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## Proposition 10, Book 2, in the *Principia*, revisited

Michael Nauenberg

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**Abstract** In Proposition 10, Book 2 of the *Principia*, Newton applied his geometrical calculus and power series expansion, to calculate motion in a resistive medium under the action of gravity. In the first edition of the *Principia*, however, he made an error in his treatment which lead to a faulty solution that was noticed by Johann Bernoulli and communicated to him while the second edition was already at the printer. This episode has been discussed in the past, and the source of Newton's initial error, which Bernoulli was unable to find, has been clarified by Lagrange and is reviewed here. But there are also problems in Newton corrected version in the second edition of the *Principia* that have been ignored in the past, which are discussed in detail here.

### 1 Introduction

In his guide to Newton's *Principia*, I. B. Cohen writes that "anyone studying the history of the *Principia* will find Book 2, Proposition 10 to be of special interest. The problem is to find the density of the medium that makes a body move in any given curve under the supposition that gravity is uniform and of constant direction, and that the resistance of the medium varies jointly as the density of that medium and the square of the velocity. . . The proposition is notable, among other things, for a display of Newtonian fluxions (or "moments")" (Cohen 1999, 167). In an unpublished preface to the *Principia*, written after the publication of the second edition, and during his priority dispute with Leibniz on the development of the calculus (Hall 1980),

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20 Newton called attention to this proposition and related ones,<sup>1</sup> to give explicit examples  
 21 where he had applied his method of analysis.<sup>2</sup> It turns out, however, that his approach  
 22 was solely based on his geometrical approach to the calculus and on his power series  
 23 expansion, without any reference to fluxions.<sup>3</sup> Moreover, in the first edition of the  
 24 *Principia*, he made a mistake in this proposition, which came back to haunt him dur-  
 25 ing this dispute. On September 1712, Newton received Niklaus Bernoulli, who came  
 26 with some upsetting news from his uncle Johann Bernoulli. The elder Bernoulli had  
 27 found a “serious error” in the first example that Newton gave for Proposition 10, Book  
 28 2 in the case that the motion occurred along a circular path. For this case, Newton’s had  
 29 found that the component of gravity along the motion exactly balanced the resistance,  
 30 which implied that the velocity is a constant. But this result is inconsistent with his  
 31 calculation that the velocity decreases as the square root of the vertical distance along  
 32 the circular path. Both Bernoullis obtained the correct solution to this problem by  
 33 applying the differential calculus of Leibniz, but they concluded that Newton’s failure  
 34 demonstrated that he had not mastered the application of higher-order differentials  
 35 in his fluxional calculus. They were unable, however, to find the source of Newton’s  
 36 error which later was pointed out by Lagrange (1797) who showed that Newton had  
 37 treated incorrectly the third-order differentials that appeared in his approach.

38 After frantically working on this problem, Newton was able to find the correct solu-  
 39 tion by modifying the diagram that he had used in his original approach. Although  
 40 the second edition of the *Principia* was already in the printers hand, he managed  
 41 to insert the revised solution, but he failed to acknowledge Bernoulli’s contribution  
 42 which offended the latter. Later on, this episode played a significant role in the pri-  
 43 ority dispute that emerged between Newton and Leibniz on the discovery of the calcu-  
 44 lus, and Johann Bernoulli argued, anonymously, that the failure of Proposition 10,  
 45 Book 2, demonstrated that at the time Newton wrote the first edition of the *Principia* he  
 46 had not yet mastered the differential calculus. In a letter printed in the *Acta Eruditorum*  
 47 for July 1716, Bernoulli wrote

48 It is firmly established that Newton at the time he wrote his *Principia Philos.*  
 49 *Mathematica*, still had not understood the method of differentiating differentials  
 50 (Bernoulli 1767)

51 It will be shown that there is some validity in Bernoulli’s statement, and that even in  
 52 the revised version of Proposition 10, Book 2, Newton continued to make errors in the

<sup>1</sup> For example, in Propositions 15–17, Book 2, Newton considered the effect of resistance on the motion of a body under the action of inverse square force (Brackenridge and Nauenberg 2002).

<sup>2</sup> For a detail discussion of Newton’s mathematical methods see (Guicciardini 2009).

<sup>3</sup> In a letter to Keill, “who was puzzled to know what Bernoulli was getting at” in his contention that Newton had not mastered the differential calculus, Newton wrote

It appears thereby that I did not understand  $y^e$  2nd fluxions when I wrote that Scholium [i.e. to Proposition 10] because (as he thinks) I take the second terms of the series of the first fluxions, the third terms for the second fluxions & so on. But he [Bernoulli] is mightily mistaken when he thinks that I here make use of the method of fluxions. Tis only a branch of  $y^e$  method of converging series that I there make uses of (Hall 1958, 296).

53 application of third-order differentials, but fortuitously these errors did not prevent  
54 him from obtaining the correct solution to this proposition.

55 In a section of his textbook *Théorie des Fonctions Analytiques*, Lagrange (1797)  
56 presented an analysis of Proposition 10, Book 2, applying Leibniz's differential calcul-  
57 ulus to describe the source of Newton's error. He concluded that

58 We believe that it was not useless to show how the method of series is applied,  
59 and that it would be granted to us to shed light at the same time on a point of  
60 analysis on which the greatest geometers made mistakes, and which may be of  
61 interest to the history of the birth of the new calculus.<sup>4</sup>

62 Newton's problems with Proposition 10, Book 2, illustrate some of the limitations of  
63 Newton's geometrical-fluxional method, and some of the advantages of the algorithmic  
64 approach of Leibniz.

65 In Proposition 6, Book 1, Newton gave a mathematical description of the motion  
66 of a body under the influence of a central force. For a small but finite interval of time  
67  $t$ , he postulated that the motion can be composed of an inertial or constant velocity  
68 contribution along the tangent of the orbit proportional to  $t$  and an accelerated motion  
69 along the instantaneous direction of the force proportional to  $t^2$ . It is important to  
70 recognize, however, that in a series expansion in powers of  $t$ , Newton's composition  
71 is valid only up to quadratic terms in  $t$ . In Proposition 10, Book 2, Newton considered  
72 an additional force due to the resistance of the medium acting along the tangent of the  
73 orbit. In this case, the motion on a line element along tangent is not inertial leading to  
74 the complications in his treatment of this problem discussed here.

75 In Sect. 2, Proposition 10, Book 2, is presented in the form given in the first edition  
76 of the *Principia*, and the source of Newton's error is described. In Sect. 3, the revised  
77 version of this proposition in the second edition of the *Principia* is discussed follow-  
78 ing closely Newton's verbal description of this proposition, but applying his fluxional  
79 calculus in a novel form that avoids his inconsistent treatment of cubic terms in his  
80 power expansions. In "Appendix I", the relation of Proposition 6, Book 1, to Propo-  
81 sition 10, Book 2, is presented, and following Lagrange's work (Lagrange 1797), in  
82 "Appendix II" Newton's missing term in cubic order in powers of the time interval  $t$   
83 is derived.

## 84 **2 Proposition 10, Book 2, in the first edition of the *Principia***

85 This section describes Newton's discussion of motion in a resisting medium under the  
86 action of a constant gravitational force given in Proposition 10, Book 2 of the first  
87 edition of the *Principia*, when the geometrical curve for the trajectory is given, and  
88 it explains why this description is flawed along the lines first discussed by Lagrange  
89 (1797).

<sup>4</sup> Nous avons cru qu'il n'était pas inutile de faire voir comment la méthode des séries pouvait y conduire, et qu'on nous saurait gré d'éclaircir, en même temps, un point d'analyse sur lequel les plus grands géomètres s'étoient trompés, et qui peut intéresser l'histoire de la naissance des nouveaux calculs (Lagrange 1797, 251).

90 Referring to the curve  $ACK$  and tangent line  $hCH$  at  $C$ , Newton started:

91 let  $CH$  and  $Ch$  be equal rectilinear lengths which bodies moving away from  
92 place  $C$  would describe in these times [equal time intervals  $t$ ] without the action  
93 of the medium or gravity (Cohen 1999, 655).

94 Without the action of the medium or gravity, a body moves with constant velocity, and  
95 therefore during an interval of time  $t$

$$96 \quad CH = Ch = vt, \quad (1)$$

97 where  $v$  is the velocity along the tangent line at  $C$ . Newton continued:

98 Through the resistance of the medium it comes about that the body as it moves  
99 forward describes instead of length  $CH$ , only length  $CF$ , and through the force of  
100 of gravity the body is transferred from  $F$  to  $G$ , and thus line element  $HF$  and  
101 line element  $FG$  are generated simultaneously, the first by the force of resistance  
102 and the second by the force of gravity (Cohen 1999, 656).

103 Let  $g$  be the acceleration of gravity, acting along the vertical direction, and  $r$  de-accel-  
104 eration due to the resistance of the medium acting along the tangent of the motion.  
105 Then, Newton asserted that

$$106 \quad HF = CH - CF = (1/2)rt^2, \quad (2)$$

$$107 \quad FG = IG - IF = (1/2)gt^2, \quad (3)$$

108 and concluded

109 And hence the resistance comes to be as  $HF$  directly and  $FG$  inversely, or as  
110  $HF/FG$ .

111 Indeed, according to the Eqs. 2 and 3

$$112 \quad HF/FG = r/g \quad (4)$$

113 Newton's decomposition of motion into a tangential and vertical component can  
114 be verified from the solution of the differential equations of motion in Cartesian coor-  
115 dinates  $x$ ,  $y$  to second order in the time interval  $t$ . We have

$$116 \quad \ddot{x} = -r\dot{x}/v \quad (5)$$

$$117 \quad \ddot{y} = -r\dot{y}/v - g \quad (6)$$

118 where  $v = \sqrt{\dot{x}^2 + \dot{y}^2}$  is the velocity,  $r$  is the resistance,<sup>5</sup> and  $g$  is the gravitational  
119 constant. Taking the origin at  $O$ , the Taylor series expansion of the solution of  
120 Eqs. 5 and 6 to second order in  $t$  is

<sup>5</sup> In the second edition of the *Principia* Newton assumed that  $r$  is proportional to  $v^2$ , but this condition is not relevant to his derivation.

$$x = OB + (vt - (1/2)rt^2) \cos(\alpha), \tag{7}$$

and

$$y = BC - (vt - (1/2)rt^2) \sin(\alpha) - (1/2)gt^2 \tag{8}$$

where  $v$  is the velocity at  $C$ , and  $\tan(\alpha) = -\dot{y}/\dot{x}$ . The angle  $\alpha$  gives the inclination of the tangent line at  $C$  relative to the horizontal, i.e.  $\tan(\alpha) = IF/CI$ . These two relations demonstrate the validity, to second order in the time interval  $t$ , of Newton's decomposition of the motion along the arc  $CG$ , namely, a component along the tangential line

$$CF = vt - (1/2)rt^2, \tag{9}$$

where  $CH = vt$ , and another vertical component

$$FG = (1/2)gt^2, \tag{10}$$

These two relations, however, cannot be applied to determine the ratio  $r/g$  from Newton's relation, Eq. 4, because although the line element  $FG$  is determined by the geometry of the curve  $AGK$ , this is not the case for the line element  $HF$ . Therefore, Newton proceeded to obtain a second expression for  $r/g$ . But it will be shown that the resulting expression is not valid because it depended on quantities that are of cubic order in  $t$  that could not be calculated from Newton's geometrical construction.

For this purpose, Newton considered the *time-reversed* motion in which case the medium *accelerates* the body along the tangent. Taking a segment  $Ck$  of the line element  $Cf$  to be equal to  $CF$ ,

$$Ck = vt' + (1/2)rt'^2 = vt - (1/2)rt^2 \tag{11}$$

where  $t'$  is the time interval to traverse the line element  $Ck$  in the absence of gravity. Hence, to *second order* in powers of  $t$ .

$$t' - t = -(r/v)t^2, \tag{12}$$

and the vertical line element  $kl$  is then

$$kl = (1/2)gt'^2. \tag{13}$$

It follows that

$$FG - kl \approx g(t - t')t = (gr/v)t^3. \tag{14}$$

It is clear, however, that this relation cannot be valid, because Newton had determined the line elements  $FG$ , Eq. 10, and  $kl$ , Eq. 13, only up to second order in  $t$ , while the



167 Likewise, the ordinate  $il$  is

$$168 \quad il = P + Qo - Ro^2 + So^3, \quad (19)$$

169 and

$$170 \quad ik = P + Qo. \quad (20)$$

171 Hence

$$172 \quad kl = ik - il = Ro^2 - So^3, \quad (21)$$

173 and

$$174 \quad FG - kl = 2So^3. \quad (22)$$

175 Finally, substituting in Newton's relation, Eq. 15,  $CF = o\sqrt{1 + Q^2}$  and  $FG^2 = R^2o^4$ ,  
 176 one obtains

$$177 \quad r/g = (S/2R^2)\sqrt{1 + Q^2}. \quad (23)$$

178 For the first application of this relation, Newton consider the case that the given  
 179 curve  $ACK$ , Fig. 1, is a semicircle of radius  $n$ . Introducing the symbol  $a$  for the abscissa  
 180 and  $e$  for the ordinate of this curve, he obtained  $\sqrt{1 + Q^2} = n/e$ ,  $R = n^2/2e^3$ , and  
 181  $S = an^2/2e^3$ , which

$$182 \quad r/g = a/n. \quad (24)$$

183 Since the tangential component of gravity is  $ga/n$ , Newton's result implied that the  
 184 total tangential acceleration  $r - ga/n$  vanishes. Hence, the velocity must be a *constant*,  
 185 but he found, on the other hand, that the velocity varied with the square root of the  
 186 ordinate of the semicircle:

187 departing from  $C$  along the straight line  $CF$ , could subsequently move in a  
 188 parabola having diameter  $CB$  and latus rectum  $(1 + Q^2)/R \dots$  (Cohen 1999,  
 189 660)

190 For parabolic motion, the velocity  $v$  is given by  $v^2 = ga/2$ , where  $a$  is the latus  
 191 rectum, and therefore  $v = \sqrt{(g/2R)(1 + Q^2)}$  which implies in this case that

$$192 \quad v = \sqrt{ge}. \quad (25)$$

193 This relation for the velocity turns out to be correct, but it contradicts Newton's previ-  
 194 ous result, Eq. 24, that the acceleration due to gravity along a circular path is exactly  
 195 compensated by the de-acceleration due to the resistance of the medium. Evidently  
 196 this contradiction escaped Newton's attention, but it was noticed by Johann Bernoulli  
 197 when he read Proposition 10, Book 2.

In Theorem VI of his paper in the *Mémoires* of 1711 and the *Acta Eruditorum* of 1713, Johann Bernoulli wrote Hall (1958)

In order that a body moving in a resisting medium may describe a circle  $LCK$ , on the supposition that a uniform gravity tends directly towards the horizontal, I say that the resistance in each point  $C$  will be to the gravity as  $3OG$  to  $2OK$

Here  $OK$  is the radius  $n$  of the circle, and  $OG$  is the abscissa  $a$  in Newton's notation. Hence Bernoulli's correct result<sup>7</sup> for the ratio of resistance to gravity,  $r/g = (3/2)OG/OK$ , differs from Newton's, Eq. 24, by a factor  $3/2$ .

Newton's analysis is flawed because the *difference* in the Galilean time intervals  $t$  and  $t'$  to fall under the action of gravity along the line elements  $FG$  and  $kl$ , respectively, depends on the third power of  $t$ , but he could calculate these line elements only up to second power of  $t$ . Applying the differential equations of motion, Eq. 5 and 6 to obtain the power series expansion of  $x$ ,  $y$  to cubic order in  $t$ , first carried out by Lagrange (1797), yields (see "Appendix II")

$$FG = (1/2)gt^2 - (1/6)(rg/v)t^3. \quad (26)$$

Correspondingly

$$kl = (1/2)gt^2 + (1/6)(rg/v)t^3, \quad (27)$$

which together with Eq. 12, now gives the correct value

$$FG - kl = (2/3)(gr/v)t^3. \quad (28)$$

valid to third order in  $t$ . Hence, instead of Newton's relation for  $r/g$ , Eq. 15, we have

$$r/g = (3/2)(FG - kl)CF/4FG^2 \quad (29)$$

and substituting  $FG - kl = 2So^3$ , Eq. 22,  $CF = o\sqrt{1 + Q^2}$  and  $FG^2 = R^2o^4$  yields the correct value<sup>8</sup>

$$r/g = (3S/4R^2)\sqrt{1 + Q^2}. \quad (30)$$

<sup>7</sup> In a scholium following his theorem, Bernoulli criticized Newton's result as follows:

Lest any one who is unable to examine these matters more deeply should wonder whether perhaps we were mistaken in confuting what has been disclosed with so much labour by this most acute man, I will demonstrate here that this Newtonian ratio leads to a contradiction. For if the resistance be to the gravity as  $OG$  to  $OK$  as Newton has it, then since the gravity itself is to the tangential force as  $OC$  or  $OK$  to  $OG$ , equally the resistance would be to the tangential or motive force as  $OG$  to  $OG$ , therefore the resistance will be equal to the motive force, and the velocity at any point  $C$  uniform, whereas we previously showed it to be  $\sqrt{CG}$ , and consequently non-uniform, as Newton himself agrees (Hall 1958).

<sup>8</sup> Johann Bernoulli suggested that Newton obtained the wrong result for the ratio  $r/g$ , 23, because he had erroneously assumed that  $R = -d^2y/dx^2$ , and  $S = -d^3y/dx^3$  (Whiteside 1981). But Bernoulli's suggestion is incorrect, and it is a coincidence that replacing in Eq. 23,  $2R$  for  $R$ , and  $6S$  for  $S$ , yields the missing factor  $3/2$ .

222 Newton, however, could not have obtained this result, which differs from his result  
 223 by an extra factor  $3/2$ , because it required the expansion of line elements to third order  
 224 in the time interval  $t$ . But he could not have calculated the coefficient of this term from  
 225 his geometrical—fluxional approach, which was confined to the second order in  $t$ .

226 After obtaining his first relation for  $r/g$ , Eq. 4, Newton warned the reader that

227 This is so in the case of nascent line elements. For in the case of line elements  
 228 of finite magnitude these ratios are not accurate (Cohen 1999, 656).

229 But then he proceeded to ignore his own warning, and derived a second relation for  
 230  $r/g$ , Eq. 23, that is incorrect and leads to a result for  $r/g$  that misses a factor  $3/2$ .

231 **3 Proposition 10, Book 2, in the second and third edition of the *Principia***

232 After receiving Niklaus Bernoulli warning from his uncle Johann Bernoulli that his  
 233 result in Proposition 10 in Book 2 for motion in a medium under the action of gravity  
 234 led to an inconsistent result for circular motion, Newton approached the problem in a  
 235 different manner which he managed to insert, in the last moment, in the second edition  
 236 of the *Principia*. He abandoned his previous approach based on time-reversed motion,  
 237 and instead, he now determined the required differential changes in line elements and  
 238 arcs at two adjacent points on a given curve describing the motion of a body moving  
 239 only forward in time. He described his new diagram, Fig. 2, as follows:

240 Let  $PQ$  be the plane of the horizon perpendicular to the plane of the figure;  
 241  $PFHQ$  a curved line meeting this plane in points  $P$  and  $Q$ ;  $G, H, I$ , and  $k$  four  
 242 places of the body as it goes in the curve from  $F$  to  $Q$ ; and  $GB, HC, ID$  and  
 243  $KE$  four parallel ordinates dropped from these points to the horizon and standing  
 244 upon the horizontal line  $PQ$  at points  $B, C, D$ , and  $E$ ; and let  $BC, CD$  and  
 245  $DE$  be distances between the ordinates equal to one another. From the points  $G$   
 246 and  $H$  draw the straight lines  $GL$  and  $HN$  touching the curve at  $G$  and  $H$  and

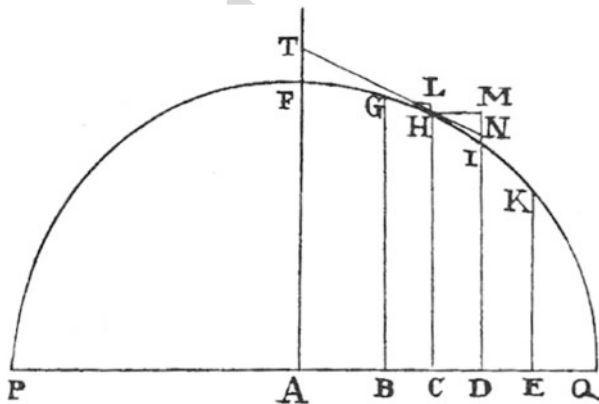


Fig. 2

247 meeting at  $L$  and  $N$  the ordinates  $CH$  and  $DI$  produced upwards and complete  
 248 the parallelogram  $HCDM$  (Cohen 1999, 655).

249 Newton proceeded again to describe analytically the arcs and the line elements  
 250 associated with the curve in his new diagram by expanding locally the ordinates, sep-  
 251 arated equally by a small distance  $o$ , in a power series up to *cubic order* in  $o$ . But he  
 252 continued to assert, as before, that the different time intervals associated with these  
 253 space intervals are proportional to the square root of the deviations from tangential  
 254 motion due to gravity, although this proportionality is correct only to *first order* in  $o$ .

255 Then the times  $[T, t]$  in which the body describes arcs  $GH$  and  $HI$  will be as  
 256 the square roots of the distances  $LH$  and  $NI$  which the body would describe in  
 257 those times by falling from the tangents (Cohen 1999, 656).

258 The “distances”  $LH$  and  $NI$  replaced the analogous line elements  $FG$  and  $kl$  in his  
 259 previous diagram, Fig. 1, but again Newton set these line elements proportional to the  
 260 square of the time intervals  $t$  and  $T$ , although his expansion of these line elements  
 261 in powers of the space interval  $o$  required also cubic terms. Moreover, Newton also  
 262 assumed that these distances were proportional to the corresponding sagittas associ-  
 263 ated with the description of this curve, that he evaluated to cubic order in  $o$ . But this  
 264 equivalence<sup>9</sup> is valid only up to second order in  $o$ .

265 Therefore, the question arises why, in spite of requiring the expansion of line ele-  
 266 ments to cubic powers in  $o$ , Newton nevertheless succeed in obtained the correct  
 267 expression for the resistance of the medium. It will be shown that the answer to this  
 268 question is that Newton’s geometrical descriptions can be implemented by restricting  
 269 the calculation of line elements to power series expansions up to *second order* in  $o$ ,  
 270 and that Newton’s application of cubic terms in  $o$ , erroneously linked in this order to  
 271 his time intervals  $t$  and  $T$ , fortuitously did not affect his calculations.

272 Referring to his diagram, Fig. 2, Newton explained that

273 the velocities will be directly as  $GH$  and  $HI$  and inversely as the times. Represent  
 274 the times by  $T$  and  $t$ , and the velocities by  $GH/T$  and  $HI/t$ , and the decrement  
 275 of the velocity occurring in time  $t$  will be represented by  $GH/T - HI/t$  (Cohen  
 276 1999, 656).

277 The ratios  $GH/T$  and  $HI/t$ , however, are the *mean* velocities over the arcs  $GH$  and  
 278  $HI$ , while for his calculation, Newton required only the *instantaneous velocities* at  
 279  $G$  and  $H$ . These velocities are given by the ratios  $GL/T$  and  $HN/t$ , respectively.  
 280 Expanding the ordinate  $DI$  in a power series to cubic order in  $o$

$$281 \quad DI = P - Qo - Ro^2 - So^3, \quad (31)$$

<sup>9</sup> For motion under the action of a central force, Newton described in Proposition 1, Book 1 of the *Principia* the deviation from inertial motion by the sagitta of small arcs of a curve that can be associated with this proposition, while in Proposition 6, Book 1, the corresponding deviation was described by a small line element from the tangent to the orbit. But these two deviations are equal, only to *second order* in a series expansion in powers of the differential time interval associated with these two deviations.

282 where  $P = CH$ , and the coefficients  $Q$ ,  $R$  and  $S$  define the nature of the curve at  $H$ ,

283 
$$HN = o\sqrt{1 + Q^2}, \tag{32}$$

284 and

285 
$$NI = Ro^2 + So^3. \tag{33}$$

286 Up to first order in  $o$ , we have

287 
$$t = \sqrt{(2/g)NI} = o\sqrt{(2R/g)}, \tag{34}$$

288 where  $g$  is the constant of gravity, and the velocity at  $H$  is<sup>10</sup>

289 
$$v = HN/t = \sqrt{(g/2R)(1 + Q^2)}. \tag{35}$$

290 Newton obtained this result which he phrased as follows:

291 And the velocity is that with which a body going forth from any place  $H$  along  
 292 the tangent  $HN$  can then move in a vacuum in a parabola having a diameter<sup>11</sup>  
 293  $HC$  and a latus rectum  $HN^2/NI$  or  $(1 + Q^2/R)$  (Cohen 1999, 658).

294 Likewise, the velocity at the adjacent point  $G$  is

295 
$$v' = GL/T = \sqrt{(g/2R')(1 + Q'^2)} \tag{36}$$

296 where, to first order in  $o$ , the coefficients  $Q'$  and  $R'$  are

297 
$$Q' = Q - 2Ro, \tag{37}$$

298 
$$R' = R - 3So, \tag{38}$$

299 Newton wrote:

300 and the decrement of the velocity occurring in time  $t$  will be represented by  
 301  $GH/T - HI/t$  (Cohen 1999, 657).

<sup>10</sup> Newton had shown that the square of the velocity satisfies the relation  $v^2/\rho = g_n$ , where  $\rho$  is the radius of curvature at  $H$ , and  $g_n$  is the component of gravity normal to the tangent (Proposition 6, Corollary 3). Moreover, he had obtained the general mathematical expression for  $\rho$  in Cartesian coordinates,  $\rho = (1 + Q^2)^{3/2}/2R$ , and  $g_n = g \cos(\alpha)$ , where  $\alpha$  is the angle of the tangent line  $HN$  with the abscissa, i.e.  $\cos(\alpha) = 1/\sqrt{1 + Q^2}$ . These relations lead to an alternative derivation of Eq. 35.

<sup>11</sup> The latus rectum for parabolic motion in a vacuum under the action of gravity is  $2v^2/g$ , and setting its value equal to  $(1 + Q^2)/R$  leads to Eq. 35. But the diameter of this parabola is not  $HC$ , but is directed, instead, along the line from  $H$  perpendicular to the tangent line  $HN$ .

302 Actually, this decrement of the velocity can be obtained more readily by replacing the  
 303 arcs  $GH$  and  $HI$  by the tangent lines  $GL$  and  $HN$ , respectively.<sup>12</sup> Hence, applying  
 304 Eqs. 35, 36, 37 and 38, one obtains.<sup>13</sup>

$$305 \quad v' - v = o\sqrt{2gR} \left[ -Q/\sqrt{(1+Q^2)} + (3S/4R^2)\sqrt{(1+Q^2)} \right]. \quad (39)$$

306 where  $v' - v = GL/T - HN/t$

307 This is an example of the “the method of differentiating differentials”, which  
 308 Johann Bernoulli late asserted, that Newton had not understood (Bernoulli 1676).  
 309 Instead, Newton continued as follows:

310 This decrement arises from the resistance retarding the body and from the grav-  
 311 ity accelerating the body . . . but in a body describing arc  $HI$ , gravity increases  
 312 the arc by only the length  $HI - HN$  or  $MI \times NI/HI$ , and thus generates only  
 313 the velocity  $2MI \times NI/t \times HI$ , (Cohen 1999, 657).

314 Substituting

$$315 \quad MI/HI \approx MN/HN = Q/\sqrt{1+Q^2}, \quad (40)$$

316 and

$$317 \quad NI/t = o\sqrt{gR/2} \quad (41)$$

318 one obtains, to first order in  $o$ ,

$$319 \quad 2MI \times NI/t \times HI = o\sqrt{2gR} \left( Q/\sqrt{(1+Q^2)} \right), \quad (42)$$

320 which corresponds to the first term on the right hand side of Eq. 39.

321 Add this velocity to the above decrement and the result is the decrement of the  
 322 velocity arising from the resistance alone namely  $GH/T - HI/t + 2MI \times$   
 323  $NI/(t \times HI)$ , (Cohen 1999, 657).

<sup>12</sup> To obtain his results, Newton had to determine the arcs  $HI$  and  $GH$  to second order in  $o$ :  $HI = o\sqrt{1+Q^2} - QRo^2/\sqrt{1+Q^2}$ , and  $GH = o\sqrt{1+Q^2} + QRo^2/\sqrt{1+Q^2}$ . But the difference between these arc lengths  $GH - HI = 2QRo^2/\sqrt{1+Q^2}$  is equal to the difference between the tangent lines  $GL - HN$ , where  $HN = o\sqrt{1+Q^2}$ , Eq. 32, and  $GL = o\sqrt{1+Q^2}$ , with  $Q' = Q - 2Ro$ , Eq. 37.

<sup>13</sup> This geometrical relation can be verified by applying the equations of motion in differential form, Eqs. 5 and 6. We have  $\dot{x}\ddot{x} + \dot{y}\ddot{y} = v\dot{v} = -rv - g\dot{y}/v$ , and substituting  $\dot{y} = v \sin(\alpha)$ , one obtains  $\dot{v} = -r + g \sin(\alpha)$ , the total acceleration along the tangent. In Eq. 39,  $\dot{v} = (v - v')/t$ ,  $t = o\sqrt{2R/g}$ , Eq. 34, and  $\sin(\alpha) = Q/\sqrt{1+Q^2}$ . Hence  $r = g(3S/4R^2)\sqrt{1+Q^2}$ , which is Newton's result for the resistance of the medium in terms of the coefficients  $Q$ ,  $R$  and  $S$  that determine the local properties of the given curve.

324 Hence, this decrement of the velocity “arising from the resistance alone” is the second  
 325 term on the right hand side of Eq. 39, and

$$326 \quad GL/T - HN/t + 2MN \times NI/(t \times HN) = o\sqrt{g/2R} \left[ (3S/2R)\sqrt{(1+Q^2)} \right].$$

327 (43)

328 And accordingly since gravity generates the velocity  $2NI/t$  in the same time in  
 329 a fallen body, the resistance will be to the gravity as  $GH \times t/T - HI + 2MI \times$   
 330  $NI/HI$  to  $2NI$  (Cohen 1999, 657).

331 The velocity generated by gravity in a fallen body is

$$332 \quad gt = 2NI/t = o\sqrt{2gR}, \quad (44)$$

333 and dividing Eq. 43 by this velocity yields Newton’s relation for the ratio  $r/g$  of the  
 334 resistance  $r$  of the medium to the acceleration  $g$  of gravity

$$335 \quad r/g = (3S/4R^2) \sqrt{1+Q^2} \quad (45)$$

336 where  $r/g = (GL \times t/T - HN)/2NI + MN/HN$ .

337 Newton, however, proceeded in a different manner to evaluate the ratio  $r/g$  by  
 338 expressing analytically his geometrical line elements in a power series expansion up  
 339 to *cubic* powers in  $o$ . But his relations for the time intervals  $T$  and  $t$  in term of the  
 340 line elements  $LH$  and  $NI$ , e.g. Eq. 34, are only valid to *first* order in  $o$ . Moreover,  
 341 Newton calculated these time intervals from the corresponding *sagittas* of the arcs  $GI$   
 342 and  $HK$  which in third order in  $o$  differ from the expansion of the line elements  $LH$   
 343 and  $NI$ . He stated,

344 Furthermore, if from ordinate  $CH$  half the sum of ordinates  $BG$  and  $DI$  are  
 345 subtracted and from ordinate  $DI$  half the sum of ordinates  $CH$  and  $EK$  are  
 346 subtracted, the remainder will be sagittas  $Ro^2$  and  $Ro^2 + 3So^3$  of arcs  $GI$  and  
 347  $HK$ . And these are proportional to the line elements  $LH$  and  $NI$ , and thus as  
 348 the square of the infinitesimal small time  $T$  and  $t$  (Cohen 1999, 658).

349 But to third order in  $o$

$$350 \quad LH = Ro^2 - 2So^3 \quad (46)$$

351 and

$$352 \quad NI = Ro^2 + So^3 \quad (47)$$

353 which, contrary to Newton’s assertion, are not proportional to the corresponding  
 354 sagittas when the expansion is carried out to the cubic order in  $o$ . This discrepancy  
 355 raises the concern that, like in his earlier version of Proposition 10, Book 2, Newton’s  
 356 would have obtained incorrectly cubic terms in the expansion of  $T^2$  and  $t^2$  in power

357 of  $o$ . But it turns out that the quantity required for Newton's new calculation for  $r/g$ ,  
 358 Eq. 45, requires only the ratio  $T/t$  up to first order in  $o$  which is proportional to the  
 359 square root of ratio between the line elements  $NI/LH$ . And to first order in  $o$ , this  
 360 ratio is also equal to the ratio between the corresponding sagittas, namely<sup>14</sup>

$$361 \quad T/t = \sqrt{NI/LH} = 1 - (3/2)So. \quad (48)$$

#### 362 4 Concluding remarks

363 In a private memorandum, written on 7 June 1713, Newton accounted for the mistake  
 364 in the first version of Proposition 10, Book 2 as follows:

365 Mr. Newton corrected the error himself, shewed him [Nickolas Bernoulli] the  
 366 correction & told him that the Proposition should be reprinted in the new Edi-  
 367 tion which was then coming abroad. The Tangents [ $GL$  and  $HN$ ] of the Arcs  
 368  $GH$  and  $HI$  are first moments of the arcs  $FG$  &  $GH$  [that] should have been  
 369 drawn the same way with the motion describing those arcs, whereas through  
 370 *inadvertency* [my italics] one of them had been drawn the contrary way, & this  
 371 occasioned the error in the conclusion (Hall 1958)

372 Actually, it has been shown here that in this memorandum, Newton gave a misleading  
 373 account of the origin of his error in the first version of this proposition. There was  
 374 nothing "inadvertent" in his drawing a tangent line, labelled  $Cf$  in the first edition  
 375 of the *Principia*, Fig. 1, in the "contrary way" to the tangent line  $CF$ , because his  
 376 drawing, introduced specifically to solve his proposition, described the time-reversed  
 377 motion at  $C$ . Evidently, at the time Newton failed to realize that a solution by his  
 378 geometrical calculus approach required that the power series expansion associated  
 379 with the trajectory curve had to be taken to cubic order in the time interval. But the  
 380 fundamental principles of orbital motion under the action of external forces that he  
 381 had enunciated in Propositions 1 and 6, Book 1, were based on such expansions only  
 382 up to second order in differential time and spatial intervals. After obtaining his first  
 383 relation for the ratio of resistance to gravity,  $r/g$ , Eq. 4, Newton warned the reader  
 384 that

385 This is so in the case of nascent line elements. For in the case of line elements  
 386 of finite magnitude these ratios are not accurate (Cohen 1999, 656).

387 But then he proceeded to ignore his own warning to derive a relation for  $r/g$ , Eq. 23,  
 388 that is incorrect by a factor  $3/2$ . In the revised version of Proposition 10, Book 2, he  
 389 also applied power series expansions to cubic order in the space interval  $o$ , but as has  
 390 been shown here, in this case only expansions to second order in  $o$  are required, which  
 391 explains why, somewhat fortuitously, he obtained the missing factor  $3/2$ .

<sup>14</sup> A simple way to see how this relation saved Newton from repeating his previous error is to suppose that  $t^2 = Ro^2 + Co^3$  where  $C$  is an undetermined coefficient. Then the expansion of  $T^2$  must be calculated with the corresponding value  $R' = R - 3So$ , while the change in  $C$  gives a fourth order change in  $T^2$  that can be neglected. Hence  $T/t = 1 - (3S/2R)o$ , Eq. 48, independent of the value of  $C$  that Newton evaluated incorrectly.

392 During the bitter priority dispute with Leibniz on the invention of the calculus,  
 393 Newton called attention to Proposition 10, Book 2, to exhibit an example of an  
 394 early application of his fluxion version of the calculus to solve problems solution  
 395 *Principia*. But this proposition was a poor choice, because he had made a “seri-  
 396 ous error” in it, and after he corrected his mistake, he failed to acknowledge Johann  
 397 Bernoulli’s contribution, who was bitter and became, albeit anonymously, his most  
 398 effective critic.

### 399 Appendix I: Relation of Proposition 10, Book 2, to Proposition 6, Book 1

400 In Proposition 6, Book 1 in the *Principia* Newton gave a geometrical expression for  
 401 the acceleration of a body moving under the action of a central force in terms of a line  
 402 element,  $QR$ , directed towards the centre of force, that describes the deviation from  
 403 inertial motion along a line element  $PR$ , tangent at  $P$ , Fig. 3. The magnitude of  $PR$   
 404 is proportional to the elapse time interval  $t$ , and the deviation  $QR$  is proportional  $t^2$ ,  
 405 in accordance with Lemma 10, Book 1. Moreover, in Proposition 1, Book 1, Newton  
 406 had shown that for central forces  $t$  is proportional to the area element  $SP \times QT$  swept  
 407 by the radial line  $SP$ , leading to the geometrical expression for the force or acceler-  
 408 ation,  $QT/(SP \times QT)^2$ . Such a description, however, not valid in the presence of  
 409 non-central Newton was able to generalize his description for non-central forces, but  
 410 in the presence of a tangential force however, such a description is incomplete, because  
 411 the resulting acceleration is directed along the instantaneous direction of motion. In  
 412 the original version of the Proposition 10, Book 2, the line elements  $FG$  and  $CF$ ,  
 413 Fig. 1, are the analogs of  $QR$  and  $PR$ , respectively, while in the revised version in  
 414 the second edition of the *Principia*, the corresponding line elements are  $NI$  and  $HN$ .  
 415 Newton continued to assert that the line elements  $FG$  and  $NI$  are proportional to  $t^2$ ,  
 416 and to take into account the resistance of the medium, Newton had to assume that  
 417 the motion along the tangent was the sum of an inertial component proportion to  $t$   
 418 and a component proportional to  $t^2$ . But the resulting geometrical relations were incom-  
 419 plete and Newton searched for additional conditions that would enable him to obtain a  
 420 geometrical relation for the motion. In the first edition of the *Principia*, he obtained an  
 421 additional constraint by considering the time-reversed motion starting from the same  
 422 initial point. In this case, the medium accelerates the motion along the tangent, but  
 423 unfortunately, his application of Lemma 10 for this problem was insufficient, because  
 424 as Lagrange, later demonstrated, a contribution to the lime element of cubic powers  
 425 of  $t$  is required for a correct solution of this problem. Newton’s neglect of this con-  
 426 tribution led him to an incorrect solution which was noticed by Johann Bernoulli and  
 427 pointed out by his nephew, Nikolaus Bernoulli to Newton. Newton’s evaluation for  
 428 the instantaneous velocity at  $H$ ,  $v = \sqrt{(g/2R)(1 + Q^2)}$ , Eq. 62, corresponds to the  
 429 relation  $g_n = v^2/\rho$ , where  $g_n = g \cos(\alpha)$  is the component of gravity normal to the  
 430 tangent, and  $\rho = (1 + Q^2)^{3/2}/R$  is the curvature at  $H$ . In Corollary 3 of Proposition 6,  
 431 Book 1, this relation was given in a related form  $g = l^2/SY^2 \times PV$  for a central force  
 432  $g$ , where  $PV = 2\rho \cos(\alpha)$  and  $l = SPv \cos(\alpha)$  are the angular momentum about the  
 433 centre of force.

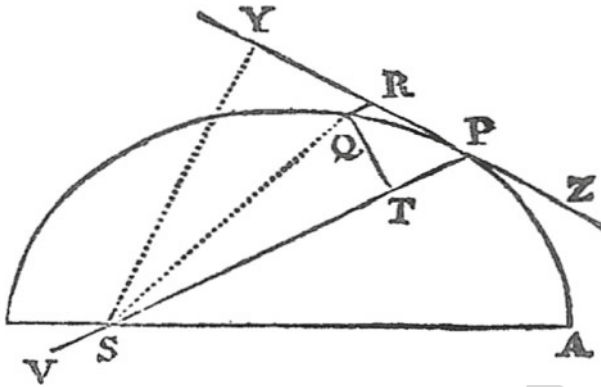


Fig. 3

### Appendix II: Lagrange's analysis of Proposition 10, Book 2

Lagrange was the first mathematician to obtain the correct explanation for Newton's failure to obtain the correct relation for motion in a resistive medium (Lagrange 1797). Lagrange's analysis did not deal directly with Newton's geometrical approach, but a similar analysis is presented here evaluating Newton's line element  $NI$  in Proposition 10, Book 2, to cubic order in the time interval  $t$ .

The equations of motion for this problem in differential form are

$$\ddot{x} = -r \cos(\alpha) \quad (49)$$

$$\ddot{y} = r \sin(\alpha) - g, \quad (50)$$

where  $\ddot{x} = d^2x/dt^2$ ,  $\ddot{y} = d^2y/dt^2$ , and  $\tan(\alpha) = -dy/dx$ . For the power series expansion, we require also the third-order derivative

$$d^3x/dt^3 = r \sin(\alpha)\dot{\alpha} - \cos(\alpha)\dot{r}, \quad (51)$$

and

$$d^3y/dt^3 = r \cos(\alpha)\dot{\alpha} + \sin(\alpha)\dot{r}, \quad (52)$$

where  $\dot{\alpha} = d\alpha/dt$ , and  $\dot{r} = dr/dt$ .

Expanding Newton's line elements in powers of a small interval  $t$  up to cubic order, we have

$$HM = x = v \cos(\alpha)t - (1/2)r \cos(\alpha)t^2 + (1/6)(r \sin(\alpha)\dot{\alpha} - \cos(\alpha)\dot{r})t^3, \quad (53)$$

and

$$MI = HC - y = v \sin(\alpha)t - (1/2)(r \sin(\alpha) - g)t^2 - (1/6)(r \cos(\alpha)\dot{\alpha} + \sin(\alpha)\dot{r})t^3. \quad (54)$$

455 Since  $MN = HM \tan(\alpha)$ ,

456 
$$MN = v \sin(\alpha)t - (1/2)r \sin(\alpha)t^2 + (1/6)(r \sin(\alpha) \tan(\alpha)\dot{\alpha} - \sin(\alpha)\dot{r})t^3, \quad (55)$$

457 and  $NI = MI - MN$

458 
$$NI = (1/2)gt^2 - (1/6)(r/\cos(\alpha))\dot{\alpha}t^3, \quad (56)$$

459 which is independent of  $\dot{r}$ . We have

460 
$$d \tan(\alpha)dt = (1/\cos^2(\alpha))\dot{\alpha} = (\dot{y}\ddot{x} - \dot{x}\ddot{y})/\dot{x}^2 = g/\dot{x} \quad (57)$$

461 which yields

462 
$$\dot{\alpha} = (g/v) \cos(\alpha) \quad (58)$$

463 Hence, to cubic order in  $t$ <sup>15</sup>

464 
$$NI = (1/2)gt^2 - (1/6)(rg/v)t^3, \quad (59)$$

465 **Appendix III: Proposition 10, Book 2, expressed in terms of differential**  
 466 **equations, and Johann Bernoulli's solution**

467 The problem of motion in a resisting medium under the action of a constant gravita-  
 468 tional force  $g$  is treated by expressing the equations of motion in differential form. In  
 469 Cartesian coordinates  $x, y$ ,

470 
$$\ddot{x} = -r\dot{x}/v, \quad (60)$$

471 and

472 
$$\ddot{y} = -r\dot{y}/v - g, \quad (61)$$

473 where  $r$  is the resistance, and  $v$  is the velocity

474 
$$v = \sqrt{\dot{x}^2 + \dot{y}^2}. \quad (62)$$

475 Hence

476 
$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = v\dot{v} = -rv - g\dot{y}, \quad (63)$$

<sup>15</sup> Following Lagrange, this result was also obtained by Whiteside; see footnote (6) in (Whiteside 1981, 375).

477 and

$$478 \quad r/g = -(\dot{v}/g + \dot{y}/v) \quad (64)$$

479 To obtain an expression for this ratio in terms of the geometrical curve for the  
480 trajectory, the time must be replaced by  $x$  as an independent variable. Since

$$481 \quad \dot{y} = y'\dot{x} \quad (65)$$

482 and

$$483 \quad \ddot{y} = y'\ddot{x} + y''\dot{x}^2 \quad (66)$$

484 then, according to Eqs. 60 and 61, we have

$$485 \quad \ddot{y} - y'\ddot{x} = -g = y''\dot{x}^2 \quad (67)$$

486 or

$$487 \quad \dot{x} = \sqrt{-g/y''}. \quad (68)$$

488 Hence, the velocity  $v = \dot{x}\sqrt{1+y'^2}$  can be expressed in terms of the first- and sec-  
489 ond-order spatial derivatives  $y'$  and  $y''$ ,

$$490 \quad v = \sqrt{-(g/y'')(1+y'^2)}. \quad (69)$$

491 Hence

$$492 \quad \dot{v} = g \left( -\frac{y'}{\sqrt{1+y'^2}} + \frac{1}{2} \frac{\sqrt{1+y'^2}}{y''^2} y'' \right), \quad (70)$$

493 and

$$494 \quad \frac{\dot{y}}{v} = \frac{y'}{\sqrt{1+y'^2}}. \quad (71)$$

495 Finally, substituting Eqs. 70 and 71 in Eq. 64,

$$496 \quad \frac{r}{g} = -\frac{\sqrt{1+y'^2}}{2y''^2} y''', \quad (72)$$

497 In Newton's notation  $y' = -Q$ ,  $y'' = -2R$ ,  $y''' = -6S$ , and

$$498 \quad v = \sqrt{(g/2R)(1+Q^2)}, \quad (73)$$

$$\frac{r}{g} = \frac{3S}{4R^2} \sqrt{1 + Q^2}, \quad (74)$$

For a solution to be possible,  $R$  and  $S$  must be positive numbers.

*Example 1* When the curve is a semicircle of radius  $a$

$$Q = \frac{x}{y}, \quad R = 2\frac{a^2}{y^3}, \quad S = 18\frac{a^2x}{y^5} \quad (75)$$

Hence

$$\frac{r}{g} = \frac{3x}{2a}, \quad (76)$$

and

$$v = \sqrt{gy} \quad (77)$$

In particular,  $ra/gx = 3/2$  is the constant ratio of frictional force to the tangential gravitational force, which accounts for the decrease in velocity as  $y$  decreases.

In the first edition of the *Principia*, the factor  $3/2$  was absent, and Johann Bernoulli observed that in this case, the resistance force was equal to the gravitational acceleration along the curve, indicating that the velocity should be a constant—“la vitesse de ce mobile seroit ici toujours la mesme & uniforme”—along the curve, leading to an inconsistency in Newton’s result—“Ce que est la contradiction que j’avois à demontrer”<sup>16</sup>

Johann Bernoulli treated the motion in a resisting medium somewhat differently, by separating the differential equations of motion into its tangential and normal components. He considered only the special case of circular motion, but I will treat also the motion along a general curve.

Let

$$dv = (F_T - r)dt \quad (78)$$

and

$$\frac{v^2}{\rho} = F_N \quad (79)$$

where  $F_T = -g\dot{y}/v = gy'/\sqrt{1 + y'^2}$ , and  $F_N = g\dot{x}/v = g/\sqrt{1 + y'^2}$  are the tangential and normal components of force, respectively, and  $r$  is the resistance force.

<sup>16</sup> Extrait d’ une Lettre de M. Bernoulli, écrite de Basle le 10. Janvier 1711, touchant la maniere de trouver les force centrales me milieux resistans en raisons composée de leurs densités & des puissances quelconques de vitesses du mobile. in Memoires de L’ Academie Royale des Sciences.

526 Bernoulli writes  $r$  in the form  $r = \gamma v^n$ , but his treatment does not depend on this  
527 particular form for  $r$ . Setting

$$528 \quad dt = \frac{ds}{v} \quad (80)$$

529 where  $ds$  is a differential arc length, Bernoulli obtains an equation between spatial  
530 differential

$$531 \quad r ds = -v dv - g dy \quad (81)$$

532 Assuming that the motion is along a semicircle of radius  $a$ ,  $\rho = a$  and Eq. 79 takes  
533 the form

$$534 \quad v^2 = gy. \quad (82)$$

535 Hence

$$536 \quad v dv = \frac{1}{2} g dy, \quad (83)$$

537 which substituted in Eq. 81 leads to

$$538 \quad r ds = -\frac{3}{2} g dy, \quad (84)$$

539 and since  $ds/dy = -x/a$ ,

$$540 \quad \frac{r}{g} = \frac{3x}{2a} \quad (85)$$

541 —“ce qu’il falloit encore a demontrer”—in accordance with Newton’s result in Exam-  
542 ple 1 in the second edition of the *Principia*.

543 For a general curve, substituting for  $\rho$  in Eq. 79,  $\rho = -(1 + y^2)^{3/2}/y''$ , I obtain

$$544 \quad v^2 = -g \frac{(1 + y^2)}{y''} \quad (86)$$

545 which is equal to the expression for  $v$ , Eq. 69, obtained previously from Newton’s  
546 formulation of this problem.

547 Hence

$$548 \quad v dv = -g \left( dy - \frac{1}{2} \frac{(1 + y^2)y'''}{y''^2} dx \right) \quad (87)$$

549 and substituting this expression in Eq. 81 with  $ds/dx = \sqrt{1 + y'^2}$  yields

$$550 \quad \frac{r}{g} = \frac{\sqrt{1 + y'^2}}{2y''^2} y''' \quad (88)$$

551 which corresponds to Eq. 72.

552 **Acknowledgments** I would like to dedicate this paper to the memory of Derek (Tom) Whiteside, whom  
 553 I had the privilege to know and benefit from his numerous criticisms of my earlier work. Tom devoted  
 554 over hundred pages of his last volume of *Newton's Mathematical Papers* (Whiteside 1980) to describe the  
 555 manuscripts in which Newton made frantic efforts to correct the initial errors he had made in Proposition  
 556 10, Book 2. Recently, he warned against creating “monsters” when framing Newton’s ideas in vector form,  
 557 “which felled even the mighty Lagrange in his attempt wholly to understand it” (Whiteside 2002), but such  
 558 monsters are still being created up to the present time. I also would like to thank Niccolo Guicciardini for  
 559 bringing to my attention the controversies associated with Newton’s formulation of Proposition 10, Book  
 560 2, and for an uncountable number of exchanges that helped to clarify my ideas on this subject, and to Gary  
 561 Weisel for calling my attention to Whiteside’s above apropos comment.

## 562 References

- 563 Bernoulli, J. 1716. In *The correspondence of Isaac Newton*, vol. 6, ed. A. Rupert Hall and Laura Tilling, p.  
 564 303. Cambridge: Cambridge University Press.
- 565 Brackenridge, J.B., and M. Nauenberg 2002. The Cambridge Companion to Newton. In *Curvature in New-*  
 566 *ton's dynamics*, ed. I.B. Cohen and G.E. Smith, pp. 85–137. Cambridge: Cambridge University Press.
- 567 Cohen, I.B. 1999. *Isaac Newton the Principia*. A new translation by Cohen, I.B., and Anne Whitman,  
 568 Preceded by a guide to Newton’s *Principia*, p. 167. Berkeley: University of California Press.
- 569 Guicciardini, N. 2009. *Isaac Newton on mathematical certainty and method*. Cambridge: MIT Press.
- 570 Hall, A.R. 1958. Correcting the Principia. *Osiris* 13: 291–326.
- 571 Hall, A.R. 1980. *Philosophers at war, the quarrel between Newton and Leibniz*. Cambridge: Cambridge  
 572 University Press.
- 573 Lagrange, J.L. 1797. *Théorie des Fonctions Analytiques, contenant Les Principes du Calcul Différentiel*,  
 574 pp. 244–251. Paris: De l’Imprimerie de la République.
- 575 Whiteside, D.T. 1976. *The Mathematical Papers of Isaac Newton* vol. 7, ed. T. Whiteside, pp. 96–97.  
 576 Cambridge: Cambridge University Press.
- 577 Whiteside, D.T. 1970. The mathematical principles underlying Newton’s Principia mathematica. *Journal*  
 578 *for the History of Astronomy* 1: 116–138.
- 579 Whiteside, D.T. 1981. *The mathematical papers of Isaac Newton*, vol. 8, ed. Derek T. Whiteside,  
 580 pp. 312–419. Cambridge: Cambridge University Press
- 581 Whiteside, D.T. 2002. The math forum @ drexel. Posted July 5 <http://mathforum.org>.