PHYSICS 110A

Homework 6

Due in class, Tuesday, February 17.

1. The **Clausius-Mossotti** formula.
   In a dielectric the polarization is proportional to the field,

$$\mathbf{P} = \epsilon_0 \chi_e \, \mathbf{E}, \tag{1}$$

where $\mathbf{E}$ is the *total, macroscopic* field, and χ_e is the susceptibility. The average dipole moment of a single atom (or molecule) is also proportional to the field

$$\mathbf{p} = \alpha \, \mathbf{E}_{\text{local}}, \tag{2}$$

but here $\mathbf{E}_{\text{local}}$ is the *local* field at the atom, including the field from charges outside the dielectric and the fields set up by *other* atoms in the dielectric but *neglecting* the field due to the atom itself. The parameter α is the atomic polarizability.

Question: What is the relation between the atomic polarizability α and the susceptibility χ_e?

Since the polarization is the dipole moment per unit volume, i.e. $\mathbf{P} = N\mathbf{p}$ where N is the number of atoms per unit volume, one would naively think that

$$\epsilon_0 \, \chi_e = N\alpha \quad \text{so} \quad \boxed{\chi_e = \frac{N}{\epsilon_0} \alpha} \quad \text{and hence} \quad \mathbf{P} = \frac{N}{\epsilon_0} \alpha \, \mathbf{E}. \tag{3}$$

In fact Eq. (3) turns out to be a good approximation when χ_e is small (remember χ_e is dimensionless), but is not right in general because the fields in Eqs. (1) and (2) are not the same.

To find the relationship between $\mathbf{E}$ and $\mathbf{E}_{\text{local}}$ we assume that the space "allotted" to each atom is a sphere of radius R. We want to calculate the field at the center due to the charges outside the sphere. We therefore separate the field $\mathbf{E}$ into a contribution from the atom at the center of the sphere, and a contribution from everything else. Remembering that the field $\mathbf{E}$ represents an average over the rapidly varying fields of the atoms, we write

$$\mathbf{E} = \mathbf{E}_{\text{self}} + \mathbf{E}_{\text{rest}}$$

where $\mathbf{E}_{\text{rest}}$ is the *average* field from the charges outside and $\mathbf{E}_{\text{self}}$ is the *average* field from the atom at the center. From Qu. 7 of HW 4 we have

$$\mathbf{E}_{\text{self}} = -\frac{1}{\frac{4}{3}\pi R^3} \frac{\mathbf{p}}{3\epsilon_0} = -N \, \frac{\mathbf{p}}{3\epsilon_0},$$

where N is the number of atoms per unit volume (the same as one over the volume per atom).

Now $\mathbf{E}_{\rm rest} = \mathbf{E} - \mathbf{E}_{\rm self}$ is the same as the the field at the *center* of the sphere (i.e. at the atom), $\mathbf{E}_{\rm local}$, a result obtained by an extension of the results in Qu. 7 of HW 4, see Qu. 3.41 of Griffiths. Hence

$$\mathbf{E}_{\rm local} = \mathbf{E} - \mathbf{E}_{\rm self}.$$

Show that

$$\mathbf{E} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right) \mathbf{E}_{\rm local}$$

and hence

$$\mathbf{P} = \frac{N\alpha}{1 - N\alpha/3\epsilon_0}\mathbf{E} \quad \text{so} \quad \boxed{\chi_e = \frac{N\alpha/\epsilon_0}{1 - N\alpha/3\epsilon_0}\,,} \tag{4}$$

rather than Eq. (3). Note, though, that if the polarization is weak, i.e. $N\alpha/\epsilon_0 \ll 1$ (which is equivalent to $\chi_e \ll 1$) Eq. (4) reduces to Eq. (3).
Incidentally Eq. (4) can be reexpressed in terms of the dielectric constant $\epsilon_r \equiv 1 + \chi_e$ as

$$\frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{N\alpha}{3\epsilon_0}\,,$$

which is known as the **Clausius-Mossotti** formula. In the context of optics, where ϵ_r is replaced by n^2 with n the refractive index, it is called the **Lorentz-Lorenz** equation.

2. We are determining a solution to Laplace's equation numerically by putting defining the potential on a uniform mesh of points. We cycle through the mesh, and at each mesh point set the new value of the potential to be the average of that at the neighboring mesh points. The electrostatic energy is (proportional to)

$$U = \sum_i \sum_{j \in \text{neighbor of } i} (V_i - V_j)^2\,.$$

Show that the energy always *decreases* (or remains the same) each time the value of V at one of the mesh points is updated. (It only remains the same if the potential at the mesh point is correct (relative to its neighbors).
Note: This shows that this "relaxation" method eventually converges to a minimum of the energy.
Hint: Consider all the terms in the energy involving a particular mesh point i and determine the change in this when V_i is replaced by the average of V on the neighbors.

3. Suppose that the magnetic field in some region has the form

$$\mathbf{B} = k\, z\, \hat{\mathbf{x}}.$$

(where k is a constant). Find the force on a square loop of side a, lying in the yz plane and centered at the origin, if it carries a current I flowing counterclockwise when looking down the x axis.

4. (a) A phonograph record (remember them!) carries a uniform charge density σ. If it rotates at angular velocity ω, what is the surface current density K at a distance r from the center.

(b) A uniformly charged solid sphere of radius R and total charge Q is rotating at an angular velocity ω about the z axis. Find the current density $\mathbf{J}$ a point (r, θ, ϕ) within the sphere.

5. For a configuration of charges and currents confined in a volume $\mathcal{V}$ show that

$$\int_{\mathcal{V}} \mathbf{J} \, d\tau = \frac{d\mathbf{p}}{dt} \,,$$

where $\mathbf{p}$ is the total dipole moment.
Hint: Evaluate $\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot (x\mathbf{J}) \, d\tau$.