PHYSICS 231

Homework 6

The final exam will be in class, Wednesday December 7, 4:00 - 7:00 pm. The exam will be closed book but you can bring one sheet of notes, if you wish. The topics on the exam will be closely related to the topics covered in the homework assignments. Hence part of your preparation for the exam should be to go over the HW questions. Solutions are available on the class website (except for the last assignment). However, you are also expected to read related material in one of the recommended books, Mardar or Ashcroft and Mermin.

This homework must be handed in, at the latest, at the final exam.

1. **Statistics of Donor Levels**
   In the standard treatment of donor levels one assumes that the energy to have two electrons (of opposite spin) in a donor level is very high (because of Coulomb repulsion), so this state is neglected. Here we do not neglect this state but assign it an energy $2\epsilon_d + \Delta$.

   (a) Show that the mean number of electrons in the donor levels (per unit volume), n_d is given by
   $$n_d = N_d \frac{1 + e^{-\beta(\epsilon_d - \mu + \Delta)}}{\frac{1}{2}e^{\beta(\epsilon_d-\mu)} + 1 + \frac{1}{2}e^{-\beta(\epsilon_d-\mu+\Delta)}}, \tag{1}$$
   where N_d is the concentration of donor impurities.

   (b) Show that this reduces to the expected result for independent electrons as $\Delta \to 0$ and to the Eq. (28.32) of AM for $\Delta \to \infty$.

   (c) In reality a donor impurity may have many bound states rather than just one as assumed above (think of the hydrogen atom). Show that the appropriate generalization of Eq. (28.32) of AM is
   $$\frac{N_d}{1 + \frac{1}{2}\left(\sum_i e^{-\beta(\epsilon_i-\mu)}\right)^{-1}}. \tag{2}$$

   (d) Indicate how (if at all) this result alters the results described on pages 582-584 of AM.

2. **Impurity Orbits**
   Indium Antimonide has an energy gap $E_g = 0.23$ ev; dielectric constant $\varepsilon = 18$ and conduction band effective mass $m_c = 0.015m$. Calculate

   (a) the donor ionization energy;

   (b) the radius of the ground state orbit.

   (c) At what minimum donor concentration will appreciable overlap effects between the orbits of adjacent impurity atoms occur?

   This overlap will produce an *impurity band*, *i.e.* a band of energy levels which permits conductivity through the electrons hopping directly from one impurity level to another without needing to be excited to the conduction band.

3. **Interpretation of Cyclotron Resonance Data**

(a) Compare the cyclotron resonance signal from silicon, AM Fig. 28.9b, with the geometry of the conduction band ellipsoids in Fig. 28.5, and explain why there are only two electron peaks although there are six pockets of electrons.
Note: The direction of the field is

$$\left(\frac{\sin\psi}{\sqrt{2}}, \frac{\sin\psi}{\sqrt{2}}, \cos\psi\right), \tag{3}$$

with $\psi = 30°$.
Note also that if θ is the direction of the field with respect to the direction of the major axis of the constant energy ellipsoid, then the cyclotron effective mass m_c is given by

$$\left(\frac{1}{m_c}\right)^2 = \frac{\cos^2\theta}{m_t^2} + \frac{\sin^2\theta}{m_l m_t}, \tag{4}$$

where m_t and m_l are the transverse and longitudinal band masses, see *e.g.* Kittel Ch. 8. This result can be derived from the result of Qu. 1 Homework 5.

(b) Verify that the positions of the electron resonances in Fig. 28.9b are consistent with the electron effective masses for silicon on AM page 569 and the expression for the resonance frequency.

4. **Constraint on Carrier Densities**

(a) Consider a doped semiconductor with more donor than acceptor levels, *i.e.* $N_d \geq P_a$. Describe the electronic configuration at $T = 0$.

(b) Show that for $T \neq 0$, the carrier concentrations are given, in the notation of AM (except that I find it more consistent to use P_a than N_a) by

$$n_c + n_d - N_d = p_v + p_a - P_a. \tag{5}$$

(c) Describe the electronic configuration at $T = 0$ of a doped semiconductor where $N_d < P_a$.

(d) Explain why Eq. (5) continues to hold in this limit.

5. **Carrier Density in a Doped Semiconductor**
Consider a doped semiconductor with a concentration of N_d donor impurities and no acceptor impurities. Assume that the gap from the valence to the conduction band is sufficiently large that a negligible number of electrons are excited out of the valence band, which can therefore be neglected in what follows. (At *really* high temperatures a significant density of carriers is excited from the valence to the conduction band, leading to another regime with $n_c(T) \simeq N_c(T)\exp(-E_g/(2T))$, as discussed in class.) Also assume that $\epsilon_c - \mu(T) \gg k_B T$. It can be shown that this is a good approximation in the two limiting cases in §5d and §5e below, but may not be adequate in the intermediate regime, where $x(T)$, defined below, is of order unity.

(a) Show that if the concentration of electrons in the conduction band is $n_c(T)$ and in the impurity states is $n_d(T)$ then

$$n_c(T) = N_d - n_d(T). \tag{6}$$

(b) Hence show that if $\mu(T)$ is the chemical potential and $N_c(T)$ is given by

$$N_c(T) = \frac{1}{4}\left(\frac{2m_c k_B T}{\pi\hbar^2}\right)^{3/2}, \tag{7}$$

where m_c is the effective mass of the electrons in the conduction band,[1] then Eq. (6) can be written

$$N_c(T)e^{-\beta(\epsilon_c-\mu(T))} = N_d\left(1-\frac{1}{\frac{1}{2}e^{\beta(\epsilon_d-\mu(T))}+1}\right), \tag{8}$$

where ϵ_c is the energy of the bottom of the conduction band and ϵ_d is the energy of the impurities.

(c) Show from this that $\mu(T)$ is given by

$$\mu(T) = \epsilon_d + k_B T \ln\left[\frac{-1+\sqrt{1+8x(T)}}{4}\right], \tag{9}$$

where

$$x(T) = \frac{N_d}{N_c(T)}e^{\beta(\epsilon_c-\epsilon_d)}. \tag{10}$$

Let us define a temperature T_d by

$$N_c(T) = N_d\left(\frac{T}{T_d}\right)^{3/2}, \tag{11}$$

where

$$T_d = \frac{\pi\hbar^2}{2m_c k_B}(4N_D)^{2/3}, \tag{12}$$

so

$$x(T) = \left(\frac{T_d}{T}\right)^{3/2} e^{\beta(\epsilon_c-\epsilon_d)}. \tag{13}$$

Note that *two* temperature scales enter: T_d which is determined by the *density* of impurities and the curvature of the conduction electron band, and $(\epsilon_c - \epsilon_d)/k_B$ which is determined by the *energy* of the impurity states. The existence of two temperature scales complicates somewhat the analysis.

Note also that $x(T)$ is much less than unity at high temperatures, monotonically increases as T decreases, and diverges as $T \to 0$. It is therefore also convenient to define another temperature scale, T_0 (related to T_d and $\epsilon_c - \epsilon_d$) by $x(T_0) = 1$, *i.e.*

$$\left(\frac{T_0}{T_d}\right)^{3/2} = e^{(\epsilon_c-\epsilon_d)/(k_B T_0)}. \tag{14}$$

Note that $T_0 > T_d$. If $\epsilon_c-\epsilon_d \ll k_B T_0$ then $T_0/T_d \simeq 1$, but in the opposite limit, $\epsilon_c-\epsilon_d \gg k_B T_0$, one has $T_0 \gg T_d$.

(d) Show that at high temperatures, $T \gg T_0$, *i.e.* $x(T) \ll 1$,

$$\mu(T) = \epsilon_c - \frac{3}{2}k_B T \ln\left(\frac{T}{T_d}\right) \tag{15}$$

$$n_c(T) = N_d, \tag{16}$$

i.e. the donors are fully ionized so the concentration of electrons in the conduction band is independent of T (in this range).

[1] For a precise definition of m_c see the comment in AM after Eq. (28.15).

(e) Show that in the low temperature limit, $T \ll T_0$, *i.e.* $x(T) \gg 1$

$$\begin{aligned} \mu(T) &= \left(\frac{\epsilon_c + \epsilon_d}{2}\right) + k_B T \left[\frac{3}{4} \ln\left(\frac{T_d}{T}\right) - \frac{1}{2} \ln 2\right] && (17) \\ n_c(T) &= \frac{N_d}{\sqrt{2}} \left(\frac{T}{T_d}\right)^{3/4} e^{-\beta(\epsilon_c - \epsilon_d)/2}, && (18) \end{aligned}$$

i.e. only a small fraction of the impurity levels are thermally ionized and the concentration of electrons in the conduction band decreases rapidly with decreasing temperature.

(f) Consider a semiconductor with $N_d = 10^{15}/\text{cm}^3, \epsilon_c - \epsilon_d = 2\,\text{mev}$ and $m_c = 0.01m$. Consider the system at (i) $T_1 = 300\,\text{K}$, and (ii) $T_2 = 4\,\text{K}$. Which of the limits of the last two sections is the system in for $T = T_1$ and $T = T_2$? Hence determine $n_c(T_1)$ and $n_c(T_2)$.
Hint: The numerical expression in Eq. (28.16) of AM may be useful.