary to Prop. 41 that the quadratures for inverse-square forces yield conic sections, it is likely that some of the controversies, which continue to the present time, would not have occurred. Although this might have been an oversight in the first edition of the *Principia*, which Newton could have easily corrected in the second or third edition, I suspect that he chose not to add such a corollary to avoid having to acknowledge Bernoulli's result, [**15**].
25. D.T. Whiteside, ed., *The Mathematical Papers of Isaac Newton 1670–1673*, Cambridge University Press, 1969. A table of integrals related to conic sections appears in pp. 244–255.
26. C. Wilson, Newton's orbit problem: A historian's response (in this issue).

Robert Weinstock's Response to Nauenberg

The late-submitted fourth critique is little more than an effort to bestow validity upon Newton's purported proof that (isf) ⇒ (cso) as it appears in Props. 11–13 cum Corollary 1 in the third edition of *Principia* Book I. The effort, epitomized in a sequence {(a), (b), (c)} of three declarations led by "In conclusion, Newton proved that," appears early in the second half of the critique. I attend overbriefly below to some of the content of what precedes and follows it; here I examine the sequence itself so as to establish its failure to outline a valid proof that (isf) ⇒ (cso):

Declaration (a) is a paraphrase of what is accomplished in Props. 11–13 as presented in (α) of my article: to wit, a proof that *if* a particle moves in a "Kepler orbit," *then* it must undergo an inverse-square acceleration. (Nauenberg's alternative wording tends to mask the role of the given Kepler orbit *as hypothesis*.)

Decl. (b) is a restatement of the first sentence of (γ)—which my case against {(α),(β),(γ)} allows to pass without proof: i.e., without need to invoke, as Nauenberg does, Prop. 17 and Corollary 4 to Prop. 6.

Decl. (c) is a paraphrase of the second sentence of (γ)—which my case against {(α), (β), (γ)} also allows to pass without proof: i.e., without need to invoke, in the manner of Nauenberg, either Prop. 1 or Prop. 41.

It should be transparent, then, that whatever argument is present in {(a), (b), (c)} —of which Nauenberg states, "This constitutes a valid proof that under an inverse-square force every orbit is a conic section"—is the same as that of the content of {(α), (β), (γ)}. In order to refute Nauenberg's claim, therefore, I merely offer to the reader once again my several arguments under "Inverse-square orbits" that refute the validity of {(α), (β), (γ)} as proof that (isf) ⇒ (cso).

Ending his "In conclusion" paragraph, Nauenberg asserts that my "claim #3 that... (γ) *assumes* that an inverse-square orbit is a conic section is clearly incorrect." I have never made the claim he attributes to me. The reader can verify the fact by reexamining my article's characterization of (γ): "What (γ) says in the context of its appearance..."—or by reading, under "Claim 3" in his critique, Nauenberg's *initial* version of my characterization of (γ): identical to my own!

Of the three "claims" of mine enumerated by Nauenberg, neither #1 nor #2 can be found anywhere in my current article—despite his clear implication of the contrary. Only #3 (in its *original* statement by Nauenberg) appears therein; but what is attacked by him is his gross distortion of it, as seen directly above.

No reader should be misled by any of the several additional misrepresentations employed in Nauenberg's critique in its effort to discredit my case against the *Principia*'s purported proof that (isf) $\Rightarrow$ (cso)—of which I cite a subset:

Nauenberg is wrong to assert—in his attempt to refute my (1982) "Claim 1"—that in Prop. 6 Newton describes, in *any* fashion, "the construction of the *orbit* of a particle." Prop. 6 establishes merely that *if* a body moves along a *given* arc under the sole influence of a force directed toward a particular point, *then* that force must be proportional to a definite ratio of certain geometric quantities related to the arc. (In his long antepenultimate paragraph ("To summarize..."), Nauenberg writes, surprisingly, "Prop. 6 shows how to calculate the forces for a given orbit"—an essentially *correct* representation of the proposition!)

In his attempt to demolish my (1982) "Claim 2," Nauenberg offers Prop. 2 as proving the existence of "a central force law requiring a particle to move in a conic section... ." Yet Prop. 2 uses *as hypothesis* motion along a *given* curve; for it can, without change of meaning, be reworded: "If a body moves along any curve..., and if it describes...areas proportional to the times, then it is urged... ." His use of Prop. 6 for the same demolition purpose exhibits, as shown in the preceding paragraph, the very flaw immanent in his use of Prop. 2.

In quoting—without contradiction—Newton's late-in-life reference to Prop. 17, Nauenberg encourages propagation of a long-enduring misrepresentation: namely, in the words quoted, "the 17^{th} proposition where it is proved that a body in going from any place with any velocity [under the sole influence of an inverse-square central attraction] will in all cases describe a conic Section." This spurious representation of Prop. 17 has had a fascinating career, in which not only Newton, but also L. Euler and I. B. Cohen, for example, are incriminated. For details and references, see my reference [**33**].

The offer tendered above relative to the three earlier critiques of my article applies also to the fourth: To receive a relatively complete reply to all *four* critics, each interested reader is invited to apply to me by mail; I shall endeavor to respond promptly.

Another County Heard From

The view of science that [Newton] exhibits, the clarity with which he writes, the number of new things he finds, manifest a physical and mathematical insight of which there is no parallel in science at any time.

S. Chandrasekhar, *Scientific American*, March 1994, p. 33.