Physics 116C
Solution of inhomogeneous ordinary differential equations using Green's functions

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1 Homogeneous Equations

We have studied, especially in a long HW problem, second order linear *homogeneous* differential equations which can be written as an eigenvalue problem of the form

$$\boxed{\mathcal{L} y_n(x) = \lambda_n y_n(x)\,,} \tag{1}$$

where $\mathcal{L}$ is an operator involving derivatives, λ_n is an eigenvalue and $y_n(x)$ is an eigenfunction (which satisfies some specified boundary conditions). The general case that we are interested in is called a "Sturm-Liouville" problem, for which one can show that the eigenvalues are real, and the eigenfunctions are orthogonal, i.e.

$$\int_a^b y_n(x) y_m(x)\, dx = \delta_{n\,m}\,, \tag{2}$$

where a and b are the upper and lower limits of the region where we are solving the problem, and we have also "normalized" the solutions[1].

A simple example, which we will study in detail, will be[2]

$$y'' + \tfrac{1}{4} y = \lambda y\,, \tag{3}$$

in the interval $0 \le x \le \pi$, with the boundary conditions $y(0) = y(\pi) = 0$. This corresponds to

$$\mathcal{L} = \frac{d^2}{dx^2} + \frac{1}{4}\,. \tag{4}$$

This is just the simple harmonic oscillator equation, and so the solutions are $\cos kx$ and $\sin kx$. The boundary condition $y(0) = 0$ eliminates $\cos kx$ and the condition $y(\pi) = 0$ gives $k = n$ a positive integer. (*Note:* For $n = 0$ the solution vanishes and taking $n < 0$ just gives the same solution as that for the corresponding positive value of n because $\sin(-nx) = -\sin(nx)$. Hence we only need consider positive integer n.) The normalized eigenfunctions are therefore

$$y_n(x) = \sqrt{\frac{2}{\pi}}\, \sin nx, \qquad (n = 1, 2, 3, \cdots), \tag{5}$$

and the eigenvalues in Eq. (3) are

$$\lambda_n = \tfrac{1}{4} - n^2\,, \tag{6}$$

since the equation satisfied by $y_n(x)$ is $y_n'' + n^2 y_n = 0$.

[1] The general Sturm-Liouville problem has a "weight function" $w(x)$ multiplying the eigenvalue on the RHS of Eq. (1) and the same weight function multiplies the integrand shown in the LHS of the orthogonality and normalization condition, Eq. (2). Furthermore the eigenfunctions may be complex, in which case one must take the complex conjugate of either y_n or y_m in in Eq. (2). Here, to keep the notation simple, we will just consider examples with $w(x) = 1$ and real eigenfunctions.

[2] Different books adopt different sign conventions for the definition of $\mathcal{L}$, and hence of the Green's functions.

2 Inhomogeneous Equations

Green's functions, the topic of this handout, appear when we consider the *inhomogeneous* equation analogous to Eq. (1)

$$\boxed{\mathcal{L}y(x) = f(x)\,,} \tag{7}$$

where $f(x)$ is some *specified* function of x. The idea of the method is to determine the "Green's function", $G(x, x')$, which is given by the solution of the equation

$$\boxed{\mathcal{L}G(x, x') = \delta(x - x')\,,} \tag{8}$$

for the specified boundary conditions. In Eq. (8), the differential operators in $\mathcal{L}$ act on x, and x' is a constant. Once G has been determined, the solution of Eq. (7) can be obtained for *any* function $f(x)$ from

$$\boxed{y(x) = \int G(x, x') f(x')\, dx'\,,} \tag{9}$$

which follows since

$$\mathcal{L}y(x) = \mathcal{L} \int G(x, x') f(x')\, dx' = \int \delta(x - x') f(x')\, dx = f(x)\,, \tag{10}$$

so $y(x)$ satisfies Eq, (7) as required. Note that we used Eq. (8) to obtain the second equality in Eq. (10). We emphasize that the *same* Green's function applies for any $f(x)$, and so it only has to be calculated once for a given differential operator $\mathcal{L}$ and boundary conditions.

3 Expression for the Green's functions in terms of eigenfunctions

In this section we will obtain an expression for the Green's function in terms of the eigenfunctions $y_n(x)$ of the homogeneous equation, Eq. (1).

We assume that the solution $y(x)$ of the inhomogeneous equation, Eq. (7), can be written as a linear combination of the eigenfunctions $y_n(x)$, obtained with the same boundary conditions, i.e.

$$y(x) = \sum_n c_n\, y_n(x)\,, \tag{11}$$

for some choice of the constants c_n. Substituting into Eq. (7) gives

$$f(x) = \mathcal{L}y(x) = \sum_n c_n \mathcal{L} y_n(x) = \sum_n c_n \lambda_n y_n(x). \tag{12}$$

To determine the c_n we multiply by one of the eigenfunctions, $y_m(x)$ say, and integrate use the orthogonality of the eigenfunctions, Eq. (2). This gives

$$\int_a^b f(x) y_m(x)\, dx = \sum_n c_n \lambda_n \int_a^b y_n(x) y_m(x)\, dx = c_m \lambda_m\,. \tag{13}$$

Substituting for c_n into Eq. (11) gives

$$y(x) = \sum_n \frac{1}{\lambda_n} \int_a^b y_n(x') f(x')\, dx'\; y_n(x)\,, \tag{14}$$

which can be written in the form of Eq. (9) with

$$\boxed{G(x,x') = \sum_n \frac{1}{\lambda_n}\, y_n(x) y_n(x')\,.} \tag{15}$$

Note that $G(x,x')$ is a symmetric function of x and x' which is a quite general result. Furthermore, it only depends on the eigenfunctions of the corresponding homogeneous equation, i.e. on the boundary conditions and $\mathcal{L}$. It is independent of $f(x)$ and so can be computed once and for all, and then applied to any $f(x)$ just by doing the integral in Eq. (9).

4 A simple example

As an example, let us determine the solution of the inhomogeneous equation corresponding to the homogeneous equation in Eq. (3), i.e.

$$\boxed{y'' + \tfrac{1}{4} y = f(x),} \qquad \text{with } y(0) = y(\pi) = 0\,. \tag{16}$$

First we will evaluate the solution by elementary means for two choices of $f(x)$

$$\text{(i) } f(x) = \sin 2x, \qquad \text{(ii) } f(x) = x/2\,. \tag{17}$$

We will then obtain the solutions for these cases from the Green's function determined according to Eq. (15).

In the elementary approach, one writes the solution of Eq. (16) as a combination of a complementary function $y_c(x)$ (the solution with $f(x) = 0$) and the particular integral $y_p(x)$ (a particular solution with $f(x)$ included). Since, for $f(x) = 0$, the equation is the simple harmonic oscillator equation, $y_c(x)$ is given by

$$y_c(x) = A\cos(x/2) + B\sin(x/2). \tag{18}$$

For the particular integral, we assume that $y_p(x)$ is of a similar form to $f(x)$. We now determine the solution for the two choices of $f(x)$ in Eq. (16).

(i) For $f(x) = \sin 2x$ we try $y_p(x) = C\cos 2x + D\sin 2x$ and substituting gives $(-4 + 1/4)D = 1$, and $C = 0$. This gives

$$y(x) = y_c(x) + y_p(x) = -\frac{4}{15}\sin 2x + A\cos(x/2) + B\sin(x/2)\,. \tag{19}$$

The boundary conditions are $y(0) = y(\pi) = 0$, which gives $A = B = 0$. Hence

$$\boxed{y(x) = -\frac{4}{15}\sin 2x,} \qquad \text{for } f(x) = \sin 2x. \tag{20}$$

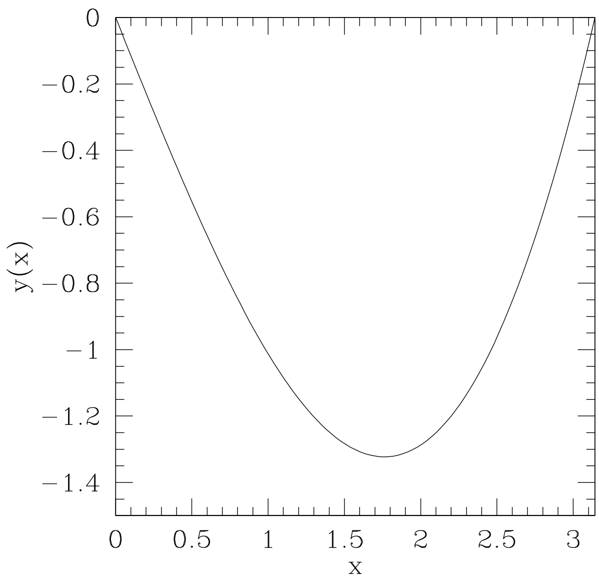
(ii) Similarly for $f(x) = x$ we try $y_p(x) = C + Dx$ and substituting into Eq. (16) gives $C = 0, D = 2$, and so

$$y(x) = y_c(x) + y_p(x) = 2x + A\cos(x/2) + B\sin(x/2)\,. \tag{21}$$

The boundary conditions, $y(0) = y(\pi) = 0$, give $A = 0, B = -2\pi$. Hence

$$\boxed{y(x) = 2x - 2\pi\sin(x/2),} \qquad \text{for } f(x) = x/2. \tag{22}$$

This is plotted in the figure below



Now lets work out the Green's function, which is given by Eq. (15). The eigenfunction are given by (5) and the eigenvalues are given by Eq. (6) so we have

$$\boxed{G(x,x') = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx\,\sin nx'}{\frac{1}{4}-n^2}\,.} \tag{23}$$

We can now substitute this into Eq. (9) to solve Eq. (16) for the two choices of $f(x)$ in Eq. (17).

(i) First of all for $f(x) = \sin 2x$ Eq. (9) becomes

$$y(x) = \frac{2}{\pi}\int_0^{\pi}\left(\sum_{n=0}^{\infty}\frac{\sin nx\,\sin nx'}{\frac{1}{4}-n^2}\right)\sin 2x'\,dx' = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin nx}{\frac{1}{4}-n^2}\int_0^{\pi}\sin nx'\sin 2x'\,dx'. \tag{24}$$

Because of the orthogonality of the $\sin nx$ in the interval from 0 to π only the $n = 2$ term contributes, and the integral for this case is $\pi/2$. Hence the solution is

$$y(x) = \frac{\sin 2x}{\frac{1}{4}-2^2} = \boxed{-\frac{4}{15}\sin 2x.} \tag{25}$$

in agreement with Eq. (20).

(ii) Now we consider the case of $f(x) = x/2$. Substituting Eq, (23) into Eq. (9) gives

$$y(x) = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin nx}{\frac{1}{4}-n^2}\int_0^{\pi}x'\sin nx'\,dx'. \tag{26}$$

There is no longer any orthogonality to simplify things and we just have to do the integral:

$$\begin{aligned}\int_0^{\pi}x'\sin nx'\,dx' &= \left[-\frac{x'\cos nx'}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx'}{n}\,dx' && (27)\\ &= -\frac{\pi\cos n\pi}{n} + \left[\frac{\sin nx'}{n^2}\right]_0^{\pi} && (28)\\ &= -(-1)^n\frac{\pi}{n}\,. && (29)\end{aligned}$$

For the particular case of $n = 0$ the integral is zero. Hence the solution is

$$\boxed{y(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n(\frac{1}{4} - n^2)}} \,. \tag{30}$$

This may not look the same as Eq. (22), but it is in fact, as one can check by expanding $2x - 2\pi \sin x/2$ as a Fourier sine series in the interval 0 to π, i.e. one writes

$$2x - 2\pi \sin x/2 = \sum_{n=1}^{\infty} c_n \sin nx, \tag{31}$$

and determines the c_n from the usual Fourier integral.

Using the Green's function may seem to be a complicated way to proceed, especially for our second choice of $f(x)$. However, you should realize that the "elementary" derivation of the solution may not be so simple in other cases, and you should note that the Green's function applies to *all* possible choices of the function on the RHS, $f(x)$. Furthermore, we will see in the next section that one can often get a *closed form* expression for G, rather than an infinite series. It is then much easier to find a closed form expression for the *solution*.

5 Closed form expression for the Green's function

In many useful cases, one can obtain a closed form expression for the Green's function by starting with the defining equation, Eq. (8). We will illustrate this for the example in the previous section for which Eq. (8) is

$$G'' + \tfrac{1}{4} G = \delta(x - x') \,. \tag{32}$$

Remember that x' is *fixed* (and lies between 0 and π) while x is a *variable*, and the derivatives are with respect to x. We solve this equation separately in the two regions (i) $0 \le x < x'$, and (ii) $x' < x \le \pi$. In each region separately the equation is $G'' + (1/4)G = 0$, for which the solutions are

$$G(x, x') = A \cos(x/2) + B \sin(x/2) \,, \tag{33}$$

where A and B will depend on x'. Since $y(0) = 0$, we require $G(0, x') = 0$ (recall Eq. (9)) and so, for the solution in the region $0 \le x < x'$, the cosine is eliminated. Similarly $G(\pi, x') = 0$ and so, for the region $x' < x \le \pi$, the sine is eliminated. Hence the solution is

$$G(x, x') = \begin{cases} B \sin(x/2) & (0 \le x < x'), \\ A \cos(x/2) & (x' < x \le \pi). \end{cases} \tag{34}$$

How do we determine the two coefficients A and B? We get one relation between them by requiring that the solution is continuous at $x = x'$, i.e. the limit as $x \to x'$ from below is equal to the limit as $x \to x'$ from above. This gives

$$B \sin(x'/2) = A \cos(x'/2) \,. \tag{35}$$

The second relation between A and B is obtained by integrating Eq. (32) from $x = x' - \epsilon$ to $x' + \epsilon$, and taking the limit $\epsilon \to 0$, which gives

$$\lim_{\epsilon \to 0} \left[\frac{dG}{dx} \right]_{x'-\epsilon}^{x'+\epsilon} + \tfrac{1}{4} \lim_{\epsilon \to 0} \int_{x'-\epsilon}^{x'+\epsilon} G(x, x')\, dx = \lim_{\epsilon \to 0} \int_{x'-\epsilon}^{x'+\epsilon} \delta(x - x')\, dx \tag{36}$$

so

$$\lim_{\epsilon \to 0} \left(\left. \frac{dG}{dx} \right|_{x'+\epsilon} - \left. \frac{dG}{dx} \right|_{x'-\epsilon} \right) + 0 \;=\; 1 \,. \tag{37}$$

Hence dG/dx has a discontinuity of 1 at $x = x'$, i.e.

$$-\frac{A}{2}\,\sin(x'/2) - \frac{B}{2}\,\cos(x'/2) = 1. \tag{38}$$

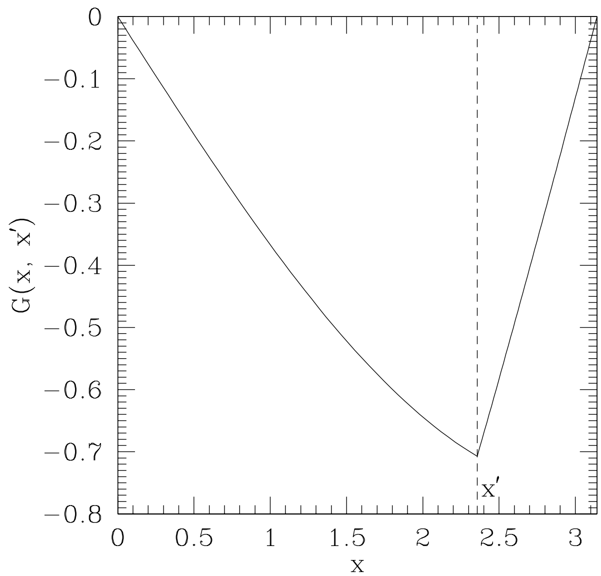
Solving Eqs. (35) and (38) gives

$$\begin{aligned} B &= -2\cos(x'/2), &(39)\\ A &= -2\sin(x'/2). &(40)\end{aligned}$$

Substituting into Eq. (34) gives

$$\boxed{G(x,x') = \begin{cases} -2\cos(x'/2)\sin(x/2) & (0 \le x < x'), \\ -2\sin(x'/2)\cos(x/2) & (x' < x \le \pi). \end{cases}} \tag{41}$$

A sketch of the solution is shown in the figure below. The discontinuity in slope at $x = x'$ (I took $x' = 3\pi/4$) is clearly seen.



It is instructive to rewrite Eq. (41) in terms of $x_<$, the smaller of x and x', and $x_>$, the larger of x and x'. One has

$$\boxed{G(x,x') = -2\sin(x_</2)\cos(x_>/2)\,,} \tag{42}$$

irrespective of which is larger, which shows that G is symmetric under interchange of x and x' as noted earlier.

We now apply the *closed form* expression for G in Eq. (41) to solve our simple example, Eq. (16), with the two choices for $f(x)$ shown in Eq. (17).

(i) For $f(x) = \sin 2x$, Eqs. (9) and (41) give

$$y(x) = -2\cos(x/2)\int_0^x \sin 2x'\sin(x'/2)\,dx' - 2\sin(x/2)\int_x^\pi \sin 2x'\cos(x'/2)\,dx'. \tag{43}$$

Using formulae for sines and cosines of sums of angles and integrating gives

$$\begin{aligned} y(x) &= -2\cos(x/2)\left(\frac{\sin(3x/2)}{3} - \frac{\sin(5x/2)}{5}\right) - 2\sin(x/2)\left(\frac{\cos(3x/2)}{3} + \frac{\cos(5x/2)}{5}\right) &(44)\\ &= -2\left(\frac{1}{3}\sin 2x\right) + 2\left(\frac{1}{5}\sin 2x\right) &(45)\\ &= -\frac{4}{15}\sin 2x\,, &(46)\end{aligned}$$

where we again used formulae for sums and differences of angles. This result is in agreement with Eq. (20).

(ii) For $f(x) = x/2$, Eqs. (9) and (41) give

$$y(x) = -\cos(x/2) \int_0^x x' \sin(x'/2)\, dx' - \sin(x/2) \int_x^\pi x' \cos(x'/2)\, dx' \tag{47}$$

Integrating by parts gives

$$\begin{aligned} y(x) &= -\cos(x/2)\,(-2x\cos(x/2) + 4\sin(x/2)) - \sin(x/2)\,(2\pi - 4\cos(x/2) - 2x\sin(x/2)) \quad (48)\\ &= 2x - 2\pi\sin(x/2)\,, \quad (49) \end{aligned}$$

which agrees with Eq. (22). Note that we obtained the closed form result explicitly, as opposed to the method in Sec. 3 where the solution was obtained as an infinite series, Eq. (30).

In general, to find a closed form expression for $G(x, x')$ we note from Eq. (8) that, for $x \neq x'$, it satisfies the homogeneous equation

$$\boxed{\mathcal{L}G(x, x') = 0, \qquad (x \neq x')\,.} \tag{50}$$

We solve this equation separately for $x < x'$ and $x > x'$, subject to the required boundary conditions, and match the solutions at $x = x'$ with the following two conditions

(i) $G(x, x')$ is continuous at $x = x'$, i.e.

$$\boxed{\lim_{x \to x'+\epsilon} G(x, x') = \lim_{x \to x'-\epsilon} G(x, x')\,.} \tag{51}$$

(ii) The derivative dG/dx has a discontinuity of 1 at $x = x'$, i.e.

$$\boxed{\lim_{x \to x'+\epsilon} \frac{dG(x, x')}{dx} - \lim_{x \to x'-\epsilon} \frac{dG(x, x')}{dx} = 1\,.} \tag{52}$$

6 Summary

We have shown how to solve linear, inhomogeneous, ordinary differential equations by using Green's functions. These can be represented in terms of eigenfunctions, see Sec. 3, and in many cases can alternatively be evaluated in closed form, see Secs. 4 and 5. The advantage of the Green's function approach is that the Green's function only needs to be computed *once* for a given differential operator $\mathcal{L}$ and boundary conditions, and this result can then be used to solve for *any* function $f(x)$ on the RHS of Eq. (7) by using Eq. (9).

In this handout we have used Green's function techniques for *ordinary* differential equations. They can also be used, in a very simple manner, for *partial* differential equations.

The advantages of Green's functions may not be readily apparent from the simple examples presented here. However, they are used in many *advanced* applications in physics.