Physics 112
An Integral

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In class we showed that the determination of the specific heat of a degenerate Fermi gas involved the following integral

$$\boxed{I = \int_{-\infty}^{\infty} \frac{x^2\, e^x}{(e^x+1)^2}\, dx\, .} \tag{1}$$

The integrand can be written as

$$\left(\frac{x}{e^{x/2} + e^{-x/2}} \right)^2 , \tag{2}$$

which is clearly an even function of x and so we can write I as an integral involving only positive values of x,

$$I = 2 \int_0^{\infty} \frac{x^2\, e^x}{(e^x+1)^2}\, dx\, . \tag{3}$$

This is not exactly of a standard type, but can be related to a more standard integral since

$$\boxed{I = -2 \left. \frac{dJ(a)}{da} \right|_{a=1} ,} \tag{4}$$

where

$$\boxed{J(a) = \int_0^{\infty} \frac{x}{e^{ax}+1}\, dx\, .} \tag{5}$$

We determine $J(a)$, initially as a series, as follows:

$$\begin{aligned} J(a) &= \int_0^{\infty} \frac{x}{e^{ax}+1}\, dx \\ &= \int_0^{\infty} x\, \frac{e^{-ax}}{1+e^{-ax}}\, dx \\ &= \int_0^{\infty} x \left[e^{-ax} - e^{-2ax} + e^{-3ax} - e^{-4ax} + \cdots \right] dx \\ &= \frac{1}{a^2} \int_0^{\infty} t e^{-t}\, dt \left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \right] , \end{aligned} \tag{6}$$

where, in the last line, we made the substitution $nax = t$ in the term involving e^{-nax}, where $n = 1, 2, 3, \cdots$. The integral $\int_0^{\infty} te^{-t}\, dt$ is equal to $1!\,(=1)$, and so

$$J(a) = \frac{1}{a^2} \left\{ \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right] - \left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots \right] \right\} . \tag{7}$$

Next, we include the terms involving the even integers which are missing in the first part of the expression, and then subtract them back out in the second term, i.e.

$$\begin{aligned} J(a) &= \frac{1}{a^2}\left\{\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right]-2\left[\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots\right]\right\} \\ &= \frac{1}{a^2}\left\{\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right]-\frac{1}{2}\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]\right\} \\ &= \frac{1}{a^2}\left(1-\frac{1}{2}\right)\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right] \\ &= \frac{1}{2a^2}\,\zeta(2)\,, \end{aligned} \tag{8}$$

where

$$\boxed{\zeta(2)\equiv 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots} \tag{9}$$

is a zeta function and has value $\pi^2/6$. (This will have been shown in 116C.) Hence, from Eqs. (8) and (9), we have

$$\boxed{J(a)=\frac{\pi^2}{12a^2}\,.} \tag{10}$$

It follows from Eq. (4) that the desired integral in Eq. (1) is given by

$$\boxed{I=\frac{\pi^2}{3}\,,} \tag{11}$$

as stated in class.