Zeroes of the Zeta Function

This handout illustrates Mathematica's wide knowledge of mathematical functions.

The zeta function \(\zeta(n)\) for positive integer \(n\) is defined by

\[
\zeta(n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots
\]

It is well known that this blows up for \(n = 1\) because the series is logarithmically divergent. Values of \(\zeta(n)\) are known exactly for \(n\) an even positive integer, e.g.

\[
\text{Zeta[2]}
\]

\[
\frac{n^2}{6}
\]

The definition given above also makes sense if one replaces \(n\) by complex \(z\) provided that \(\text{Re}[z] > 1\) but blows up if \(\text{Re}[z] < 1\). However, there are other definitions of \(\zeta(z)\), beyond the scope of the course, which agree with the above one where the latter is valid but which can be used anywhere in the complex plane (except for \(z=1\) exactly where the function has a pole).

The properties of \(\zeta(z)\) in the complex plane, thus defined, have aroused a lot of interest, especially the location of the zeroes. There are "trivial" zeroes for \(z\) a negative even integer. This is illustrated in the plot below:

\[
\text{Plot[Zeta[x], \{x, -10.5, -1\}, PlotRange -> \{\{-10.5, 0\}, \{-0.02, 0.01\}\}, AxesLabel \rightarrow \{"x", \"\zeta(x)\"\}]}
\]

It is generally believed, but not proved, that all the other zeroes have \(\text{Re}[z] = 1/2\). This is the famous Riemann hypothesis. Using Mathematica's ability to compute and plot functions in the complex plane, we can see that there are indeed zeroes along this line.
We can locate the roots accurately using the `FindRoot` command, taking the initial estimates (2 are needed because there is no analytic expression for the derivative). As an example, for the first root:

\[
y /. \text{FindRoot}[\text{Abs}[\text{Zeta}[1/2 + iy]], \{y, 14, 15\}]
\]

14.1347

It can be shown that all zeroes but the trivial ones must lie in the region \(0 < \text{Re}[z] < 1\). If you can find a zero in this region which does not have \(\text{Re}[z] = 1/2\) you will be famous!

Below is a 3-D plot of \(1/|\zeta(x + iy)|\) where the zeroes of \(\zeta\) show up as peaks. The range covered is \(-2 < x < 2, 0 < y < 30\). One sees the expected peaks at \(x = 1/2\) (three of them for this range of \(y\)) and the peak corresponding to the trivial zero at \(x = -2, y = 0\).