PHYSICS 112
Homework 9 Solutions

1. The web site is `http://physics.ucsc.edu/~peter/ising/ising.html`.

   (a) The range of the correlations is quite small. The system quickly settles down to a state which (statistically) doesn't depend on the starting state.

   (b) The correlated regions are now large. The fluctuations of these large correlated regions are quite slow.

   (c) The state is ordered, with virtually all spins down (white) and just a few flipped spins. With a random start, domains of both signs grow, and usually, for some time, there is one large white domain and one large blue domain. Eventually one of these will grow at the expense of the other, and we will have a ordered state with just a few overturned spins, as when when we started with an ordered state. This is an example of "spontaneous symmetry breaking".

2. (a) Consider

$$m = \tanh\left(\frac{J_0 m + B}{k_B T}\right) , \qquad (1)$$

   for $T > T_c = J_0/k_B$ and $B \to 0$. In this limit $m \to 0$ as shown in class. Hence we will differentiate this expression with respect to B (remembering that *both* factors of m must be differentiated) and then set $m = B = 0$. The differentiation gives (with $\chi \equiv \partial m/\partial B$)

$$\chi = \frac{J_0\chi + 1}{k_B T}\,\mathrm{sech}^2\left(\frac{J_0 m + B}{k_B T}\right) ,$$

   Setting $m = B = 0$ the sech becomes unity, and so

$$\chi = \frac{J_0\chi + 1}{k_B T} ,$$

   which can easily be arranged to

$$\boxed{\chi = \frac{1}{k_B(T - T_c)}\,.}$$

   Note: This shows that the system is very sensitive to a small magnetic field just above the transition temperature. This should not be surprising because below T_c the magnetization appears *spontaneously* without any applied field at all.

   (b) Going back to Eq. (1) and setting $T = J_0/k_B\ (= T_c)$, and using $\tanh(x) = x - x^3/3 + \cdots$ gives

$$m = (m + b) - \frac{1}{3}(m + b)^3 + \cdots ,$$

   where $b = B/k_B T$. This immediately gives

$$(m + b)^3 = 3b\,. \qquad (2)$$

   For $b \to 0$ the solution is $m \sim b^{1/3}$ so $m \gg b$ and the factor of b on the LHS of Eq. (2) can be neglected compared with m. Hence we get

$$m \sim b^{1/3}\ \boxed{\sim B^{1/3}\,.}$$

3. As discussed in class the mean field H is given, in the absence of an external field, by

$$H = zJ\langle S\rangle = zJm$$

where $m = \langle S\rangle$ is the magnetization. In a field H the energy states of a spin are, for the spin-1 case,

$$\begin{array}{ll} S = 1, & E = -H \\ S = 0, & E = 0 \\ S = -1, & E = H \end{array}$$

Hence the expectation of S is given by

$$m = \frac{e^{\beta H} + 0 - e^{-\beta H}}{e^{\beta H} + 1 + e^{-\beta H}},$$

or

$$m = \frac{2\sinh\left(\dfrac{zJm}{k_B T}\right)}{1 + 2\cosh\left(\dfrac{zJm}{k_B T}\right)}.$$

To get the temperature of the transition, assumed second order, expand the RHS to first order in m, i.e.

$$m = \frac{2zJ}{3k_B T}\, m + \cdots .$$

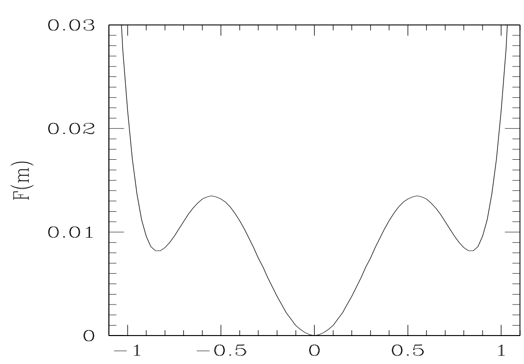
The transition is where the cofficients of m on both sides of the equation are equal, i.e.

$$\boxed{k_B T_c = \frac{2}{3}\, zJ\,.}$$

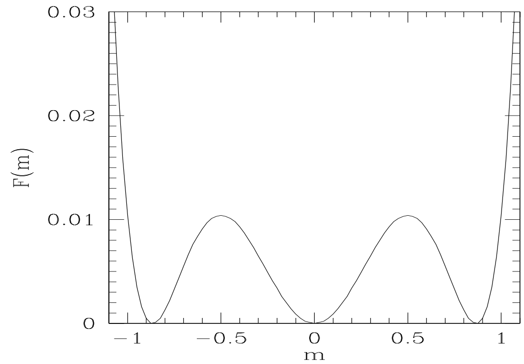
4. (Optional)
We consider the Landau free energy

$$F(m) = \frac{1}{2}\, a(T)\, m^2 + \frac{1}{4}\, c\, m^4 + \frac{1}{6}\, d\, m^6 .$$

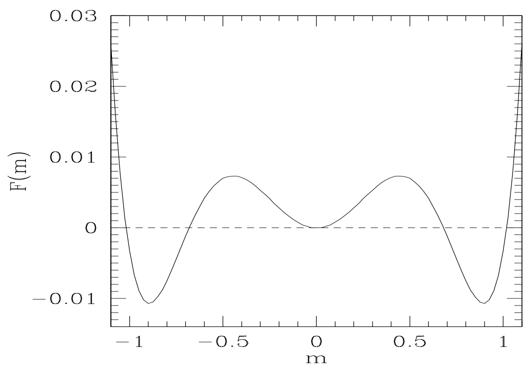
We discussed in class the case of $c > 0$, and showed that the equilibrium value of m tends to zero *continuously*. If $c < 0$ then $F(m)$ starts to develop minima at non-zero m even above T_c as shown in the figure below.



At $T = T_c$ the free energy of these two (equivalent) minima is equal to that of the $m = 0$ solution. See the figure below.



For $T < T_c$ the solutions at non-zero m have the lowest free energy, as shown in the figure below.



Let $\pm m_0$ be the value of m at the two minima at non-zero m in middle of the three figures above. The figures show that the magnetization drops discontinuously from m_0 to 0 at T_c. In other words the transition is $\boxed{\text{discontinuous}}$ if $c < 0$.