PHYSICS 219

Homework 4

Due in class, Wednesday May 24

1. *Mean field theory*
   Consider the spin-1 Ising model, in which each spin takes the values $-1, 0$ and 1. Assume nearest neighbor interactions only, and take the number of neighbors to be z.

   (a) Derive the mean field equation for the order parameter.

   (b) Determine the transition temperature in mean field theory.

2. *Mean Field Theory of a quantum model*
   Consider the spin-1/2 Ising model in a transverse field, whose Hamiltonian is given by

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i^z S_j^z - \Gamma \sum_i S_i^x \ ,$$

   where the $\mathbf{S}_i$ are spin-1/2 operators.
   (*n.b.* This model has frequently been used as a model for structural phase transitions; it is not just a theoretical oddity).

   (a) Find a self-consistent expression for the magnetization, $m = \langle S_i^z \rangle$, in the mean field approximation.
   Hint: You need to consider the magnitude and direction of the mean field.

   (b) What is the value of the critical temperature when $\Gamma = 0$, (*i.e.* the regular Ising model), and what is the critical value of Γ at $T = 0$?

   (c) Show that $\langle S_i^x \rangle$ is independent of T below T_c, in the approximations that you are using.

   (d) Give a rough sketch of the phase diagram, *i.e.* the $\Gamma - T$ plane, showing the disordered (paramagnetic) region, where $m = 0$, and the ferromagnetic region, where $m \neq 0$.

3. *Landau Theory*
   Consider a system which allows a third order term in the expansion of the free energy in powers of the order parameter.

   (a) Show that Landau theory predicts a first order phase transition.
   Hint: Sketch the free energy as a function of the order parameter for $T > T_c$, $T = T_c$ and $T < T_c$. Make sure that you explain clearly the condition that determines T_c.

   (b) Assume that there is no third order term in the free energy but that the coefficient of the quartic term is negative. Show that Landau theory predicts a first order transition.
   Note: One needs to include a higher order term in the free energy otherwise the free energy is unbounded from below. See the hint to part (a).

   (c) Above the transition, assumed now to be second order, we showed in class that the susceptibility diverges as

$$\chi = \frac{C_+}{t^\gamma} \quad (T > T_c) \ ,$$

where $t \equiv (T - T_c)/T_c$ is the reduced temperature, the (universal) critical exponent, γ, is equal to 1 in Landau theory, and C_+ is the called the critical amplitude. Note that C_+ is *not* universal. However, we can also define the corresponding critical amplitude below T_c in an analogous manner, i.e.

$$\chi = \frac{C_-}{|t|^\gamma} \quad (T < T_c) \ .$$

Show that the ratio C_+/C_- *is* universal according to Landau theory and find its universal value.

Note: Such critical amplitude ratios are also believed to be universal in an exact theory, though the universal value will, in general, be different from that predicted by Landau theory, just as for the exponents.

4. *More Landau theory*
   Consider an n-component spin system which has cubic symmetry, rather than full rotational invariance, within the n-dimensional spin space. Write down the form of the Landau free energy up to fourth order in the order parameter. What are the possible ordered states of this system?

5. *Transfer matrices*
   Consider the spin-1 Ising model in one dimension, with a "quadrupole-quadrupole" Hamiltonian given by

$$\mathcal{H} = -J \sum_i (3S_i^2 - 2)(3S_{i+1}^2 - 2) \ ,$$

   where $S_i = 1, 0$ or -1.

   (a) What is the ground state configuration(s) (i) for $J > 0$ and (ii) for $J < 0$? What is the ground state energy and entropy per spin in each case. Also, what is the entropy per spin at infinite temperature?

   (b) Compute the free energy and energy per site using transfer matrix techniques.

   (c) Check that your expresssion obtained in part (b) reproduces the results in part (a), for *both* signs of J.

6. *Duality in two dimensions*
   The q-state Potts model is defined as follows: at each site i there is a variable which can be in one of q states labeled by an integer $n_i \,(= 0, 1, \cdots, q-1)$. The energy of two neighboring sites is equal to $-J$ if the states of the two sites are the same and zero if the two sites are in different states. The Hamiltonian can therefore be written

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J \delta_{n_i, n_j} \ .$$

   Using a duality transformation, find the exact critical temperature for a square lattice.
   n.b. The case of $q = 2$ corresponds to the Ising model, apart from a redefinition of the zero of energy and a rescaling of J by a factor of 2.
   Hint: You may find it useful to define a bond variable, $n_{ij} = (n_i - n_j) \bmod q$. For more general duality transformations see Wu and Wang, J. Math. Phys, **17**, 439 (1976).