PHYSICS 231

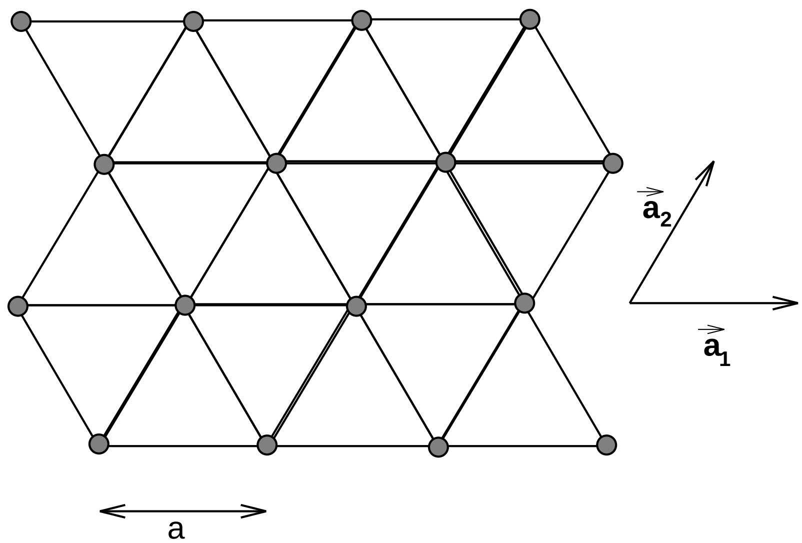
Final Examination

Wednesday December 7, 2011, 4:00–7:00 pm, in ISB 231.

This is a closed book exam but you are allowed to bring in one page of hand written notes, if you wish.
You must explain your working to get full credit.
There are questions on both sides of the sheet.

1. **(35 points)**

   Consider the triangular lattice with lattice spacing a shown in the Figure.



(a) Determine two basis vectors of the reciprocal lattice, and explain what structure is the reciprocal lattice.

(b) What is the shape of the (first) Brillouin zone? Determine the coordinates of (a) a corner of the zone, and (b) the center of an edge of the zone.

(c) Consider a tight-binding Hamiltonian which describes a single band of electrons hopping between nearest-neighbor sites on the triangular lattice,

$$\mathcal{H} = -t \sum_{\langle i,j \rangle} \left(c_i^\dagger c_j + c_j^\dagger c_i \right) , \qquad (1)$$

where the sum is over all nearest-neighbor pairs counted once. (We will neglect spin in this question.) Determine the single-particle energy levels $\epsilon(\mathbf{k})$.

(d) What is the value of $\epsilon(\mathbf{k})$ for

   i. $\mathbf{k} = 0$,

   ii. $\mathbf{k}$ at a corner of the Brillouin zone,

   iii. $\mathbf{k}$ at the center of an edge of the zone.

2. **(15 points)**

(a) By the usual counting arguments, show that the density of low energy acoustic phonon modes in d-dimensions has the form

$$\rho(\omega) \propto \omega^{d-1}. \tag{2}$$

n.b. The volume of a sphere of radius k in d-dimensions is proportional to k^d.

(b) Noting that the specific heat per atom in the harmonic approximation is given by

$$\frac{C}{k_B} = \int_0^\infty \rho(\omega) \left(\frac{\hbar\omega}{k_B T}\right)^2 \frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega} - 1)^2}\, d\omega, \tag{3}$$

(you are *not* required to derive this), show that the low temperature specific heat has the form

$$C \propto T^d. \tag{4}$$

3. **(15 points)**
Consider massless electrons in **two** dimensions with energy given by

$$\epsilon(\mathbf{k}) = \hbar v\, k \tag{5}$$

where v is their velocity.

(a) Starting from the usual result for counting of states in $\mathbf{k}$ space, derive the density of states as a function of energy $g(\epsilon)$.

(b) If the number of electrons per unit area is n, compute the Fermi energy ϵ_F.

(c) Hence show that the energy per electron at $T = 0$ is equal to $(2/3)\,\epsilon_F$.

4. **(35 points)**

(a) Show that the density of carriers in the conduction and valence bands are given by

$$n_c(T) = N_c(T) \exp[-\beta(\epsilon_c - \mu)], \qquad p_v(T) = P_v(T) \exp[-\beta(\mu - \epsilon_v)], \tag{6}$$

respectively, where ϵ_c is the energy of the bottom of the conduction band and ϵ_v is the energy of the bottom of the conduction band, and $N_c(T)$ and $P_v(T)$ are functions of T which you should evaluate.
Note: You are *given* that the density of states of the conduction and valence bands is $g_{c,v} = \sqrt{2|\epsilon - \epsilon_{c,v}|}m_{c,v}^{3/2}/\hbar^3\pi^2$ and that $\int_0^\infty x^{1/2} \exp(-x)\, dx = \sqrt{\pi}/2$.

(b) Explain why these results, as written, are valid whether or not there are impurity states.

(c) Determine n_c and p_v if there are *no* impurities.

(d) Explain qualitatively how μ, n_c, p_v and the product $n_c p_v$ change if donor impurities are added.