PHYSICS 219

Homework 1

Due in class, Wednesday April 19

1. *First, a problem in statistics.*
In the game of bridge each of the 4 players receives 13 cards. The "declarer" sees his own hand and that of "dummy". He therefore knows which cards are held by the two opposing players, but does not know how those cards are divided between them. Suppose the declarer knows that his two opponents hold 4 clubs between them. What are the probabilities that they are divided 0:4, 1:3, and 2:2?
Solve the problem in two stages. First assume that the the four cards are embedded in an infinity of irrelevant cards (this is a popular approximation). Then take into account that each opponent is only dealt 13 cards.

2. *A "random walk" problem, sometimes known as the "drunken walker" problem.*
A man walks along a straight line in unit steps, either positive or negative, starting from the origin. The probability of his taking a positive step is p and that of a negative step $1 - p$.
   (a) Show that the probability of his being a distance m from the origin after t steps is zero unless t is at least as big as m, and of the same parity. Find the probability in the case that it is non-zero.
   (b) Calculate the expectation value of the displacement from the origin after t steps. Also calculate the mean square deviation from this value.

3. *Gaussian integrals*
   (a) Evaluate
$$\int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx, \quad \int_{-\infty}^{\infty} x \exp\left(\frac{-x^2}{2\sigma^2}\right) dx, \quad \frac{\int_{-\infty}^{\infty} x^2 \exp\left(\frac{-x^2}{2\sigma^2}\right) dx}{\int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx},$$
and
$$\frac{\int_{-\infty}^{\infty} x^4 \exp\left(\frac{-x^2}{2\sigma^2}\right) dx}{\int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx}.$$
   (b) Show, by integration by parts or otherwise, that
$$\frac{\int_{-\infty}^{\infty} x^{2n} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx}{\int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2\sigma^2}\right) dx} = (2n-1)!!\, \sigma^{2n}$$
where
$$(2n-1)!! = (2n-1) \times (2n-3) \times \ldots \times 3 \times 1\ .$$

4. Consider a set of N quantum harmonic oscillators of the same frequency ω given that the total quantum number n (and hence the total energy $U \equiv n\hbar\omega$) is *fixed*, i.e.

$$\sum_{i=1}^{N} n_i = n \ .$$

Show that the number of possible states is given by

$$g(N,n) = \frac{(n+N-1)!}{n!\ (N-1)!} \ .$$

Hence determine the temperature of the system and show that

$$\frac{n}{N} = \frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1} \ .$$

This is the Planck result derived the "hard way" using the microcanonical ensemble. It is, of course, much easier to derive it using the canonical ensemble in which the total energy is not strictly fixed.

5. *Fluctuations in energy and particle number*

   (a) Consider a system of fixed volume and fixed number of particles in thermal contact with a reservoir. Derive a relation between the mean square fluctuations in the energy and the specific heat.

   (b) Now suppose that the system can also exchange particles with the reservoir. Express the mean square fluctuations in the number of particles in terms of the derivative $\partial\langle N\rangle/\partial\mu$.

   (c) For both cases (a) and (b) above discuss how the root mean square fluctuations vary with the number of particles.

6. *Zipper problem.*
   A zipper has N links; each link has a state in which it is closed with energy 0 and a state in which it is open with energy ϵ. We require, however, that that the zipper can only unzip from the left end and that the link number s can only open if all links to the left $(1, 2, \ldots, s-1)$ are already open.

   (a) Show that the partition function can be summed in the form

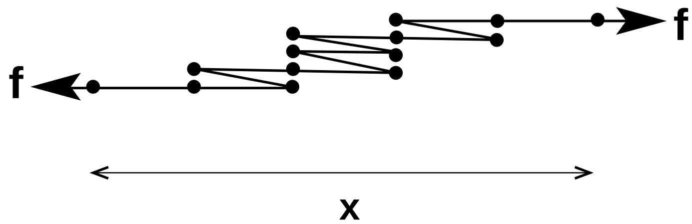
   $$Z = \frac{1 - \exp[-(N+1)\epsilon/k_B T]}{1 - \exp(-\epsilon/k_B T)}.$$

   (b) In the limit $N \to \infty$, find the average number of open links.

   n.b. This model is a very simplified model of the unwinding of two-stranded DNA molecules—see C. Kittel, Amer. J. Phys. **37**, 917 (1969).

7. *Validity of the Canonical Ensemble*
   In the derivation of the grand canonical ensemble in class we considered a system able to exchange both energy and particles with a resevoir. We expanded ΔS, the entropy difference of the resevoir when the system has (a) energy U_1 and N_1 particles, and (b) energy U_2 and N_2 particles, in powers of the differences $N_1 - N_2$ and $U_1 - U_2$. Show that, as assumed, higher order terms in this expansion are negligible when the size of the reservoir tends to infinity, even if $U_1 - U_2$ and $N_1 - N_2$ are not particularly small.

8. Consider a chain consisting of N segments of unit length maintained under tension.



   Each segment can either point to the right or to the left along a line so the problem is effectively one-dimensional. There are no forces other than those at the ends.

   (a) Calculate the entropy of the chain as a function of the end to end distance, x, for $x \ll N$.
   Note: Stirling's approximation, valid for large N, is $\ln N! \simeq N \ln N - N$.

   (b) Determine the relation between the temperature T of the chain and the force, f, (tension) which is necessary to maintain the distance x.

   Note: Your result is the reason why rubber, which consists of cross-linked polymers, has an elastic constant which *increases* with T, whereas most substances get softer as they heat up so their elastic constant *decreases*.