PHYSICS 112  
Homework 6 Solutions

1.  (a) The thermodynamic identity is 
\[ TdS = dU + PdV. \]
If the volume changes by a small amount \( dV \), and the temperatures by \( dT \) at constant entropy we have \( 0 = dU + PdV \). Now \( dU = C_VdT \) since the energy of an ideal gas only depends on \( T \), not the volume, so 
\[ C_VdT + PdV = 0. \]

(b) Writing \( P = N k_B T / V \) and recalling that \( N k_B = C_P - C_V \) we get 
\[ C_V \frac{dT}{T} + (C_P - C_V) \frac{dV}{V} = 0. \]
Dividing by \( C_V \) gives the desired result
\[ \frac{dT}{T} + (\gamma - 1) \frac{dV}{V} = 0, \]
where \( \gamma = C_P / C_V \).

(c) Integrating the last expression gives
\[ \ln T + (\gamma - 1) \ln V = \text{const}. \]
and exponentiating gives
\[ TV^{\gamma - 1} = \text{const}. \]
Substituting \( T = PV/Nk_B \) into the result of the previous part gives
\[ \frac{PV}{N k_B} V^{\gamma - 1} = \text{const}. \]
and so
\[ PV^{\gamma} = \text{const}'. \]
where const'. is another constant. (Remember we are keeping \( N \) constant here.)

(d) At constant \( T \), we have \( PV = C \) where \( C = N k_B T \) is constant, and so
\[ B_T = -V \left( \frac{\partial P}{\partial V} \right)_T = (-V) \left( -\frac{C}{V^2} \right) = \frac{C}{V} = \frac{P}{V}. \]
At constant entropy, we have \( PV^{\gamma} = C \), and so
\[ B_S = -V \left( \frac{\partial P}{\partial V} \right)_S = (-V) \left( -\frac{\gamma C}{V(1 + \gamma)} \right) = \frac{\gamma C}{V^{\gamma}} = \frac{\gamma P}{V}. \]

2. The density of states in two dimensions was worked out in HW 3, Qu. 1 and is
\[ \rho(\epsilon) = A \frac{m}{2\pi\hbar^2}, \]
where we divided that expression by 2 since we have spin = 0 here. Note that this is independent of \( \epsilon \).
As in three dimensions we have
\[ \Omega = -k_B T \lambda z^{(1)} \]
where now
\[ z^{(1)} = \sum_l e^{-\beta \epsilon_l} = A \frac{m}{2\pi \hbar^2} \int_0^\infty e^{-\beta \epsilon} d\epsilon = A \frac{m k_B T}{2\pi \hbar^2} = \frac{A}{A_Q}, \]
where \( A_Q \), the “quantum area”, is given by
\[ A_Q = \frac{2\pi \hbar^2}{mk_B T}. \]

Note that \( A_Q = V_Q^{2/3} \).

Results for the free energies for classical ideal gas in three dimensions go over with \( V \) replaced by \( A \) and \( V_Q \) replaced by \( A_Q \). In particular:

(a) \[ \mu = k_B T \ln(nA_Q) = -k_B T \ln \left( \frac{1}{n} \left( \frac{mk_B T}{2\pi \hbar^2} \right) \right), \]
where \( n = N/A \) is the areal density.

(b) Also \[ F = N k_B T \left[ \ln(nA_Q) - 1 \right]. \]

We obtain \( U \) from \( U = (\partial / \partial \beta)(\beta F) \) which gives \[ U = N k_B T, \]
noting that \( A_Q \sim T^{-1} \) (not \( T^{-3/2} \) which is the result in three dimensions).

(c) In the same way, \( S = -\partial F / \partial T \) gives
\[ S = N k_B \left[ 2 - \ln(nA_Q) \right]. \]

3. Let us define \( V_1 = V, V_2 = 2V, V_3 = 4V \), where \( V \) is the initial volume, and similarly \( T_1 = T (= 300), T_2 = T_1, \) and the final temperature is \( T_3. \)

(a) \( V_1 \rightarrow V_2 \) is isothermal. As discussed in class the heat supplied is \( N k_B T \ln(V_2/V_1) = N k_B T \ln 2. \) \( V_2 \rightarrow V_3 \) is isentropic so no heat is added. Hence the total heat added is
\[ N k_B T \ln 2 = 6.02 \times 10^{23} \times 0.693 \times 1.38 \times 10^{-23} \times 300 = 1728 \text{ J}. \]

where we used that 1 mole contains Avogadro’s number of molecules \( N = 6.02 \times 10^{23}. \)

(b) In the first process \( T \) is constant. In the second process we have \( TV^{2/3} = \text{const.} \) and so
\[ T_3 = T \left( \frac{1}{2} \right)^{2/3} = 0.63T = 189 \text{ K}. \]

(c) As discussed in the book the increase in entropy is
\[ \Delta S = N k_B \ln(V_2/V_1) = N k_B \ln 2 = 5.76 \text{ JK}^{-1}. \]
4. We are given that the distribution of speeds is

\[ P(v) = \frac{1}{\sqrt{2\pi}} \left( \frac{m}{k_B T} \right)^{3/2} v^2 \exp \left( -\frac{mv^2}{2k_B T} \right). \]  

(1)

To answer this question we will need certain results for Gaussian integrals mentioned in class

\[ \int_{0}^\infty e^{-a^2x^2/2} \, dx = \sqrt{\frac{\pi}{2 \, a}}, \]  

(2)

\[ \int_{0}^\infty x^2 e^{-a^2x^2/2} \, dx = \sqrt{\frac{\pi}{2 \, a^3}}, \]  

(3)

\[ \int_{0}^\infty x^4 e^{-a^2x^2/2} \, dx = 3 \sqrt{\frac{\pi}{2 \, a^5}}. \]  

(4)

Note that Eq. (3) shows that the distribution in Eq. (1) is correctly normalized, i.e. \( \int_{0}^\infty P(v) \, dv = 1 \). We will also need

\[ \int_{0}^\infty x e^{-a^2x^2/2} \, dx = \frac{1}{a^2}, \]  

(5)

which is easy because indefinite integral is \(-(1/a^2)e^{-a^2x^2/2}\), and

\[ \int_{0}^\infty x^3 e^{-a^2x^2/2} \, dx = \frac{2}{a^3}, \]  

(6)

which is done by integrating by parts to make it look like Eq. (5).

(a) Using Eq. (4) with \( a^2 = m/k_B T \) we get

\[ \langle v^2 \rangle = \int_{0}^\infty v^2 P(v) \, dv = \sqrt{\frac{2}{\pi}} \left( \frac{m}{k_B T} \right)^{3/2} 3 \sqrt{\frac{\pi}{2}} \left( \frac{k_B T}{m} \right)^{5/2} = \frac{3k_B T}{m}, \]

and so the rms velocity is given by

\[ v_{\text{rms}} \equiv \langle v^2 \rangle^{1/2} = \sqrt{3k_B T \over m}. \]

(b) The most probable value of the speed, \( v_{\text{mp}} \), is where \( P(v) \) in Eq. (1) has a maximum, i.e.

\[ \left( 2v_{\text{mp}} - v_{\text{mp}}^2 \frac{mv_{\text{mp}}}{k_B T} \right) \exp \left( -\frac{mv_{\text{mp}}^2}{2k_B T} \right) = 0, \]

i.e.

\[ v_{\text{mp}} = \sqrt{2k_B T \over m}. \]

(c) From Eq. (3), the mean speed is given by

\[ \langle v \rangle = \int_{0}^\infty v P(v) \, dv = \sqrt{2 \over \pi} \left( \frac{m}{k_B T} \right)^{3/2} \int_{0}^\infty v^3 \exp \left( -\frac{mv^2}{2k_B T} \right) \, dv = \sqrt{2 \over \pi} 2 \sqrt{\frac{k_B T}{m}} \int_{0}^\infty v^3 \exp \left( -\frac{mv^2}{2k_B T} \right) \, dv = \sqrt{8k_B T \over \pi m}. \]
(d) We are also given that the probability for a single component of velocity is

\[ P_z(v_z) = \sqrt{\frac{m}{2\pi k_B T}} \exp \left( -\frac{mv_z^2}{2k_B T} \right). \]

Noting that \( v_z \) can have either sign we get

\[ \langle |v_z| \rangle = \int_{-\infty}^{\infty} |v_z| P_z(v_z) \, dv_z = 2 \int_0^\infty v_z P_z(v_z) \, dv_z = 2 \sqrt{\frac{m}{2\pi k_B T}} \int_0^\infty v_z \exp \left( -\frac{mv_z^2}{2k_B T} \right) = \sqrt{\frac{2k_B T}{\pi m}}, \]

where we used Eq. (5) to get the final result. Note that \( \langle |v_z| \rangle = \frac{1}{2}\langle v \rangle \).

5. Following the discussion in class, the number of states in which the magnitude of the wavevector lies between \( k \) and \( k + dk \) is

\[ 2 \left( \frac{L}{\pi} \right)^3 \frac{4\pi k^2 \, dk}{8}. \]

We write this as \( \rho(\epsilon) \, d\epsilon = \rho(\epsilon) \left( \partial\epsilon / \partial k \right) \, dk \), and so

\[ \rho(\epsilon) = \frac{V}{\pi^2 (\hbar c)^3} \frac{1}{\left( \partial\epsilon / \partial k \right)} = \frac{V}{\pi^2 (\hbar c)^3} \frac{1}{\hbar c} = \frac{V}{\pi^2 (\hbar c)^3} \epsilon^2. \]

(a) At \( T = 0 \) we fill up all the states up to \( \epsilon_F \), i.e.

\[ N = \frac{V}{\pi^2 (\hbar c)^3} \int_0^{\epsilon_F} \epsilon^2 \, d\epsilon = \frac{V}{3\pi^2} \left( \frac{\epsilon_F}{\hbar c} \right)^3. \]  

This can be rearranged as

\[ \epsilon_F = \frac{\pi^{2/3} \hbar c (3n)^{1/3}}{3}. \]

(b) The energy is given by

\[ U = \int_0^{\epsilon_F} \epsilon \rho(\epsilon) \, d\epsilon = \frac{V}{4\pi^2} \left( \frac{\epsilon_F}{\hbar c} \right)^3 = \frac{3}{4} N \epsilon_F, \]

where we used Eq. (7).

6. (a) As shown in the book, the energy of an ideal Fermi gas at \( T = 0 \) is

\[ U = \int_0^{\epsilon_F} \epsilon \rho(\epsilon) \, d\epsilon = \frac{3}{5} N \epsilon_F = \frac{3}{10} N \frac{\hbar^2}{m} \left( 3\pi^2 N / V \right)^{2/3}. \]  

From the thermodynamic identity

\[ dU = T dS - P dV \]

we have

\[ P = -\left( \frac{\partial U}{\partial V} \right)_S. \]

However at \( T = 0 \) the entropy is zero, (Third Law) and so constant \( S \) is equivalent to constant \( T \). Hence we can obtain the pressure by differentiating Eq. (8) with respect to \( V \), i.e.

\[ P = -\frac{\partial U}{\partial V} = \frac{(3\pi^2)^{2/3} \hbar^2}{5} \frac{n^{5/3}}{m}. \]