PHYSICS 231

Homework 5

Due in class, Monday November 21

1. Consider an energy band, $\epsilon(\mathbf{k})$, which has a minimum, i.e.

$$\epsilon(\mathbf{k}) = \text{constant} + \left(\frac{\hbar^2}{2}\right)\left(\frac{k_x^2}{m_x} + \frac{k_y^2}{m_y} + \frac{k_z^2}{m_z}\right). \tag{1}$$

(If the minimum is not at $\mathbf{k} = 0$ but at $\mathbf{k} = \mathbf{k}_0$ say, then one replaces $\mathbf{k}$ above by $\mathbf{k} - \mathbf{k}_0$, and the result for m^* below will be unchanged.)

Find the time dependent solution of the equations of motion

$$m_x \frac{dv_x}{dt} = (-e)\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{H}\right)_x, \tag{2}$$

(and similarly for the other components), assuming that $\mathbf{E} = 0$ and the magnetic field is in the z direction. Hence show that the cyclotron frequency is given by

$$\omega = \frac{eH}{m^* c}, \tag{3}$$

where the cyclotron effective mass is given by

$$m^* = (m_x m_y)^{1/2}. \tag{4}$$

Note that this is different from the effective mass mass in the specific heat which is given by $(m_x m_y m_z)^{1/3}$, see Qu. 3 of Homework 4.

2. Consider a electron in a periodic potential in a magnetic field in the z direction. We showed in class that both k_z and the energy, ϵ, are conserved. We also discussed that the orbits could either be closed (i.e. the electron repeatedly follows the same closed path in k-space) or [if the orbit touches the edge of the Brillouin zone] it might be extended (i.e. k in one direction will monotonically increase).

Here we consider *closed orbits*. We showed in the handout, see also Ashcroft and Mermin, p. 231-233, that the period of the closed orbit is given by

$$T(\epsilon, k_z) = \frac{\hbar^2 c}{eH} \frac{\partial}{\partial \epsilon} A(\epsilon, k_z), \tag{5}$$

where A is the area of the closed classical orbit.

Go over that derivation. Show that if one has free electron bands, then the period is just

$$T = \frac{2\pi}{\omega_c}, \tag{6}$$

where

$$\omega_c = \frac{eH}{mc} \tag{7}$$

is the cyclotron frequency, as expected.

n.b. More generally one defines a cyclotron effective mass, m^*, by

$$m^*(\epsilon, k_z) = \frac{\hbar^2}{2\pi} \frac{\partial}{\partial \epsilon} A(\epsilon, k_z). \tag{8}$$

You calculated a cyclotron effective mass in problem 1 above.

3. Consider the Onsager quantization condition,

$$A(\epsilon_\nu(k_z), k_z) = (\nu + \lambda)\Delta A, \tag{9}$$

where

$$\Delta A = \frac{2\pi e H}{\hbar c}, \tag{10}$$

ν is an integer is λ is a real number independent of ν.

   (a) Show that if one applies this expression with $\lambda = \frac{1}{2}$ to the orbits of free electron one gets the free electron energy levels

$$\epsilon_\nu(k_z) = \frac{\hbar^2 k_z^2}{2m} + \left(\nu + \tfrac{1}{2}\right)\hbar\omega_c, \tag{11}$$

   where

$$\omega_c = \frac{eH}{mc} \tag{12}$$

   is the cyclotron frequency.

   (b) Show that the degeneracy,

$$\frac{2e}{hc} H A, \tag{13}$$

   of the free electron levels, where A is the area of the sample perpendicular to the magnetic field, is just the number of zero-field electron levels with the given k_z, and k_x and k_y within a planar region of area ΔA.

4. For the de Haas-van Alphen effect, the fundamental relation is

$$\Delta\left(\frac{1}{H}\right) = \frac{2\pi e}{\hbar c}\frac{1}{A_e}, \tag{14}$$

where A_e is any extremal cross-section area of the Fermi surface in a plane normal to the magnetic field. From this deduce the ratio of the areas of the two extremal orbits responsible for the oscillations in Fig. 14.2b of Ashcroft and Mermin.
n.b. Neither of the axes are labeled but, in this case, it does not matter!

5. In the Drude model, the probability of an electron suffering a collision in any infinitesimal interval dt is just dt/τ.

   (a) Show that an electron picked at random at a given moment had no collision during the preceding t seconds with probability $e^{-t/\tau}$. Show that it will have no collisions during the next t seconds with the same probability. Hence show that the probability that the next collision is between t and $t + dt$ is $(dt/\tau)e^{-t/\tau}$ (and similarly for the preceding collision).

   (b) Show that the probability that the time interval between two successive collisions of an electron falls in the range between t and $t + dt$ is $(dt/\tau)e^{-t/\tau}$.

   (c) Show that as a consequence of (a) that at any moment the mean time back to the last collision (or up to the next collision) averaged over all electrons is τ.

   (d) Show that as a consequence of (b) that the mean time between successive collisions of an electron is τ.

(e) Part (c) implies that at any moment the time T between the last and next collision averaged over all electrons is 2τ. Explain why this is not inconsistent with the result in (d). (You may want to discuss the probability distribution for T).
n.b. Apparently Drude did not appreciate this subtlety and got the electrical conductivity wrong by a factor of 2.

6. **Surface Plasmons**
An electromagnetic wave that can propagate along the surface of a metal complicates the observation of ordinary (bulk) plasmons. Let the metal be contained in the half space $z > 0$, with the region $z < 0$ being vacuum. Assume that the electric charge density ρ appearing in Maxwell's equations vanishes both inside and outside the metal (though a surface charge density can appear in the plane $z = 0$). The surface plasmon is a solution to Maxwell's equations where the time dependence is of the form $e^{-i\omega t}$ and the space dependence is of the form:

$$E_x = Ae^{iqx}e^{-Kz},\ E_y = 0,\ E_z = Be^{iqx}e^{-Kz}, \qquad (z > 0); \tag{15}$$

$$E_x = Ce^{iqx}e^{K'z},\ E_y = 0,\ E_z = De^{iqx}e^{K'z}, \qquad (z < 0); \tag{16}$$

where q is real and K and K' both real and positive. (Note that the amplitude decays exponentially away from the surface both for z positive and negative.)

Maxwell's equations imply that

$$\vec{\nabla} \cdot (\epsilon \vec{E}) = 0, \tag{17}$$

$$-\nabla^2 \mathbf{E} = \left(\frac{\omega}{c}\right)^2 \epsilon(\omega)\mathbf{E}, \tag{18}$$

and are to be solved with the usual boundary conditions, ($\mathbf{E}_{||}$ continuous, $(\epsilon\mathbf{E})_\perp$ continuous). Note also the Drude result

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \qquad \sigma_0 = \frac{ne^2\tau}{m}, \tag{19}$$

and the general relation between conductivity and dielectric constant,

$$\epsilon(\omega) = 1 + \frac{4\pi i \sigma(\omega)}{\omega}. \tag{20}$$

(a) Find three equations relating q, K, and K' as functions of ω. *Note*: The coefficients, A, B, C and D should not appear in these equations.

(b) Hence, eliminating K and K', show that

$$\omega^2 = c^2 q^2 \left[1 + \frac{1}{\epsilon(\omega)}\right]. \tag{21}$$

(c) Determine $\epsilon(\omega)$ in the limit $\omega\tau \gg 1$.

(d) For this limit, sketch c^2q^2 as a function of ω^2.

(e) Hence sketch ω as a function of q.

(f) In the limit as $cq \gg \omega$ show that there is a solution at frequency $\omega = \omega_p/\sqrt{2}$, where ω_p is the bulk plasmon frequency. What is $\epsilon(\omega)$ at this frequency? Show from an examination of K and K' that the wave is confined to the surface. Describe its polarization. This wave is known as a *surface plasmon*.

(g) Show that there are also solutions with $\omega > \omega_p$. However, these are unphysical. Why?