PHYSICS 231

Homework 3

Due in class, Wednesday October 26

1. **Lack of Translational Order in One and Two Dimensions**

   The intensity of Bragg scattering is reduced due to motion of the atoms about their equilibrium position by the Debye-Waller factor
$$e^{-2W}, \tag{1}$$
   where
$$W = \frac{1}{N}\sum_{\mathbf{k}}\sum_{s} \frac{\hbar}{2M\omega_s(\mathbf{k})}\left[\mathbf{G}\cdot\epsilon_s(\mathbf{k})\right]^2 \left[n(\omega_s(\mathbf{k})) + \tfrac{1}{2}\right], \tag{2}$$
   in which $\mathbf{G}$ is the reciprocal lattice vector, $n(\omega)$ is the Planck distribution and $\epsilon_s(\mathbf{k})$ is the polarization vector of the mode $(\mathbf{k}, s)$.

   (a) Show that $e^{-2W} = 0$ in one and two dimensions for $T > 0$.

   (b) Show that $e^{-2W} = 0$ in one dimension for $T = 0$.

   (c) What are the implications of these results for crystalline order of one- or two-dimensional crystals?

   Hint: Consider the contribution of long wavelength sound waves to W.

2. **Three Phonon processes in One Dimension**

   Consider a process in which two phonons combine to give a third (or one phonon decays into two others). Let all phonons be acoustic and assume that the two transverse branches lie below the longitudinal branch. Assume that all wavevectors lie along a fixed direction and assume that $d^2\omega/dk^2 \le 0$ for all three branches.

   (a) By representing the conservation laws graphically, or otherwise, show that there can be no process in which all three phonons belong to the same branch.

   (b) Show that the only possible processes are those in which the single phonon is in a branch higher than at least one of the members of the pair: *i.e.* the allowed processes are
$$\text{transverse} + \text{transverse} \leftrightarrow \text{longitudinal} \tag{3}$$
   and
$$\text{transverse} + \text{longitudinal} \leftrightarrow \text{longitudinal}. \tag{4}$$

   Note: although the constraints of energy and momentum conservation are less severe in three dimensions than in this one-dimensional example, they still reduce the rate of phonon scattering due to three phonon processes.

3. **The Free Electron Gas in Two Dimensions**

   (a) What is the relation between n and k_F in two dimensions?

   (b) What is the relation between k_F and r_s in two dimensions? Here r_s is the average spacing between electrons defined, for $d = 3$, in Ashcroft and Mermin, Eq. (1.2).

(c) Prove that in two-dimensions, the free electron density of states, $g(\epsilon)$, is a constant independent of ϵ for $\epsilon > 0$.

(d) Prove that because $g(\epsilon)$ is a constant, every term in the Sommerfeld expansion for the density n vanishes (see Ashcroft and Mermin Eq. (2.70) and Appendix C).

(e) Deduce that in two dimensions

$$\mu + k_B T \ln \left(1 + e^{-\mu/k_B T}\right) = \epsilon_F. \tag{5}$$

(f) Hence show that μ differs from ϵ_F by an exponentially small amount as T is raised from zero, and contrast this with the prediction of the Sommerfeld expansion.

n.b. Even in three dimensions there are exponentially small terms in addition to the terms proportional to powers of T given by the Sommerfeld expansion. Mathematically this is because the Sommerfeld expansion is an *asymptotic* expansion with zero radius of convergence, see *e.g.* Arfken, Mathematical Methods for Physicists. The special feature of the two-dimensional free electron gas is only that *all* the power law terms vanish because the density of states is a constant.

4. **The Classical Limit of Fermi-Dirac Statistics**
The Fermi-Dirac distribution reduces to the (classical) Maxwell-Boltzmann distribution provided that the probability of occupancy of *any* state is much less than unity. Mathematically this corresponds to

$$e^{\mu/k_B T} \ll 1, \tag{6}$$

which clearly then gives

$$f_{FD}(\epsilon) \approx f_{MB}(\epsilon) \equiv e^{-(\epsilon-\mu)/k_B T}. \tag{7}$$

(a) Show this condition for the validity of classical statistics can be written as

$$r_s \gg \left(\frac{\hbar^2}{2mk_B T}\right)^{1/2}. \tag{8}$$

(b) What is the physical significance of the length that r_s must exceed?

(c) Show that the above condition on r_s leads numerically to

$$\frac{r_s}{a_0} \gg \left(\frac{10^5 \text{ K}}{T}\right)^{1/2}, \tag{9}$$

where $a_0 \equiv \hbar^2/me^2$ is the Bohr radius.

5. **Magnitude of the Fermi Velocity**

(a) Show that the typical magnitude of the Fermi velocity is given by

$$v_F \sim \alpha c, \tag{10}$$

where c is the speed of light and $\alpha \equiv e^2/\hbar c \simeq 1/137$ is the fine-structure constant.

(b) The speed of sound in a solid is typically two orders of magnitude (or a bit more) slower than this. What is the important difference between the ions and the electrons which accounts for this difference?