PHYSICS 232

Homework 4

Due in class, Wednesday March 8.

1. *Reflectivity of simple metals*

   The optical reflectivity of a material is given by

$$R(\omega) = \left| \frac{1 - \hat{N}(\omega)}{1 + \hat{N}(\omega)} \right|^2 = \frac{[1 - n(\omega)]^2 + k(\omega)^2}{[1 + n(\omega)]^2 + k(\omega)^2}$$

   where $\hat{N}(\omega) = \sqrt{\epsilon(\omega)} = n(\omega) + ik(\omega)$ is the complex refractive index. Light has wavevector very close to zero and so we will assume $\mathbf{q} = 0$ here, and just indicate the frequency dependence in the notation. In this limit the difference between the longitudinal and transverse responses of the system is negligible, and so we will not need to specify that this is really the transverse response.

   Consider a simple metal, and assume the optical conductivity can be described by the Drude approximation

$$\sigma(\omega) = \frac{ne^2\tau}{m} \frac{1}{1 - i\omega\tau} .$$

   The dielectric constant is related to the conductivity by the usual expression

$$\epsilon(\omega) = 1 + \frac{4\pi i}{\omega}\sigma(\omega) .$$

   Assume

$$\omega_p \tau \gg 1$$

   since this is satisfied in most metals, where ω_p is the plasmon frequency given by

$$\omega_p^2 = \frac{4\pi ne^2}{m} .$$

   Consider the following three ranges of frequency

   (a)

$$0 < \omega \ll \tau^{-1},$$

   Show that, in this region,

$$R(\omega) \simeq 1 - \left(\frac{8\omega}{\omega_p^2 \tau} \right)^{1/2} .$$

   (This is known as the Hagen-Rubens regime.)

(b)

$$\tau^{-1} \ll \omega \ll \omega_p$$

Show that, in this region, the reflectivity is independent of ω:

$$R(\omega) \simeq 1 - \frac{2}{\omega_p \tau}\,.$$

(This is known as the relaxation regime.)

(c)

$$\omega_p \ll \omega\,.$$

Show that the reflectivity tends to zero in this regime and that, for $\omega \to \infty$,

$$R(\omega) = \left(\frac{\omega_p^2}{4\omega^2}\right)^2\,.$$

Show also that $k(\omega)$, which governs the rate of absorption of the wave as it propagates in the material tends to zero as $\omega \to \infty$ like

$$k(\omega) = \frac{\omega_p^2}{2\omega^3\tau}\,.$$

Hence the material becomes *transparent* above the plasmon frequency.
Note: The plasmon frequency is in the ultra violet for common metals, which is why metals with a clean surface are observed to be shiny and reflective. The the reddish color of copper is due to increase absorption above the red part of the spectrum due to interband transitions from the *d*-band to the conduction band. (The increased absortion in the orange–blue visible region, reduces the amount of light that is reflected in this region.) The explanation of the yellow color of gold is similar. Interband transitions are, of course, not included in the simple description discussed here.
Additional Note: The algebra is not difficult, but, if necessary, you can get help from Dressel and Grüner, "Electrodynamics of Solids" (on reserve in the library).

2. *Skin Depth*
As we discussed in class, a transverse excitation satisfies the condition

$$q = \frac{\omega}{c}\,\hat{N}(\omega)\,.$$

With $\hat{N} = n + ik$, the wave varies as

$$\exp\left[i\left(\frac{\omega}{c}\right)nx\right]\,\exp\left[-\left(\frac{\omega}{c}\right)kx\right]\,,$$

which shows that the wave decays within a distance

$$\delta = \frac{c}{\omega k}$$

from the surface, which is known as the "skin depth".

Determine the skin depth of a metal according to the Drude theory, in the regimes (i) $0 < \omega \ll \tau^{-1}$, and (ii) $\tau^{-1} \ll \omega \ll \omega_p$, of the previous question. Show that in case (i) the skin depth varies as $\omega^{-1/2}$, while in case (ii) it is independent of ω.

3. *Anomalous Skin Depth*
At low frequency and long relaxation time τ the mean free path of the electrons $\ell\,(= v_F\tau)$, can exceed the skin depth, δ.

(a) Show that the condition for this (neglecting numerical factors) is

$$\omega > \left(\frac{c}{v_F\omega_p}\right)^2 \frac{1}{\tau^3}\,. \qquad (1)$$

(b) If the sample is very clean, then, at low-T, the relaxation time τ can be long enough for this to be true. In this region, we expect that σ_1 will be independent of τ, since the electrons don't have time to scatter in the region (near the surface) where the electric field is non-zero. If, in addition, we are in the region where $\omega \ll v_F|q|$ (where we replace $|q|$ by δ^{-1}) then we should be able to use the expression for the low-frequency transverse conductivity discussed in class,

$$\sigma_1(\mathbf{q},0) = \frac{3}{4}\frac{ne^2\pi}{qv_Fm}\,, \qquad \sigma_2(\mathbf{q},0) = 0\,. \qquad (2)$$

Notes:

i. Eq. (2) is derived in the RPA, which does not include the relaxation time τ, so naturally the expression is independent of τ. We therefore only expect the result to be valid when σ *really* is independent of τ, i.e. when Eq. (1) is satisfied.

ii. Eq. (2) is for the region where the transverse response, needed here, is quite different from the longitudinal response. Usually though, when dealing with optical response, we are in the region where $\omega \gg v_Fq$ where the distinction between longitudinal and transverse response disappears.

iii. Because of the strong q dependence in Eq. (2) the conductivity is said to be "non-local" (i.e. the current at $\mathbf{r}$ depends on the electric field at $\mathbf{r}'$, where $\mathbf{r} \neq \mathbf{r}'$).

Take the expression for $\sigma(\mathbf{q},0)$ above, (use $q = \delta^{-1}$, which should only lead to an error of a numerical factor of order unity) and determine an expression for the "anomalous" skin depth δ as a function of ω. You should find that $\delta \sim w^{-1/3}$, as opposed to $\omega^{-1/2}$ for the normal skin effect.

(c) Show that the condition $\omega < v_F/\delta$ corresponds to

$$\omega < \frac{\omega_p v_F}{c}\,. \qquad (3)$$

Further notes:

(i) For the anomalous skin effect to occur, both Eqs. (1) and (3) must be satisfied, see Dressel and Grüner, Appendix E, particularly Fig. E2, for more details.

(ii) Since the anomalous skin depth is independent of the (hard to calculate) relaxation time, it turns out that it can be used to get information about the shape of the Fermi surface by having the electromagnetic field point in different directions relative to the crystal axes, see e.g. Dressel and Grüner, Sec. 12.1.2. However, other techniques, such as the de Haas-van Alphen effect, give results which are more straightforward to interpret, see e.g. Ashcroft and Mermin, Ch. 14.

4. *Helicon Waves*
   Read Ch. 23 of Mardar, and answer Qu. 3.