

PHYSICS 231
Electrons in a Magnetic Field

I. INTRODUCTION

The response of a metal to a magnetic field, B , in the z -direction say, has given considerable information about the Fermi surface, because electrons follow a trajectory on the Fermi surface in a plane of constant k_z . In these notes we discuss how free electron states are modified by a magnetic field, and how this affects the energy of the system in an oscillatory manner, provided certain conditions, given in Eqs. (10) and (11) below, are satisfied. It is these oscillations which give information on the Fermi surface.

As is shown in standard texts on quantum mechanics, the energy levels of free electrons in a magnetic field are modified from

$$\epsilon(\mathbf{k}) = \frac{\hbar^2 k^2}{2m} \quad (1)$$

to

$$\epsilon(n_L, k_z) = \frac{\hbar^2 k_z^2}{2m} + (n_L + \frac{1}{2}) \hbar \omega_c, \quad (2)$$

where $n_L = 0, 1, 2, \dots$ and

$$\omega_c = \frac{eB}{mc} \quad (3)$$

is the cyclotron frequency, i.e. the frequency of the classical motion. The states with a given n_L are known collectively as a Landau level. The degeneracy of a Landau level is

$$\mathcal{N} = 2 \frac{e}{hc} AB = \frac{AB}{\phi_0}, \quad (4)$$

where A is the area of the sample in the plane perpendicular to B , the factor of two in the first expression comes from spin degeneracy, and

$$\phi_0 = \frac{hc}{2e} = 2.07 \times 10^{-7} \text{G-cm}^2 \quad (5)$$

is the flux quantum for a pair of electrons.

II. TWO DIMENSIONS, FREE ELECTRONS

It is easier to consider first the case of two dimensions, where the electrons are confined to the x - y plane in a perpendicular field. Hence there is no motion in the z direction so the term in Eq. (2) involving k_z^2 does not occur. You showed in Qu. 3 of Homework 3 that the density of states in two dimensions is a constant,

$$g(\epsilon) = A \frac{m}{\pi \hbar^2}, \quad (6)$$

in the absence of a field. Here we give the total density of states, rather than the density of states per unit area, so a factor of A appears. In a field, the allowed energy values are discrete with a constant spacing between levels of $\hbar \omega_c$. Now the number of zero field states in an interval $\hbar \omega_c$ is given by

$$g(\epsilon) \hbar \omega_c = A \frac{m}{\pi \hbar^2} \hbar \frac{eB}{mc} \quad (7)$$

$$= \frac{2e}{hc} AB \quad (8)$$

$$= \mathcal{N}, \quad (9)$$

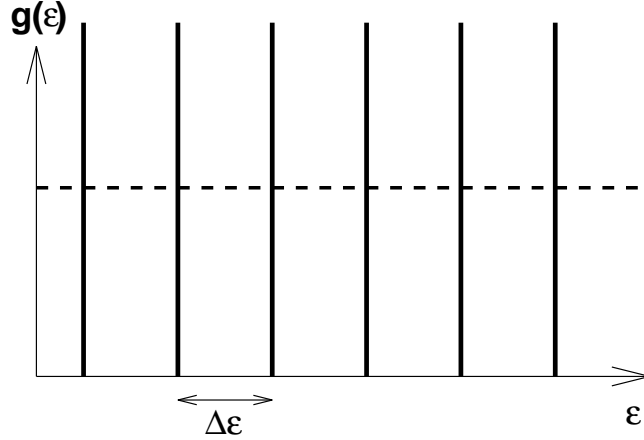


FIG. 1: The dashed line shows the density of states of the two dimensional free electron gas in the absence of a magnetic field. It has the constant value $g(\epsilon) = Am/\pi\hbar^2$. In the presence of a magnetic field the energy levels are bunched into discrete values $(n_L + 1/2)\Delta\epsilon$ where $n_L = 0, 1, 2, \dots$, and $\Delta\epsilon = \hbar\omega_c$, where $\omega_c = eH/mc$ is the cyclotron frequency. Hence the density of states is a set of delta functions, shown by the vertical lines. The weight of the delta function is equal to the zero field density of states times $\Delta\epsilon$, so the energy levels are just shifted locally, the total number of states in a region comprising a multiple of $\Delta\epsilon$ being unchanged.

the degeneracy of a Landau level. In other words, the field bunches the states into discrete levels, but the total number of states in a region much larger than the Landau level spacing is unchanged by the field. The density of states, both with and without a magnetic field is shown in Fig. 1.

At zero temperature, as we increase the magnetic field the number of occupied Landau levels will change, since the number of electrons is fixed but the degeneracy of each Landau level changes with B . This will lead to an oscillatory behavior of the energy as a function of the magnetic field, that we discuss below. At finite temperature these oscillations will be washed out if $k_B T \gg \hbar\omega_c$, because then many Landau levels will be partially filled. In order to see the oscillations, we therefore need

$$\hbar\omega_c = \frac{e\hbar}{mc} B \gg k_B T, \quad (10)$$

which requires large fields and very low temperatures, of order a few K. At higher temperatures, there is a *smooth* change in the energy with B which leads to a small diamagnetic response, see Appendix A, Ashcroft and Mermin (AM) p. 664, and Peierls, Quantum Theory of Solids, pp. 144-149.

It is also important to realize that if there are impurities, then the electron states will have a finite lifetime, τ , which will broaden the levels by an amount \hbar/τ . This will also wash out the oscillations if the level broadening is greater than the Landau level splitting. Hence, to observe the oscillations we also need a second condition,

$$\omega_c \tau = \frac{eB}{mc} \tau \gg 1, \quad (11)$$

which requires very clean samples.

Let us now determine the change in energy of our two-dimensional model at $T = 0$ as a function of field. Let

$$n = \frac{N}{A} \quad (12)$$

be the number of electrons per unit area. Hence, for $B = 0$, the Fermi energy is determined from

$$N = \int_0^{\epsilon_F} g(\epsilon) d\epsilon, \quad (13)$$

or

$$N = g(\epsilon)\epsilon_F = A \frac{m}{\pi\hbar^2} \epsilon_F, \quad (14)$$

which gives

$$\epsilon_F = n \frac{\pi \hbar^2}{m}. \quad (15)$$

Hence the ground state energy per electron in zero field is

$$\epsilon(B=0) \equiv \frac{E(0)}{N} = \int_0^{\epsilon_F} \epsilon g(\epsilon) d\epsilon = \frac{\epsilon_F}{2} = n \frac{\pi \hbar^2}{2m}, \quad (16)$$

using Eq. (15).

In the presence of the field the levels, $n_L = 0, 1, 2, \dots, p-1$ will be fully occupied with \mathcal{N} electrons and level p will be partially occupied with $\lambda \mathcal{N}$ electrons, where $0 \leq \lambda < 1$.

Counting up electrons one has

$$N = \mathcal{N} \nu \quad (17)$$

where

$$\nu = p + \lambda \quad (18)$$

is called the *filling factor* of the Landau levels. Note that λ and ν take a continuous range of values, whereas p is an integer. From Eq. (4) we have

$$\nu = n \frac{\phi_0}{B} = \frac{B_1}{B} \left(= \frac{\epsilon_F}{\hbar \omega_c} = n \frac{\pi \hbar^2}{m} \frac{1}{\hbar \omega_c} \right), \quad (19)$$

where the last equality is from Eq. (15), in which

$$B_1 = n \phi_0 \quad (20)$$

is the field required to put all the electrons in the lowest Landau level. Consequently

$$p = \left[\frac{B_1}{B} \right], \quad \lambda = \frac{B_1}{B} - \left[\frac{B_1}{B} \right] \quad (21)$$

where $[x]$ means the largest integer less than or equal to x . Note that λ is the fractional filling of the last Landau level.

The energy per electron is then just obtained by summing the energies of each Landau level, i.e.

$$\epsilon(B) = \left\{ (0 + \frac{1}{2}) + (1 + \frac{1}{2}) + \dots + (p-1 + \frac{1}{2}) + (p + \frac{1}{2}) \lambda \right\} \frac{\mathcal{N}}{N} \hbar \omega_c \quad (22)$$

$$= \left\{ \frac{p^2}{2} + (p + \frac{1}{2}) \lambda \right\} \frac{\hbar \omega_c}{\nu} \quad (23)$$

$$= \left\{ \frac{(p + \lambda)^2}{2} + \frac{\lambda - \lambda^2}{2} \right\} \frac{\hbar \omega_c}{\nu}. \quad (24)$$

Eliminating $p + \lambda \equiv \nu$ in favor of n using Eq. (19) one finds

$$\epsilon(B) = \frac{\pi \hbar^2}{2m} n + \frac{\hbar \omega_c}{2\nu} \lambda (1 - \lambda) \quad (25)$$

which, from Eqs. (3), (16) and (19)–(21), can be conveniently written as

$$\frac{\epsilon(B)}{\epsilon(0)} = 1 + \left(\frac{B}{B_1} \right)^2 \lambda (1 - \lambda). \quad (26)$$

Eq. (26) is our main result. It gives the field dependence of the ground state energy (per electron) of free electrons in two-dimensions.

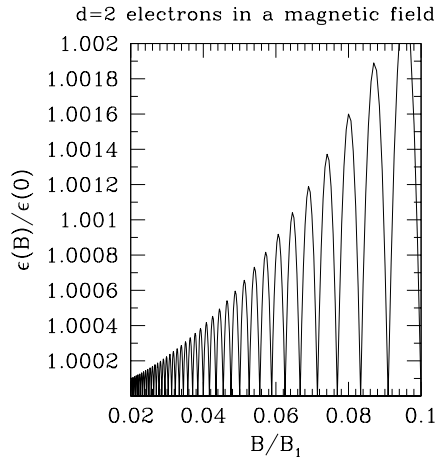


FIG. 2: The energy of the two dimensional electron gas at $T = 0$ according to Eq. (26), as a function of B/B_1 where $B_1 = n\phi_0$ is the field at which all the electrons are in a completely filled lowest Landau level. Note that the overall size of the energy change varies as B^2 in addition to the oscillations. The oscillations get closer together for small B . In fact, apart from the smooth B^2 variation, the data is a periodic function of $1/B$, see Figs. 3 and 4.

A sketch of the of energy as a function of B , according to Eq. (26), is shown in Fig. 2. One clearly sees oscillations whose amplitude increases smoothly which B , actually as B^2 . The period of the oscillations gets smaller with decreasing B , since the energy is actually a periodic function of $1/B$ not B (apart from the B^2 variation in amplitude). This is seen in Fig. 3 below. It occurs because the energy in Eq. (26) depends on the fractional filling factor λ , defined in Eq. (21), which depends periodically on $1/B$. Although the energy is a continuous function of B the derivative is discontinuous when the field is such that the Landau levels are completely filled (remember that λ is the fractional filling of the last Landau level). Physically, this is because if one decreases B below $B_p = B_1/p$, say, then it is Landau level p which starts to be occupied, whereas if one increases B above B_p then it is Landau level $p - 1$ which starts to be depleted.

Fig. 2 is to be contrasted with the situation at a temperature where $k_B T \gg \hbar\omega_c$, where thermal fluctuations average over the oscillations. Then one has only the *smooth* increase in the energy proportional to B^2 . Since the magnetization is given by $M = -\partial E/\partial B$, where E is the energy per unit volume, one then finds a small diamagnetic (i.e. negative) susceptibility, χ_{dia} , where $M = \chi_{\text{dia}} B$, see Appendix A, AM p. 664, and Peierls, Quantum Theory of Solids, pp. 144-149. This was first calculated by Landau in 1930.

Note from Eq. (21) that λ is a periodic function of $1/B$ and hence, apart from the slowly varying factor of B^2 , the energy, given by Eq. (26), is an oscillatory function of $1/B$. This is clearly seen in Fig. 3.

The oscillation in the ground state energy give rise to oscillations in the magnetization per electron M , given by

$$M = -\frac{\partial \epsilon}{\partial B}, \quad (27)$$

which, from Eqs. (5), (16), (18), (19), (21) and (26) is given by

$$\frac{M}{\mu_B} = 1 - 2\lambda - \frac{2\lambda(1-\lambda)}{p+\lambda}, \quad (28)$$

where

$$\mu_B = \frac{e\hbar}{2mc} \quad (29)$$

is the Bohr magneton. For $p \gg 1$, which is generally the case of interest, this is given, to a good approximation, by

$$\frac{M}{\mu_B} = 1 - 2\lambda. \quad (30)$$

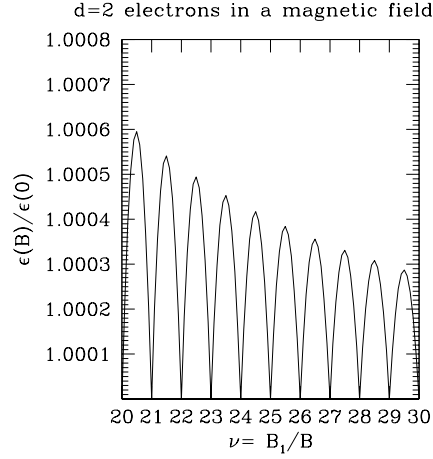


FIG. 3: The energy of the two dimensional electron gas at $T = 0$ as a function of $\nu = B_1/B$ where ν is the filling factor (the number of filled Landau levels) and $B_1 = n\phi_0$ is the field at which all the electrons are in a completely filled lowest Landau level. For metals, B_1 is much greater than a typical laboratory field so in experimental situations the filling factor is large. Note that the behavior is periodic apart from a gentle overall decrease of the amplitude which is negligible for $\nu \gg 1$.

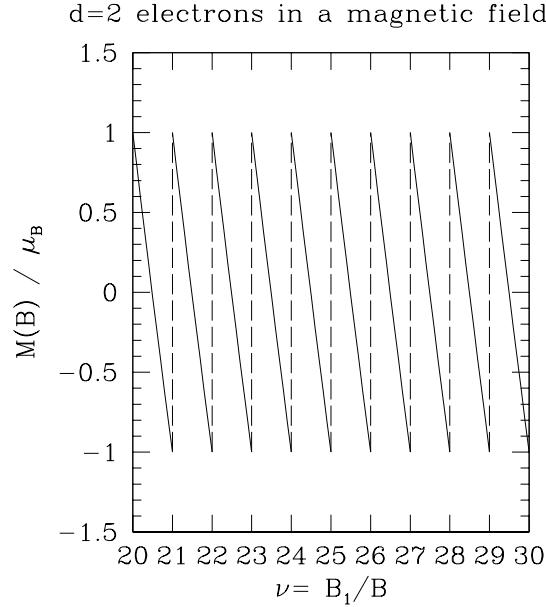


FIG. 4: The magnetization of the two dimensional electron gas at $T = 0$ (in units of μ_B) as a function of $\nu = B_1/B$ where ν is the filling factor and $B_1 = n\phi_0$, obtained from Eq. (28). Note that the magnetization is a periodic function of $1/B$.

Since λ is periodic in $1/B$, see Eq. (21), it follows that M is also periodic in $1/B$, see Fig. 4.

The main conclusion so far is that M is a periodic function of $1/B$ with period given by

$$\Delta\left(\frac{1}{B}\right) = \frac{1}{B_1} = \frac{1}{n\phi_0}. \quad (31)$$

From Eq. (15) it follows that the density is related to the Fermi wave vector k_F by

$$n = \frac{k_F^2}{2\pi}, \quad (32)$$

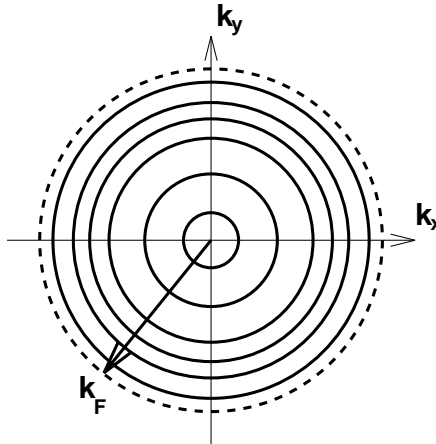


FIG. 5: The solid circles are the semi classical orbits of free electrons in two dimensions in a magnetic field. Each circle corresponds to a particular Landau level. The electrons follow orbits around these circles with a period $T = 2\pi/\omega_c$, where ω_c is the cyclotron frequency. As the strength of the field is increased the radii increase and the orbits “pop out” of the Fermi circle, denoted by a dashed circle. This leads to oscillatory behavior in the energy and magnetization provided $k_B T$ is much less than the spacing of the Landau levels, $\hbar\omega_c$.

and so, using Eq. (5),

$$\Delta \left(\frac{1}{B} \right) = \frac{2\pi e}{\hbar c} \frac{1}{A_F}, \quad (33)$$

where

$$A_f \equiv \pi k_F^2 \quad (34)$$

is the area of the Fermi surface. In Sec. III we shall show that Eq. (33), which relates the periodicity to just fundamental constants and the area of the Fermi surface, is true quite generally, not just for free electrons. In Sec. IV we will also discuss the implications of this result for de Haas-van Alphen experiments which use oscillations in the magnetization to map out Fermi surfaces of metals, see e.g. Ashcroft and Mermin, Ch. 12.

It is interesting to see how Eq. (33) emerges from the semi-classical picture. The semi classical orbits for free electrons are circles, on which the energy stays constant. Denoting by k_p the magnitude of the wave vector corresponding to Landau level p , then equating $\hbar^2 k_p^2 / 2m$ to $(p + \frac{1}{2})\hbar\omega_c$ one gets

$$k_p^2 = (p + \frac{1}{2}) \frac{2eB}{\hbar c} \quad (35)$$

and so the area in k -space swept out by an electron in Landau level p is

$$A_p = \pi k_p^2 = (p + \frac{1}{2}) \frac{2\pi e B}{\hbar c}, \quad (36)$$

which can also be written as

$$\frac{1}{B} = (p + \frac{1}{2}) \frac{2\pi e}{\hbar c} \frac{1}{A_p}. \quad (37)$$

Hence, in the semi-classical picture of free electrons in a magnetic field, the allowed states in k -space form circles whose radius increases with B , see Fig. 5. As the states go through the Fermi wave vector and become de-populated this gives rise to oscillations in the energy. From Eq. (37) the change in $1/B$ for one Landau level to go through the Fermi surface (i.e. let $p \rightarrow p - 1$ with $A_L = A_F$, the area of the Fermi surface) is given by

$$\Delta \left(\frac{1}{B} \right) = \frac{2\pi e}{\hbar c} \frac{1}{A_F}, \quad (38)$$

in agreement with Eq. (33).

III. TWO DIMENSIONS, BLOCH ELECTRONS

So far we have derived Eq. (33) only for free electrons in two dimensions. The same result is also true for Bloch electrons (i.e. electrons in a periodic potential) as we shall now show. Already the fact that it does not depend on the electron mass, which could be altered to an effective mass by band structure effects, gives one a suspicion that this might be so. In this section we still consider two dimensions.

The first stage in the proof is to derive an equation, (45) below, relating the period of an orbit of a Bloch electron in a magnetic field to the change in the area of the orbit with energy. We start with the semi-classical equations of motion:

$$\mathbf{v}(\mathbf{k}) = \frac{1}{\hbar} \frac{\partial \epsilon(\mathbf{k})}{\partial \mathbf{k}} \quad (39)$$

$$\hbar \dot{\mathbf{k}} = (-e) \frac{1}{c} \mathbf{v}(\mathbf{k}) \times \mathbf{B}, \quad (40)$$

from which it follows that

$$|\dot{\mathbf{k}}| = \frac{eB}{\hbar^2 c} \left| \left(\frac{\partial \epsilon(\mathbf{k})}{\partial \mathbf{k}} \right)_{\perp} \right|, \quad (41)$$

where $(\partial \epsilon(\mathbf{k})/\partial \mathbf{k})_{\perp}$ is the component of $\partial \epsilon(\mathbf{k})/\partial \mathbf{k}$ perpendicular to the field, i.e. its projection in the plane of the orbit. (Note this projection is not necessary in two dimensions considered here, but we include it so that our derivation of Eq. (45) below will also apply in three dimensions.) Hence the period of an orbit is given by

$$T = \oint \frac{dk}{|\dot{\mathbf{k}}|} = \frac{\hbar^2 c}{eB} \oint \frac{dk}{|(\partial \epsilon(\mathbf{k})/\partial \mathbf{k})_{\perp}|}, \quad (42)$$

where dk is the magnitude of a small element of the orbit, and we integrate around the closed orbit. If we consider two orbits whose difference in energy $\Delta \epsilon$ is small then the region between them in the k_x - k_y plane is a ribbon of width $\Delta(\mathbf{k})$, see AM Fig. 12.9, where

$$\Delta \epsilon = \left| \left(\frac{\partial \epsilon(\mathbf{k})}{\partial \mathbf{k}} \right)_{\perp} \right| |\Delta(\mathbf{k})|. \quad (43)$$

Hence the period T can be written as

$$T = \frac{\hbar^2 c}{eB} \frac{1}{\Delta \epsilon} \oint |\Delta(\mathbf{k})| dk. \quad (44)$$

The integral in the last equation is just the area between the two neighboring orbits, ΔA , and so

$$T = \frac{\hbar^2 c}{eB} \left(\frac{\Delta A}{\Delta \epsilon} \right). \quad (45)$$

Note that with a free electron band structure, $\epsilon = \hbar^2 k^2/2m$, $A = \pi k^2$, this gives $T = 2\pi/\omega_c$, where ω_c is the cyclotron frequency, as expected.

The second stage of the proof is to use another relationship involving the period of the orbits, the Bohr-Sommerfeld correspondence principle, which states that if ϵ_p and ϵ_{p+1} are two adjacent energy levels with quantum numbers p and $p+1$ where p is assumed large, then

$$\epsilon_{p+1} - \epsilon_p = \frac{h}{T}, \quad (46)$$

where T is the period of the motion of a semiclassical wavepacket on an orbit with energy centered on ϵ_p . This can be derived by noting that the wavepacket is built up out of adjacent energy levels, and its periodic motion comes from interference between different levels. This requires that the levels around ϵ_p be uniformly spaced with spacing $\hbar\omega = h/T$, where ω is the angular frequency of the orbit. (Note that the ‘‘correspondence principle’’

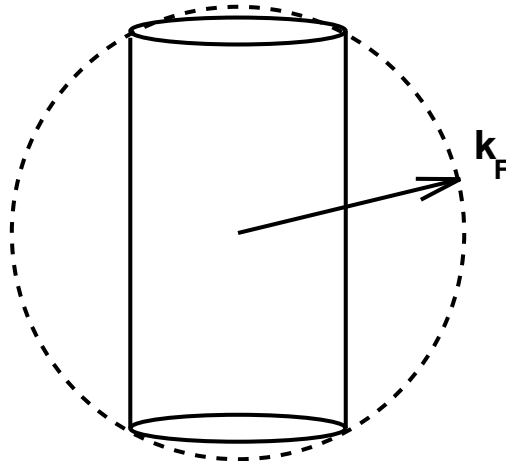


FIG. 6: The dashed circle represents the Fermi sphere in three dimensions. The cylinder represents the semi-classical orbit corresponding to a single Landau level. As the strength of the magnetic field increases, the radius of the cylinder increases and will eventually exceed the Fermi wave vector, k_F . This “popping out” of the Landau levels from the Fermi surface leads to oscillatory behavior of the energy and magnetization at sufficiently low temperature.

really only comes in with the further remark that T is also the period of a purely *classical* particle of the same energy.)

From Eq. (46) we substitute $T\Delta\epsilon = h = 2\pi\hbar$ into Eq. (45), and then find that the area between the semiclassical orbits of two adjacent Landau levels is given by

$$\Delta A = \frac{2\pi eB}{\hbar c}. \quad (47)$$

This elegant result, first obtained by Onsager, can be reexpressed by stating that at large p the area A_p inside the Landau level p is given by

$$A_p = (p + \lambda) \frac{2\pi eB}{\hbar c}, \quad (48)$$

where λ is some number independent of p . This is the same as Eq. (36) for free electrons, except that λ need not equal $1/2$. The periodicity in $1/B$ in Eq. (38), obtained from Eq. (36), follows equally from Eq. (48).

IV. THREE DIMENSIONS AND THE DE HAAS-VAN ALPHEN EFFECT

How does all this go over into three dimensions? One now needs to add k_z , so the semi-classical orbits, which were circles in two dimensions, are now cylinders. One such cylinder is shown in Fig. 6. As B increases the radius increases. However, one does not expect a dramatic change in the magnetization until the radius increases to k_F , (shown dashed in Fig. 6), beyond which the cylinder does not intersect the Fermi sphere at all. When the cylinders “pop out” of the Fermi sphere one expects oscillatory behavior in the magnetization just as in two dimensions. This can be confirmed by a mathematical analysis, see e.g. Peierls, pp. 144-149. Clearly then, the periodicity of the magnetization with $1/B$ involves the *maximum* area formed by the intersection of the Fermi sphere with a plane perpendicular to B . If one has a general Fermi surface from some complicated band structure, then it is also possible that, on varying k_z , the intersection of the Fermi surface with the plane of constant k_z might have a *minimum*. These orbits will also contribute to the oscillatory behavior because the part of the cylinder inside the Fermi sphere will decrease rapidly when the radius of the cylinder passes this extremal radius.

Thus we conclude that the magnetization will show oscillations with periods given by

$$\Delta \left(\frac{1}{B} \right) = \frac{2\pi e}{\hbar c} \frac{1}{A_{\text{ext}}}, \quad (49)$$

where A_{ext} is the area of any *extremal* orbit in the plane perpendicular to the field. If there is more than one extremal area then several periods will be superimposed.

A determination of oscillations in M as a function of $1/B$ (the de Haas-van Alphen effect) for different orientations of the field has been the most successful method for mapping out the shape of the Fermi surface of metals. It is discussed in AM, Ch. 14. Oscillations occur in other quantities as well. For example, the resistivity varies with magnetic field, an effect called magnetoresistance. At low temperature and in very clear samples, there are oscillations in the magnetoresistance as a function of $1/B$ of the same origin as the oscillations in magnetization.

APPENDIX A: LANDAU DIAMAGNETISM

In this appendix we derive the expression for Landau diamagnetism of free electrons. Our approach will be to derive the result first for 2- d from which we will easily be able to obtain the 3- d result by integrating over k_z .

Firstly, we remind ourselves of some statistical mechanics. It is convenient to calculate the *grand partition function*, \mathcal{Z} where

$$\mathcal{Z} = \text{Tr} \exp[\beta(\mu N - \mathcal{H})], \quad (\text{A1})$$

where N is the number of particles and \mathcal{H} is the Hamiltonian. The trace is over all states for a given N and over all N . This ensemble, where both N and E are allowed to vary, is called the “grand canonical ensemble”. \mathcal{Z} is related to the “grand potential”, Ω , by

$$\Omega = -k_B T \log \mathcal{Z}. \quad (\text{A2})$$

The mean number of particles is given by

$$N = -\frac{\partial \Omega}{\partial \mu}, \quad (\text{A3})$$

which implicitly determines μ . Ω is related to the free energy $F \equiv U - TS$ by

$$F = \Omega + \mu N. \quad (\text{A4})$$

Note that Ω is considered as a function of μ and so differentiating the right hand side of Eq. (A4) with respect to μ (keeping N constant) gives

$$\frac{\partial F}{\partial \mu} = 0. \quad (\text{A5})$$

This will be useful later. Note that a lot of confusion in statistical mechanics comes from a lack of understanding of what is being kept constant in partial derivatives. Once the correct value of μ has been determined for a given N , we determine the magnetization from

$$M = -\frac{\partial F}{\partial B}, \quad (\text{A6})$$

where B is the magnetic field and the derivative is at fixed N . Note that in general we need to determine M from $-(\partial F/\partial B)_N$ rather than $-(\partial \Omega/\partial B)_\mu$, because the number of particles is kept constant when the field is applied, not the chemical potential. However, as we shall see, they both give the same result to lowest order in B , the case of interest here, because of Eq. (A5). The susceptibility is then found from

$$\chi = \frac{\partial M}{\partial B} = -\frac{\partial^2 F}{\partial B^2}, \quad (\text{A7})$$

so we have to compute the B^2 term in the free energy.

A useful feature of the grand canonical ensemble is that, for non-interacting particles, the grand potential factorizes into a product of grand potentials for each single particle state, and so the grand potential is a *sum* of grand potentials for single particle states. Hence, for free electrons,

$$\Omega = -k_B T \sum_k 2 \ln[1 + e^{\beta(\mu - \epsilon_k)}], \quad (\text{A8})$$

which can be conveniently be expressed in terms of the density of states, $g(\epsilon)$ (for both spin species) by

$$\Omega = -k_B T \int_0^\infty g(\epsilon) \ln[1 + e^{\beta(\mu - \epsilon)}] d\epsilon. \quad (\text{A9})$$

Consider first $B = 0$. We are interested in the low- T limit where, as discussed in class, the difference between μ and its $T = 0$ limit, ϵ_F , is negligible. As discussed above $g(\epsilon)$ is a constant $A m/(\pi\hbar^2)$. and so, for small T and $B = 0$

$$\Omega(B = 0) = -k_B T A \frac{m}{\pi\hbar^2} \int_0^\infty \ln[1 + e^{\beta(\mu - \epsilon)}] d\epsilon \quad (\text{A10})$$

$$\simeq -k_B T A \frac{m}{\pi\hbar^2} \int_0^{\epsilon_F} \beta(\epsilon_f - \epsilon) d\epsilon \quad (\text{A11})$$

$$= -A \frac{m}{\pi\hbar^2} \frac{\epsilon_F^2}{2} = -\frac{N\epsilon_F}{2}, \quad (\text{A12})$$

where the last equality is from Eq. (15). For $T \rightarrow 0$, where TS is negligible, Ω is just $U + \mu N$ with $\mu = \epsilon_F$, and so

$$U = \frac{N\epsilon_F}{2} \quad (\text{A13})$$

in agreement with Eq. (16), which gives the energy per electron rather than the total energy as here.

Now we add a field, which changes Ω in two ways. Firstly it changes the density of states, as we have discussed in detail in the main part of the text. Secondly it changes the chemical potential, so $\mu(B) = \mu(0) + \text{const.} B^2 + O(B^4)$. However, we shall see that the change in Ω due to the change in μ with B does not affect the free energy F , so we just focus here on the change in Ω due to the modification of the density of states.

In the presence of a field B , the energy levels take the discrete values $\epsilon_n \equiv (n + \frac{1}{2})\hbar\omega_c$ and so

$$\Omega(B) = -k_B T \frac{Am}{\pi\hbar^2} \hbar\omega_c \sum_n \ln[1 + e^{\beta(\mu - \epsilon_n)}], \quad (\text{A14})$$

which is to be contrasted with Eq. (A10) for $B = 0$. The difference is that Eq. (A14) can be thought of as a discretized approximation to Eq. (A10) of the sort that is often used in numerical analysis. The integral over a range of width $\hbar\omega_c$ is replaced by $\hbar\omega_c$ times the value of the the integrand at the *midpoint* of the interval. The difference between the integral and the approximation to it using this ‘‘midpoint rule’’ is well known (see e.g. Numerical Recipes in C (or in Fortran) by Press et al, Eq. (4.4.1)), and can be easily derived by replacing the function in each interval by a polynomial, doing the integral with the first few terms of the polynomial, and summing over intervals. The result can be expressed as

$$\int_{x_0}^{x_N} f(x) dx = h[f_{1/2} + f_{3/2} + \dots + f_{N-1/2}] + h^2 \frac{(f'_N - f'_0)}{24} + O(h^4), \quad (\text{A15})$$

where $x_n = x_0 + nh$ and $f_n = f(x_n)$. This approximation is good provided the function $f(x)$ varies smoothly over a single interval of width h . For the present problem, h is replaced by $\hbar\omega_c$, and the integrand varies rapidly on a scale of $k_B T$. Hence use of Eq. (A15) will be valid for $k_B T \gg \hbar\omega_c$. In this situation, many Landau levels are partially occupied and the oscillations found in the main part of the text are washed out. In our case the integral goes to infinity but both the function and its derivative vanish in this limit, and so, evaluating the derivative of the integrand at $\epsilon = 0$, we get, to order B^2 ,

$$\Omega(B) = \Omega(0) + \frac{e^2}{24\pi m c^2} B^2 + \frac{\partial\Omega}{\partial\mu} \delta\mu(B), \quad (\text{A16})$$

where $\Omega(0)$ corresponds to the integral in Eq. (A15) and $\Omega(B)$ to the sum, and, from now on, we work per unit area. The last term in Eq. (A16) is the change in Ω from $\delta\mu(B)$, the change in the chemical potential due to the field, which is also of order B^2 .

Substituting Eq. (A16) into in Eq. (A4) gives

$$F(B) = F(0) + \frac{e^2}{24\pi m c^2} B^2, \quad (\text{A17})$$

where, from Eq. (A3), we have noted that the contribution from $\delta\mu(B)$ cancels. This cancellation occurs because $\partial F/\partial\mu = 0$, as noted in Eq. (A5).

The magnetization is given by $M = -\partial F/\partial B$ which yields $M = \chi_{dia}^{(2d)} B$ with $\chi_{dia}^{(2d)}$, the diamagnetic susceptibility of free electrons per unit area in two dimensions, given by

$$\chi_{dia}^{(2d)} = -\frac{e^2}{12\pi mc^2}. \quad (\text{A18})$$

To get the corresponding result in 3- d we need to add the motion in the z -direction, specified by k_z , and integrate k_z from $-k_F$ to k_F . This is easy because Eq. (A18) is a constant independent of the (2- d) density of electrons (which would vary with k_z) and hence the diamagnetic susceptibility of free electrons in 3- d per unit *volume* is given by

$$\chi_{dia}^{(3d)} = -\frac{e^2}{12\pi mc^2} \int_{-k_F}^{k_F} \frac{dk_z}{2\pi} = -\frac{e^2 k_F}{12\pi^2 mc^2}. \quad (\text{A19})$$

This is the expression first found by Landau. It is equivalent to Eqs. (31.72) and (31.69) of Ashcroft and Mermin.