PHYSICS 232

Homework 3

Due in class, Wednesday February 22

1. *A Ferrimagnet*
   Consider a magnetic structure made up of two types of spins that occupy two interpentrating sublattices. Let neighboring spins with sublattice 1 be coupled by exchange constant $J^{(1)}$, within sublattice 2 by $J^{(2)}$, and between sublattices 1 and 2 by $J^{(3)}$.

   (a) Generalize the mean field theory for a simple ferromagnet to this situation, showing that one obtains two coupled equations for the two sublattice magnetizations of the form

   $$m_1 = M_0\left(\frac{h + J^{(1)}(0)m_1 + J^{(3)}(0)m_2}{k_B T}\right) \tag{1}$$

   $$m_2 = M_0\left(\frac{h + J^{(2)}(0)m_2 + J^{(3)}(0)m_1}{k_B T}\right) \tag{2}$$

   where $J^{(1)}(0) = \sum_j J^{(1)}_{ij}$ etc, h is the external field, and $M_0(H/k_BT)$ is the magnetization of a free spin in a field H.

   (b) Deduce from this that the suscptibility in the limit of zero field, above any transition, is the ratio of a polynomial linear in T to one quadratic in T.

   (c) Verify that the susceptibility reduces back to the Curie-Weiss form when the two ions in the two sublattices are identical and ferromagnetically coupled ($J^{(1)}(0) = J^{(2)}(0) > 0, J^{(3)}(0) > 0$.

   (d) Verify that when the ions in the two sublattices are identical ($J^{(1)}(0) = J^{(2)}(0) > 0$) and antiferromagnetically coupled ($J^{(3)}(0) < 0$), the Curie-Weiss temperature becomes negative.

2. *Spin wave theory for the anisotroptic Heisenberg model*
   Consider an *anisotropic* Heisenberg model

   $$\mathcal{H} = -\sum_{\langle ij\rangle}\left[J^z_{ij}S^z_iS^z_j + J^{\perp}_{ij}\left(S^x_iS^x_j + S^y_iS^y_j\right)\right] ,$$

   where $J^z_{ij} > J^{\perp}_{ij} > 0$. The system is therefore a ferromagnet which orders along the z direction.

   (a) Show that the ground state considered in class is still a ground state, and the 1-spinwave excitations considered in class are also still elementary excitations. However, show that there is now a minimum excitation energy equal to

   $$S\sum_j\left(J^z_{ij} - J^{\perp}_{ij}\right).$$

(b) Show that the magnetization now deviates from its saturation value only exponentially in $-1/T$ at low-T, (rather than $T^{3/2}$ for the isotropic case).

(c) Show that in the isotropic case, the leading spinwave correction for the magnetization diverges in two-dimensions, suggesting (correctly) that there is no magnetization at finite T, but that this divergence does not happen in the anisotropic case (suggesing, again correctly, that there *can* be magnetization at finite T).

3. *The one-dimensional Ising model.*
Consider the one-dimensional Ising model with Hamiltonian

$$\mathcal{H} = -J \sum_i S_i S_{i+1} \ ,$$

where J is the nearest neighbor interaction, $i = 1, 2, \ldots, N$, $S_i = \pm 1$, and there are periodic boundary conditions. Determine the free energy per spin and the energy per spin as a function of T in the limit of $N \to \infty$. Calculate the spin-spin correlation function $C(n) = \langle S_i S_{i+n} \rangle$. Determine the zero field susceptibility and show that it only diverges at $T = 0$.
(*n.b.* You may find that the method known as the "transfer matrix" helps. This is discussed in standard books on statistical mechanics, e.g. Wannier, Plischke and Bergersen, Huang, ... Your results show that there is no transition at a finite temperature. This is a general result for one-dimensional systems with short range interactions, see the above books and Landau and Lifshitz, Statistical Mechanics.)

4. *Transverse magnetic response; a symmetry relation*
Consider an isotropic Heisenberg ferromagnet. You may, for simplicity, take it to be classical (i.e. $S = \infty$), though this will not affect the results. Apply a magnetic field h and then define longitudinal and transverse susceptibilities, χ_L and χ_T respectively, as the response to an additional infinitesimal field parallel to and transverse to h respectively. First give a symmetry argument which shows that *in general*

$$\chi_T = \frac{m}{h}$$

where m is the magnetization per atom. (Use units where $g\mu_B = 1$). Next compute the *wavevector-dependent* transverse susceptibility close to T_c at long wavelengths in mean field theory. How does the transverse susceptibility vary with k as $k \to 0$ in the limit of $h \to 0$ below T_c? How does the transverse spin-spin correlation function vary with distance as a result?

5. *A "quantum" Ising model*
Consider the Ising model in a transverse field with Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - \Gamma \sum_i \sigma_i^x$$

where the σ are Pauli spin matrices. *n.b.* This is a model for a magnetic system with a crystal field splitting which leaves just two low energy levels, split by an energy 2Γ, which is much smaller than the energy to the next excited state. The σ are then fictitious spins not the real spins of the ions. The model is also used to describe systems with Hydrogen atoms, which are in a potential with two minima. Here Γ is the tunneling frequency between the two minima.

Determine Mean Field equations for this model. Deduce the location of the phase transition for *(a)* $\Gamma = 0$ and *(b)* $T = 0$. Sketch the phase diagram in the Γ–T plane.

6. *Magnetic behavior at high-T*
   Consider the high-temperature series expansion for the spin-S Heisenberg model. Show that the first two terms are

$$\chi(T) = \frac{(g\mu_B)^2}{3k_BT} S(S+1) \left[1 + \frac{\theta}{T} + O\left(\frac{\theta}{T}\right)^2 \right]$$

   where

$$\theta = \frac{S(S+1)}{3} \frac{J_0}{k_B} , \quad J_0 = \sum_{\mathbf{R}} J((\mathbf{R}) .$$

   Deduce the mean field theory for this model, and show that you obtain the *same* first two terms in the expansion (though higher order terms will be different). Hence show that in a plot of χ^{-1} against T, the exact curve approaches the mean field curve at large T. *n.b.* If you need help, you might look at Ashcroft and Mermin p. 711.

7. *Bulk Plasmons* (*easy*)
   Consider a sample in the shape of a slab consisting of positive and negative charges. Suppose the positive charges are uniformly displaced relative to the negative charges by an amount x. Show that the resulting electric field provides a restoring force, proportional to x, which leads to simple harmonic motion (of the relative displacement of the positive and negative charges) with frequency ω_p, the plasmon frequency, where

$$\omega_p^2 = \frac{4\pi n e^2}{m}$$

8. *Surface Plasmons* (*harder*)
   An electromagnetic wave that can propagate along the surface of a metal complicates the observation of ordinary (bulk) plasmons. Let the metal be contained in the half space $z > 0$, with the region $z < 0$ being vacuum. Assume that the electric charge density ρ appearing in Maxwell's equations vanishes both inside and outside the metal (though a surface charge density can appear in the plane $z = 0$). The surface plasmon is a solution to Maxwell's equations where the time dependence is of the form $e^{-i\omega t}$ and the space dependence is of the form:

$$E_x = Ae^{iqx}e^{-Kz}, \; E_y = 0, \; E_z = Be^{iqx}e^{-Kz}, \qquad (z > 0); \tag{3}$$

$$E_x = Ce^{iqx}e^{K'z}, \; E_y = 0, \; E_z = De^{iqx}e^{K'z}, \qquad (z < 0); \tag{4}$$

   where q is real and K and K' both real and positive. (Note that the amplitude decays exponentially away from the surface both for z positive and negative.)

Maxwell's equations imply that

$$\vec{\nabla} \cdot (\epsilon \vec{E}) = 0, \tag{5}$$

$$-\nabla^2 \mathbf{E} = \left(\frac{\omega}{c}\right)^2 \epsilon(\omega) \mathbf{E}, \tag{6}$$

where $\epsilon(\omega)$ is the frequency dependent dielectric constant, and are to be solved with the usual boundary conditions, ($\mathbf{E}_{\parallel}$ continuous, $(\epsilon \mathbf{E})_{\perp}$ continuous). (At the low-q region of interest here, the longitudinal and transverse dielectric constants are equal.) Assume the Drude result

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \qquad \sigma_0 = \frac{ne^2\tau}{m}, \tag{7}$$

and note the general relation between conductivity and dielectric constant,

$$\epsilon(\omega) = 1 + \frac{4\pi i \sigma(\omega)}{\omega}. \tag{8}$$

(a) Find three equations relating q, K, and K' as functions of ω. *Note*: The coefficients, A, B, C and D should not appear in these equations.

(b) Hence, eliminating K and K', show that

$$\omega^2 = c^2 q^2 \left[1 + \frac{1}{\epsilon(\omega)}\right]. \tag{9}$$

(c) Determine $\epsilon(\omega)$ in the limit $\omega\tau \gg 1$.

(d) For this limit, sketch c^2q^2 as a function of ω^2.

(e) Hence sketch ω as a function of q.

(f) In the limit as $cq \gg \omega$ show that there is a solution at frequency $\omega = \omega_p/\sqrt{2}$, where ω_p is the bulk plasmon frequency. What is $\epsilon(\omega)$ at this frequency? Show from an examination of K and K' that the wave is confined to the surface. Describe its polarization. This wave is known as a *surface plasmon.*

(g) Show that there are also solutions with $\omega > \omega_p$. However, these are unphysical. Why?