

Physics 115/242; Peter Young

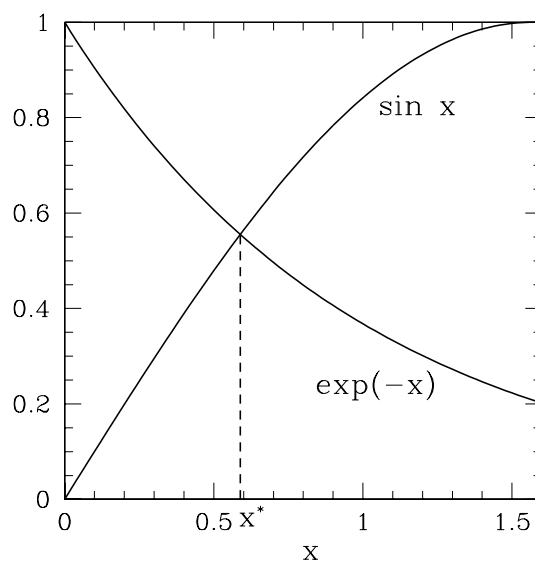
Some output from root finding algorithms.

We consider the problem of finding the root of

$$\boxed{f(x) \equiv e^{-x} - \sin x = 0} \quad (1)$$

for  $x$  in the interval from 0 to  $\pi/2$ . This corresponds to the intersection of the curves  $e^{-x}$  and  $\sin x$  shown in the figure. We denote the root by  $x^*$ . It's value is is

$$\boxed{x^* = 0.5885327439818611 \dots} \quad (2)$$



### I. BISECTION METHOD

We take the desired accuracy to be  $10^{-4}$ , and use the starting values  $x_l = 0.4, x_g = 0.8$ . Successive estimates are then

$n$	$x_l$	$x_g$
1	0.40000	0.60000
2	0.50000	0.60000
3	0.55000	0.60000
4	0.57500	0.60000
5	0.58750	0.60000
6	0.58750	0.59375
7	0.58750	0.59063
8	0.58750	0.58906
9	0.58828	0.58906
10	0.58828	0.58867
11	0.58848	0.58867
12	0.58848	0.58857

The program stopped when  $|x_g - x_l| < 10^{-4}$  and gives

$$\boxed{x = 0.5885}, \quad (3)$$

which is correct to 4 decimal places. The uncertainty in the value of  $x^*$  after  $n$  interactions,  $\epsilon_n \equiv x_g - x_l$ , varies as

$$\boxed{|\epsilon_n| = \frac{1}{2} |\epsilon_{n-1}|}, \quad (4)$$

so the number of decimal places of accuracy increases proportional to  $n$ , which we call (in this context) *linear convergence*.

## II. SECANT METHOD

We take the desired accuracy to be  $10^{-10}$  and use take starting values  $x_0 = 0.4$  and  $x_1 = 0.8$ . Subsequent values are:

$n$	$x_n$	$ x_n - x_{n-1} $	$x_n - \text{exact}$	bracketed?
2	0.604690800056	0.195309200	$0.161580561 \times 10^{-1}$	not bracketed
3	0.586997111757	$0.176936883 \times 10^{-1}$	$-0.153563222 \times 10^{-2}$	bracketed
4	0.588542746639	$0.154563488 \times 10^{-2}$	$0.100026570 \times 10^{-4}$	bracketed
5	0.588532750126	$0.999651316 \times 10^{-5}$	$0.614386464 \times 10^{-8}$	not bracketed
6	0.588532743982	$0.614388918 \times 10^{-8}$	$-0.245359288 \times 10^{-13}$	bracketed
7	0.588532743982	$0.245359288 \times 10^{-13}$	0	

The program stopped when  $|x_n - x_{n-1}| < 10^{-10}$ , and gives

$$\boxed{x = 0.5885327440}, \quad (5)$$

correct to 10 decimal places. The fourth column, which gives the error in  $x_n$  shows that the final value of  $x$  actually agrees with the correct value to machine precision (about  $10^{-16}$ ), much more accurate than the precision specified. The last column indicates whether the last two values for  $x_n$  bracket the root or not. There is apparently no simple pattern to this.

In class, we stated without proof that the number of decimal places of accuracy typically increases by a factor of the golden ratio,  $1.618 \dots$  on each iteration, i.e., if  $\epsilon_{n-1}$  is small,

$$\boxed{|\epsilon_n| = C |\epsilon_{n-1}|^{1.618}}, \quad (6)$$

where  $C$  is a constant and  $\epsilon_n$  is the error in  $x_n$ . For a derivation see

<http://www.math.uic.edu/~leykin/mcs471/NOTES/secant.pdf>

and

<http://www.mathpath.org/Algor/squareroot/secant.pdf>.

One can see that the data is indeed consistent with a convergence that is faster than linear but slower than quadratic. Note, however, that the root does not remain bracketed by the last two values of  $x_n$ , and convergence is not guaranteed if the initial values,  $x_0$  and  $x_1$ , are far from the root even if they bracket it.

### III. NEWTON-RAPHSON METHOD

We take the desired accuracy to be  $10^{-14}$  and the starting values are  $x_0 = 0.8$ . Subsequent values are

$n$	$x_n$	$ x_n - x_{n-1} $
1	0.5661267157835728	0.233873284216E+00
2	0.5883360593267145	0.222093435431E-01
3	0.5885327285001999	0.196669173485E-03
4	0.5885327439818610	0.154816610642E-07
5	0.5885327439818611	0.111022302463E-15

The program stopped when  $|x_n - x_{n-1}| < 10^{-14}$ , and gives

$$x = 0.58853274398186, \quad (7)$$

correct to 14 decimal places.

Note that the data is consistent with the expected result that the number of decimal places of accuracy doubles on each iteration,

$$|\epsilon_n| = C |\epsilon_{n-1}|^2, \quad (8)$$

if  $\epsilon_{n-1}$  is small. In other words we have *quadratic convergence*, which is very rapid. On the other hand, depending on the problem and the starting value of  $x$ , convergence may not occur at all!