PHYSICS 110A

Homework 2 Solutions

1. (a)

$$\nabla \cdot (r^n\,\hat{\mathbf{r}}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,r^n\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\,r^{n+2} = \boxed{(n+2)r^{n-1}}$$

except for $n = 2$ for which there is a δ function contribution at the origin. To determine this, as discussed in class, compute the integral of $\nabla \cdot (r^{-2}\,\hat{\mathbf{r}})$ over a sphere of radius R centered at the origin. This is equal to $\oint r^{-2}\,\hat{\mathbf{r}} \cdot d\mathbf{a}$ over the surface of the sphere, which is $R^{-2}4\pi R^2 = 4\pi$. This is independent of R so all the contribution must come from $R = 0$ and so we get $\boxed{\nabla \cdot (r^{-2}\,\hat{\mathbf{r}}) = 4\pi\,\delta^{(3)}(\mathbf{r}).}$

(b) Using the definition of the curl in spherical polars given in the inside cover of the book, we see that it is zero for a function with only a radial component which only depends on r. Hence $\boxed{\nabla \times (r^n\,\hat{\mathbf{r}}) = 0.}$

One might worry that there is a delta function at the origin, as occured in part (a) for the divergence with $n = -2$. However, this does not occur. To see this note that $\int(\nabla \times \mathbf{v})d\tau = -\oint \mathbf{v} \times d\mathbf{a}$ (see problem 1.60(b) of Griffiths. The result follows from the divergence theorem in which $\mathbf{v}$ is replaced by $(\mathbf{v} \times \mathbf{c})$.)

But $\mathbf{v} = r^n\,\hat{\mathbf{r}}$ and $d\mathbf{a}$ are both in the $\hat{\mathbf{r}}$ direction, so $\mathbf{v}\times d\mathbf{a} = 0$ and hence $-\oint \mathbf{v}\times d\mathbf{a} = 0$ so there is $\boxed{\text{no delta function contribution at the origin.}}$

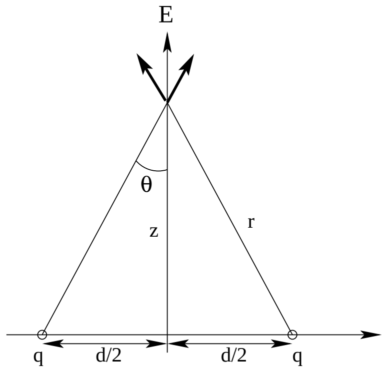
2. (a) $\boxed{\text{Zero.}}$ The reason is that the contributions to the electric field at the origin are vectors of equal length which form the four sides of a square and so the vector sum is zero. Alternatively we could say, in this case, that there is a cancellation of the fields from pairs of charges in opposite directions. (However, this alternative argument won't work for part (c).)

(b) $\boxed{F = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2},}$ where r is the distance from the origin to each charge. **F** points *towards* the missing charge. The reason is *superposition*. This situation is equivalent to part (a) but with an extra charge $-q$ at one of the charges. Because of part (a) the whole contribution comes from this extra charge.

(c) $\boxed{\text{Zero.}}$ The reason is the same as the first argument in part (a).

(d) $\boxed{F = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2},}$ where r is the distance from the origin to each charge. **F** points *towards* the missing charge. The reason is the same as in (b).

3. (a)

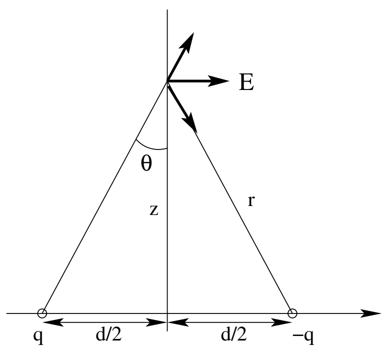


The horizontal components cancel, see the figure. The net vertical field is $E_z = \frac{1}{4\pi\epsilon_0}\frac{2q}{r^2}\cos\theta$ where $r^2 = z^2 + (d/2)^2$ and $\cos\theta = z/r$. Hence

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left(z^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}\,\hat{\mathbf{z}}\,.}$$

When $z \gg d$ you are so far away that it looks like a single charge $2q$. The field should therefore reduce to $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{z^2}\,\hat{\mathbf{z}}$ and it does, as you can see by putting $d = 0$ in the formula.

(b)

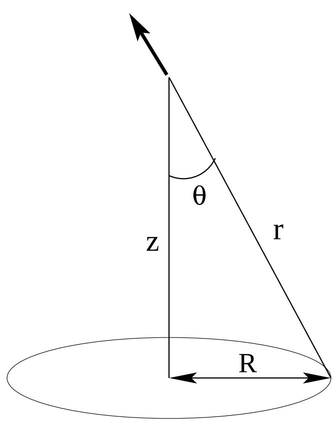


This time the vertical components cancel, see the figure. The net horizontal field is $E_x = \frac{1}{4\pi\epsilon_0}\frac{2q}{r^2}\sin\theta$ where $\sin\theta = d/2r$. Hence

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{qd}{\left(z^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}\,\hat{\mathbf{x}}\,.}$$

Far away ($z \gg d$) the field goes like $\mathbf{E} \simeq \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3}\,\hat{\mathbf{x}}$ which, as we shall see, is the field of a *dipole*. (If we put $d = 0$ we get $\mathbf{E} = 0$. This appropriate; to the extent that this configuration looks like a single point charge the net charge is zero, so $\mathbf{E} \to 0$.)

4.



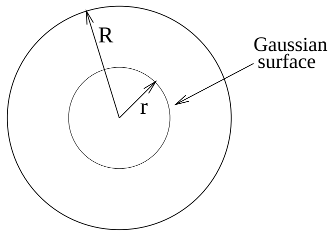
The horizontal components cancel and the net vertical field is given by $E_z = \frac{1}{4\pi\epsilon_0}\left\{\int \frac{\lambda dl}{r^2}\cos\theta\right\}\hat{\mathbf{z}}$, where $r^2 = z^2 + R^2$ and $\int dl \;\; 2\pi R$. Hence

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda\, 2\pi R\, z}{(z^2 + R^2)^{3/2}}\,\hat{\mathbf{z}}\,.}$$

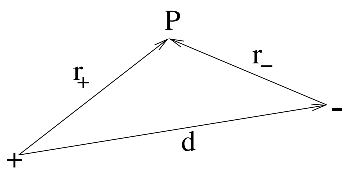
5. (a) $\rho = \epsilon_0 \boldsymbol{\nabla}\cdot\mathbf{E} = \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k r^3\right) = \boxed{5\epsilon_0 k r^2.}$

(b) *By direct integration:* $Q_{\text{enc}} = 5\epsilon_0 k \int_0^R r^2\, 4\pi r^2\, dr = \boxed{4\pi\epsilon_0 k R^5.}$

By Gauss's law: $Q_{\text{enc}} = \epsilon_0 \oint \mathbf{E}\cdot d\mathbf{a} = \epsilon_0 k R^3\, 4\pi R^2 = \boxed{4\pi\epsilon_0 k R^5,}$ in agreement with direct integration.

6. 

Inside the sphere apply Gauss's law to the Gaussian surface of radius $r\,(< R)$: $\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 4\pi r^2 = \frac{1}{4\pi\epsilon_0}\,Q_{\rm enc} = \frac{1}{4\pi\epsilon_0}\,\frac{4}{3}\pi r^3\rho$. Hence $\boxed{\mathbf{E} = \dfrac{1}{3\epsilon_0}\,r\,\rho\,\hat{\mathbf{r}}.}$
Outside the sphere, all the charge, $Q_{\rm tot} = \frac{4}{3}\pi R^3\rho$, is inside the Gaussian surface, so $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\,\frac{4\pi}{3}\,\frac{R^3\rho}{r^2}\,\hat{\mathbf{r}} = \boxed{\dfrac{1}{3\epsilon_0}\,\dfrac{R^3}{r^2}\,\rho\,\hat{\mathbf{r}}.}$

7. 

Using the solution for Qu. 6 the field inside the positive sphere is $\mathbf{E}_+ = \frac{\rho}{3\epsilon_0}\,\mathbf{r}_+$, where $\mathbf{r}_+$ is a vector from the center of the positive sphere to the point P in question. Similarly, the field inside the negative sphere is $\mathbf{E}_- = -\frac{\rho}{3\epsilon_0}\,\mathbf{r}_-$. Hence the total field in the region inside *both* spheres is $E = \frac{\rho}{3\epsilon_0}\,(\mathbf{r}_+ - \mathbf{r}_-)$. However, $\mathbf{r}_+ - \mathbf{r}_- = \mathbf{d}$, the vector from the positive center to the negative center (see the figure). Hence $\boxed{\mathbf{E} = \dfrac{\rho}{3\epsilon_0}\,\mathbf{d}\,,}$ which is *constant.*

8. We need to compute

$$\boldsymbol{\nabla}\times\mathbf{E} = \frac{1}{4\pi\epsilon_0}\,\boldsymbol{\nabla}\times\int_{\mathcal{V}}\frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\,d\tau' = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\rho(\mathbf{r}')\,\left[\boldsymbol{\nabla}\times\,\frac{1}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}}\right]\,d\tau',$$

since ρ depends only on $\mathbf{r}'$ and not on $\mathbf{r}$. Since $\mathbf{r}'$ is a constant as far as the derivative with respect to the components of $\mathbf{r}$ is concerned we can equivalently differentaie with respect to the components of $\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'$. However, according to Qu. 1b,

$$\boldsymbol{\nabla}_{\mathscr{r}}\times\,\frac{1}{\mathscr{r}^2}\,\hat{\boldsymbol{\mathscr{r}}} = 0$$

and so $\boxed{\boldsymbol{\nabla}\times\mathbf{E} = 0.}$

9. (a) $\boldsymbol{\nabla}\times\mathbf{E}_1 = (0-2y)\,\hat{\mathbf{x}} + (0-3z)\,\hat{\mathbf{y}} + (0-x)\,\hat{\mathbf{z}}$, which is not zero. Hence $\mathbf{E}_1$ cannot be an electric field.

(b) $\boldsymbol{\nabla}\times\mathbf{E}_2 = (2z-2z)\,\hat{\mathbf{x}} + (0-0)\,\hat{\mathbf{y}} + (2y-2y)\,\hat{\mathbf{z}}$, which is zero and so E_2 could be an electric field.
Since $E_2 = -\boldsymbol{\nabla}V_2$, we have $\frac{\partial V_2}{\partial x} = -y^2$, $\frac{\partial V_2}{\partial y} = -(2xy+z^2)$, $\frac{\partial V_2}{\partial z} = -2yz$. Integrating gives, successively, $V_2 = -xy^2 + f(y,z)$, $V_2 = -xy^2 - yz^2 + g(x,z)$, $V_2 = -yz^2 + h(x,y)$. Requiring that these three expressions are consistent gives $\boxed{V_2 = -x\,y^2 - y\,z^2,}$ (plus an arbitrary constant). Taking (minus) the gradient of this does indeed give $\mathbf{E}_2$ since, for example, $-\frac{\partial}{\partial x}\,(-x\,y^2 - y, z^2) = y^2$ which is the x-component of $\mathbf{E}_2$. Similarly for the y and z components.