PHYSICS 110A

Final Examination, 2009, Solutions

1. [**20 points**]

   (a) $\boxed{\text{A point charge at the origin.}}$

   $$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = \boxed{4\pi\delta^{(3)}(\mathbf{r}).}$$

   (b) i.

   $$\mathbf{E} = -\nabla V = -A\frac{\partial}{\partial r}\left(\frac{e^{-r}}{r}\right)\hat{\mathbf{r}} = \boxed{Ae^{-r}\left(1+r\right)\frac{\hat{\mathbf{r}}}{r^2}.}$$

   ii.

   $$\rho = \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 A\left\{e^{-r}\left(1+r\right)\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) + \frac{\hat{\mathbf{r}}}{r^2}\cdot\nabla\left[e^{-r}\left(1+r\right)\right]\right\}.$$

   But

   $$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^{(3)}(\mathbf{r}),$$

   from part (a), and

   $$\nabla\left[e^{-r}\left(1+r\right)\right] = \hat{\mathbf{r}}\,\frac{\partial}{\partial r}\left[(1+r)\,e^{-r}\right] = \hat{\mathbf{r}}\left(-r\,e^{-r}\right).$$

   Hence

   $$\boxed{\rho = \epsilon_0 A\left[4\pi\delta^{(3)}(\mathbf{r}) - \frac{e^{-r}}{r}\right].}$$

   iii. The total charge is given by

   $$\begin{aligned} Q = \int \rho(\mathbf{r})\,d\tau &= \epsilon_0 A\left\{4\pi\int\delta^{(3)}(\mathbf{r})\,d\tau - \int_0^\infty \frac{e^{-r}}{r}\,4\pi r^2\,dr\right\} \\ &= \epsilon_0 A\left\{4\pi - 4\pi\int_0^\infty re^{-r}\,dr\right\} \\ &= \epsilon_0 A\left(4\pi - 4\pi\right) = \boxed{0.} \end{aligned}$$

2. [**20 points**]
   For $r > R$ the solution is

   $$V(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}\,P_l(\cos\theta), \quad (r > R),$$

   and for $r < R$ the solution is

   $$V(r,\theta) = \sum_{l=0}^{\infty} A_l\,r^l\,P_l(\cos\theta).$$

For $r = R$ they must both give

$$V(R,\theta) = V_0(\theta) = k\left(2\cos^2\theta - 1\right) = \frac{k}{3}\left[4P_2(\cos\theta) - P_0(\cos\theta)\right].$$

Matching the solutions gives $\boxed{A_l = B_l = 0 \text{ except for } l = 0 \text{ and } 2.}$ Also

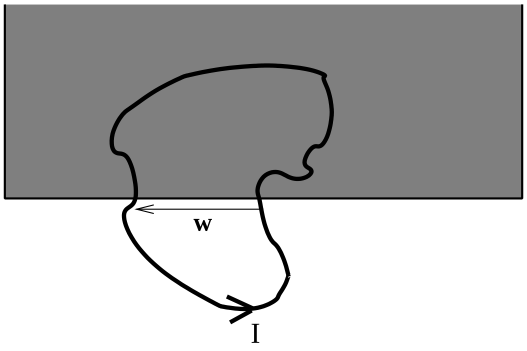
$$A_0 = -\frac{1}{3}k, \quad A_2 = \frac{4}{3R^2}k, \text{ so } \boxed{V_{\text{in}} = \frac{k}{3}\left[-1 + 4\left(\frac{r}{R}\right)^2 P_2(\cos\theta)\right]},$$

$$B_0 = -\frac{1}{3}Rk, \quad B_2 = \frac{4}{3}R^3k, \text{ so } \boxed{V_{\text{out}} = \frac{k}{3}\left[-\left(\frac{R}{r}\right) + 4\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right]}.$$

The surface charge density is equal to ϵ_0 times the discontinuity in the normal component of the electric field, i.e.

$$\begin{aligned}\sigma(\theta) &= \epsilon_0\left(\mathbf{E}_{r,\text{out}} - \mathbf{E}_{r,\text{in}}\right) \\ &= \epsilon_0\left(-\frac{\partial V_{\text{out}}}{\partial r} + \frac{\partial V_{\text{in}}}{\partial r}\right), \\ &= \epsilon_0\frac{k}{3R}\left[-1 + 20P_2(\cos\theta)\right] = \boxed{\epsilon_0\frac{k}{R}\left[10\cos^2\theta - \frac{11}{3}\right]}.\end{aligned}$$

3. [**15 points**]



(a) From the magnetic force law we have

$$\mathbf{F} = I\int\left(d\boldsymbol{\ell} \times \mathbf{B}\right).$$

However, $\mathbf{B}$ is constant and can be taken out of the integral so we have

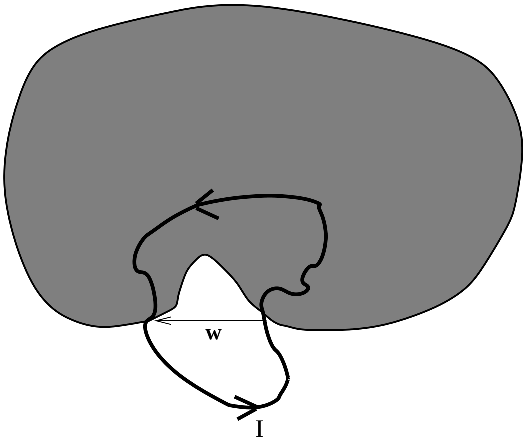
$$\mathbf{F} = I\mathbf{w} \times \mathbf{B},$$

where $\mathbf{w}$ is a vector of length w (see figure) starting at the point on the loop where the current enters the field, and ending where the current leaves the field. In magnitude we have $\boxed{F = IwB.}$

(b) Even if the region where the field is non-zero is not a straight line the answer is exactly the same, namely

$$\mathbf{F} = I\mathbf{w} \times \mathbf{B},$$

where $\mathbf{w}$ is a vector of length w (see figure) starting at the point on the loop where the current enters the field, and ending where the current leaves the field, see the figure below:



(c) From the right hand rule the direction is $\boxed{\text{downwards}}$.

4. [**15 points**]

(a) Using Ampère's law for $\mathbf{H}$, $\oint \mathbf{H}\cdot d\boldsymbol{\ell} = I_{f,\text{enc}}$, we have $2\pi s H = \pi s^2 J = \frac{\pi s^2}{\pi a^2}\, I$ so

$$\boxed{\mathbf{H} = \frac{I}{2\pi a^2}\, s\, \widehat{\boldsymbol{\phi}}.}$$

(b)

$$\mathbf{B} = \mu\mathbf{H} = \mu_0(1+\chi_m)\mathbf{H} = \boxed{\mu_0(1+\chi_m)\,\frac{I}{2\pi a^2}\, s\, \widehat{\boldsymbol{\phi}}.}$$

(c) $\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M} = \boldsymbol{\nabla}\times(\chi_m\mathbf{H}) = \chi_m\boldsymbol{\nabla}\times\mathbf{H} = \boxed{\chi_m\mathbf{J}_f,}$ where we used Ampère's law.

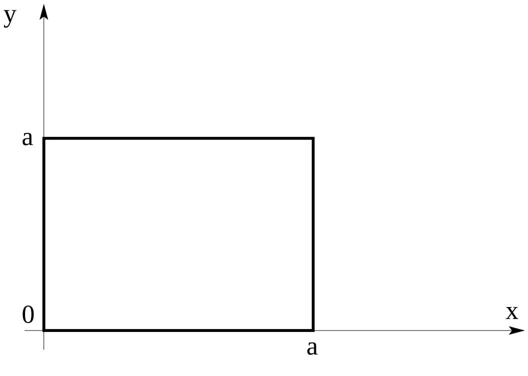
Hence $\boxed{\mathbf{J}_b = \chi_m\mathbf{J}_f = \chi_m\dfrac{I}{\pi a^2}.}$

(d) Ampère's law for $\mathbf{B}$ is $2\pi s B = \pi s^2(J_b + J_f) = \frac{\pi s^2}{\pi a^2}\, I(1+\chi_m)$, so

$$\mathbf{B} = \boxed{\mu_0(1+\chi_m)\,\frac{I}{2\pi a^2}\, s\, \widehat{\boldsymbol{\phi}},}$$

in agreement with part (b).

5. [**10 points**]



The loop has resistance R and is subjected to a non-uniform time-dependent magnetic field

$$\mathbf{B}(y,t) = \begin{cases} 0, & (t<0), \\ k\, y^2\, t\,(10-t)\,\hat{\mathbf{z}}, & (0<t<10), \\ 0, & (t>10), \end{cases}$$

where k is a constant.

(a) $\boxed{\text{For } t < 0 \text{ and } t > 10}$ there is no flux and hence no induced emf and $\boxed{\text{no current.}}$ For $0 < t < 10$ the flux is $\Phi = kt(10-t)\int_0^a dx \int_0^a y^2\, dy = \frac{1}{3}a^4 kt(10-t)$. Hence the emf is

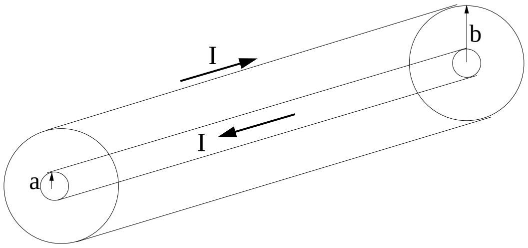
$$\mathcal{E} = -\frac{d\Phi}{dt} = \frac{1}{3}a^4 k\,(10-2t), \quad \Rightarrow \boxed{I = \frac{a^4 k}{3R}\,(2t-10), \quad (0 < t < 10).}$$

(b) The total charge transported past a point on wire is

$$Q = \int I(t)\, dt = \frac{a^4 k}{3R}\int_0^{10}(2t-10)\, dt = \boxed{0.}$$

Note: The answer *must* be zero because $\mathcal{E} = -\frac{d\Phi}{dt}$ tells us that $\int \mathcal{E}\, dt = -\left(\Phi_{\text{final}} - \Phi_{\text{initial}}\right)$. Here the initial and final fluxes are both zero so the integrated emf (and hence the integrated current) is zero.

6. [**20 points**]



The coaxial cable has an inner radius a and an outer cylindrical shell of radius b (see figure). A current I flows as shown,

(a) By Ampère's law, $B\, 2\pi s = \mu_0 I_{\text{enc}}$ so $\boxed{\mathbf{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}.}$

(b) The flux between the inner wire and the outer shell is $\Phi = \frac{\mu_0 I}{2\pi}\int_a^b \frac{ds}{s} = \frac{\mu_0 I}{2\pi}\ln\left(\frac{b}{a}\right)$, per unit length. Hence the self-inductance per unit length is

$$\boxed{\mathcal{L} = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right).}$$

(c) By Gauss' law, $\mathbf{E} = \frac{Q}{2\pi\epsilon_0 s}\,\hat{\mathbf{r}}$ so the potential difference is given by $V \equiv V_a - V_b = -\int_b^a \mathbf{E}\cdot d\mathbf{r} = \frac{Q}{2\pi\epsilon_0}\int_a^b \frac{ds}{s} = \frac{Q}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$. The capacitance per unit length is given by

$$\mathcal{C} = \frac{Q}{V} = \boxed{\frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}.}$$

(d)

$$\boxed{\mathcal{L}\,\mathcal{C} = \epsilon_0\mu_0} = \frac{1}{c^2},$$

independent of a and b.