Physics 232

Longitudinal and Transverse Dynamical Response of the Electron Gas in the Random Phase Approximation (RPA)

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Contents

TODO

1. Equations of motion derivation of the RPA density response function.

2. Improve the sections on current response functions.

3. More on diagrams. In particular explain that when dealing with the screened response, it is convenient to also use the screened potential (sums up an infinite series of diagrams. Also emphasize, for the transverse case, what is the effective interaction (photon propagator) for the unscreened and also screened case. Point out that the screened interaction is, in practice very small compared with the Coulomb interactions, and can be neglected. (Magnetic interactions weak compared with Coulomb interactions.)

I. INTRODUCTION

In this handout we study the electromagnetic response of a a conductor with a view to understanding absorption and reflection of electromagnetic radiation. We start by reviewing Maxwell's equations, pointing out the utility of separating the fields into longitudinal and transverse components. We then consider the longitudinal and transverse responses of the electron gas in the simplest approximation, known as the random phase approximation (RPA).

In this handout we work in units where $\hbar = 1$. We will assume the sample has unit volume. For the most part, we also assume zero temperature.

II. MAXWELL'S EQUATIONS IN VACUUM; LONGITUDINAL AND TRANSVERSE PARTS

We start with Maxwell's equations[1, 2] in Gaussian units, in standard notation:

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E} &= 4\pi\rho && (1)\\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 && (2)\\ \boldsymbol{\nabla}\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} &= 0 && (3)\\ \boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{J}\,. && (4) \end{aligned}$$

If we add $(1/c)$ times the divergence of Eq. (4) to the time derivative of Eq. (1), and recognize that $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{B})=0$, we obtain the continuity equation

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{J}=0, \tag{5}$$

which arises from conservation of electric charge.

We recall that the electric and magnetic fields can be obtained from scalar and vector potentials, V and $\mathbf{A}$, according to

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla}V-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},\\ \mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A}\,. \end{aligned} \tag{6}$$

The fields $\mathbf{E}$ and $\mathbf{B}$ are unchanged if V and $\mathbf{A}$ are altered by a "gauge transformation" where

$$\begin{aligned} V\to V' &= V-\frac{1}{c}\frac{\partial\chi}{\partial t},\\ \mathbf{A}\to\mathbf{A}' &= \mathbf{A}+\boldsymbol{\nabla}\chi \end{aligned} \tag{7}$$

where the gauge function $\chi(\mathbf{r},t)$ is a function of $\mathbf{r}$ and t.

It will be very convenient to separate the vector fields in Maxwell's equations into their longitudinal and transverse components. A longitudinal field $\mathbf{E}^L(\mathbf{r})$ has zero has zero curl and a transverse field $\mathbf{E}^T(\mathbf{r})$ has zero divergence, *i.e.*

$$\boldsymbol{\nabla}\times\mathbf{E}^L=0,\qquad \boldsymbol{\nabla}\cdot\mathbf{E}^T=0\,. \tag{8}$$

An important theorem states that any field which vanishes at infinity can be uniquely written as the sum of a longitudinal and a transverse field

$$\mathbf{E}(\mathbf{r})=\mathbf{E}^L(\mathbf{r})+\mathbf{E}^T(\mathbf{r}). \tag{9}$$

If the divergence and curl of $\mathbf{E}$ are given to us, *i.e.*

$$\boldsymbol{\nabla} \cdot \mathbf{E} = f(\mathbf{r}), \qquad \boldsymbol{\nabla} \times \mathbf{E} = \mathbf{g}(\mathbf{r}), \tag{10}$$

then clearly

$$\boldsymbol{\nabla} \cdot \mathbf{E}^L = f(\mathbf{r}), \qquad \boldsymbol{\nabla} \times \mathbf{E}^T = \mathbf{g}(\mathbf{r}). \tag{11}$$

Explicit expressions (which vanish at infinity) can be obtained[1] for $\mathbf{E}^L$ and $\mathbf{E}^T$ which solve Eq. (11) in terms of of the given functions f and $\mathbf{g}$. (These can be most simply written for the Fourier transforms of the fields, see Eqs. (31)–(36).) An example of a longitudinal field is an electrostatic field. This illustrates that longitudinal field lines begin and end of points, lines or surfaces. An example of a transverse field is a magnetic field, which illustrates that transverse field lines form closed loops.

The point is that these solutions *uniquely* determine the vector field $\mathbf{E}$ according to Eq. (9). The reason is that if we write $\mathbf{E} \to \mathbf{E}' = \mathbf{E} + \mathbf{E}_1$, then we must have $\boldsymbol{\nabla} \cdot \mathbf{E}_1 = \boldsymbol{\nabla} \times \mathbf{E}_1 = 0$. But the second equality implies $\mathbf{E}_1 = \boldsymbol{\nabla}\phi$, where ϕ is a scalar function, and the first equality shows that ϕ satisfies Laplace's equation, *i.e.* $\nabla^2\phi = 0$. Since the boundary condition is $\mathbf{E}_1 \to 0$ for $r \to \infty$, it follows that $\nabla\phi \to 0$ in that limit. A solution is obviously $\nabla\phi = 0$ *everywhere*. However, according to a well known theorem[1, 2], a solution of Laplace's equation with specified values of $\boldsymbol{\nabla}\phi$ on the boundary is *unique* apart from an additive constant. Hence we must have $\mathbf{E}_1 = 0$ *everywhere*. Consequently a vector field $\mathbf{E}$ is uniquely determined by the sum of its longitudinal and transverse components, see Eq. (10), where longitudinal part is given in terms of the divergence of $\mathbf{E}$, and the transverse part in terms of the curl of $\mathbf{E}$, by the solutions of Eq. (11) which vanish at ∞.

Let us therefore now rewrite Maxwell's equations, indicating whether the longitudinal or transverse parts of the fields are being used:

$$\boldsymbol{\nabla} \cdot \mathbf{E}^L = 4\pi\rho \tag{12}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B}^L = 0 \tag{13}$$

$$\boldsymbol{\nabla} \times \mathbf{E}^T + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \tag{14}$$

$$\boldsymbol{\nabla} \times \mathbf{B}^T - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J}. \tag{15}$$

Eq. (13) tells that the longitudinal part of $\mathbf{B}$ is zero; the magnetic field is purely transverse. Hence $\mathbf{B}$ in Eq. (14) can be replaced by $\mathbf{B}^T$. Eq. (12) tells us that the longitudinal part of $\mathbf{E}$ is given by the charge density. In Eq. (15), the longitudinal part of the term involving $\partial\mathbf{E}/\partial t$ cancels the longitudinal part of the term involving $\mathbf{J}$ because, if we take the time derivative of Eq. (12),

$$\boldsymbol{\nabla} \cdot \frac{\partial \mathbf{E}^L}{\partial t} = 4\pi \frac{\partial \rho}{\partial t}, \tag{16}$$

and use the continuity equation, Eq, (5) (which we now realize involves $\mathbf{J}^L$), we get

$$\boldsymbol{\nabla} \cdot \frac{\partial \mathbf{E}^L}{\partial t} = -4\pi \boldsymbol{\nabla} \cdot \mathbf{J}^L . \tag{17}$$

Since we are dealing with longitudinal fields, the fact that they have the same divergence means that they must be equal (since we know that their curls both vanish), and so

$$\frac{\partial \mathbf{E}^L}{\partial t} = -4\pi \mathbf{J}^L . \tag{18}$$

Hence the longitudinal parts of Eq. (15) cancel.

We therefore see that two of Maxwell's equations only refer to the longitudinal components

$$\boldsymbol{\nabla} \cdot \mathbf{E}^L = 4\pi\rho, \tag{19}$$

$$\mathbf{B}^L = 0, \tag{20}$$

and the other two only refer to the transverse components

$$\boldsymbol{\nabla} \times \mathbf{E}^T + \frac{1}{c}\frac{\partial \mathbf{B}^T}{\partial t} = 0, \tag{21}$$

$$\boldsymbol{\nabla} \times \mathbf{B}^T - \frac{1}{c}\frac{\partial \mathbf{E}^T}{\partial t} = \frac{4\pi}{c}\mathbf{J}^T . \tag{22}$$

The gauge transformation, Eq. (7), only affects the longitudinal part of the vector potential, *i.e.*

$$\begin{aligned} V \rightarrow V' &= V - \frac{1}{c}\frac{\partial \chi}{\partial t}, \\ \mathbf{A}^L \rightarrow \mathbf{A}^{L\prime} &= \mathbf{A}^L + \boldsymbol{\nabla}\chi, \\ \mathbf{A}^T \rightarrow \mathbf{A}^{T\prime} &= \mathbf{A}^T . \end{aligned} \tag{23}$$

We will be particularly interested in fields which oscillate at a given frequency and wavevector so it is convenient to Fourier transform with respect to space and time. We describe the electric field as a complex quantity

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \exp[i(\mathbf{q} \cdot \mathbf{r} - \omega t)] , \tag{24}$$

(though, of course, it is the real part that we are interested in at the end of the calculation). Hence $\mathbf{E}(\mathbf{r}, t)$ is related to $\mathbf{E}(\mathbf{q}, \omega)$ by the inverse transform

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \iint_{-\infty}^{\infty} \mathbf{E}(\mathbf{q}, \omega) \, \exp[i(\mathbf{q} \cdot \mathbf{r} - \omega t)] \, d\omega \, d\mathbf{q}, \tag{25}$$

while the Fourier transform is

$$\mathbf{E}(\mathbf{q},\omega) = \iint_{-\infty}^{\infty} \mathbf{E}(\mathbf{r},t)\, \exp[-i(\mathbf{q}\cdot\mathbf{r} - \omega t)]\, dt\, d\mathbf{r}. \tag{26}$$

A great advantage of Fourier transformed quantities is that the Fourier transform of a derivative is simply the *product* of the Fourier transform of the function times a factor of ω or $\mathbf{q}$ (depending on which derivative is taken). For example, the Fourier transform of $\partial\mathbf{E}(\mathbf{r},t)/\partial t$ is given by

$$\iint_{-\infty}^{\infty} \frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t}\, \exp[-i(\mathbf{q}\cdot\mathbf{r} - \omega t)]\, dt\, d\mathbf{r} = -i\omega \iint_{-\infty}^{\infty} \mathbf{E}(\mathbf{r},t)\, \exp[-i(\mathbf{q}\cdot\mathbf{r} - \omega t)]\, dt\, d\mathbf{r} = -i\omega\mathbf{E}(\mathbf{q},\omega)\,, \tag{27}$$

where we integrated by parts and assumed a "pulse" of $\mathbf{E}$ which is localized in time, as well as in space, so $\mathbf{E} \to 0$ for $t \to \pm\infty$. Hence we find

$$\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t} \xrightarrow{FT} -i\omega\mathbf{E}(\mathbf{q},\omega)\,, \tag{28}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{r},t) \xrightarrow{FT} i\mathbf{q}\cdot\mathbf{E}(\mathbf{q},\omega)\,, \tag{29}$$

$$\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r},t) \xrightarrow{FT} i\mathbf{q}\times\mathbf{E}(\mathbf{q},\omega)\,. \tag{30}$$

We can now solve Eqs. (11) in terms of the Fourier transformed fields $\mathbf{E}^L(\mathbf{q})$ and $\mathbf{E}^T(\mathbf{q})$. For the longitudinal part we have $i\mathbf{q}\cdot\mathbf{E}^L(\mathbf{q}) = f(\mathbf{q})$, $\mathbf{q}\times\mathbf{E}^L(\mathbf{q}) = 0$. The solution is

$$i\mathbf{E}^L(\mathbf{q}) = \frac{\mathbf{q}}{q^2}\, f(\mathbf{q})\,. \tag{31}$$

Similarly for the transverse part we have $i\mathbf{q}\times\mathbf{E}^T(\mathbf{q}) = \mathbf{g}(\mathbf{q})$, $\mathbf{q}\cdot\mathbf{E}^T(\mathbf{q}) = 0$ which has solution

$$i\mathbf{E}^T(\mathbf{q}) = -\frac{\mathbf{q}\times\mathbf{g}(\mathbf{q})}{q^2}\,, \tag{32}$$

where we used $\mathbf{q}\times(\mathbf{q}\times\mathbf{E}^T) = (\mathbf{q}\cdot\mathbf{E}^T)\mathbf{q} - q^2\mathbf{E}^T$, and noted that $\mathbf{E}^T$ is transverse so $\mathbf{q}\cdot\mathbf{E}^T(\mathbf{q}) = 0$. Alternatively, one can write $\mathbf{E}^L(\mathbf{q})$ and $\mathbf{E}^T(\mathbf{q})$ in terms of the electric field itself $\mathbf{E}(\mathbf{q})$, rather than its divergence and curl. Since $f(\mathbf{q}) = i\mathbf{q}\cdot\mathbf{E}(\mathbf{q})$, we get, from Eq. (31), that

$$\boxed{\mathbf{E}^L(\mathbf{q}) = \frac{\mathbf{q}\cdot\mathbf{E}(\mathbf{q})}{q^2}\, \mathbf{q}\,,} \tag{33}$$

or

$$E^L_\mu(\mathbf{q}) = \mathcal{P}^L_{\mu\nu}(\mathbf{q}) E_\nu(\mathbf{q}), \tag{34}$$

where μ and ν are cartesian coordinates and

$$\boxed{\mathcal{P}^L_{\mu\nu}(\mathbf{q}) = \frac{q_\mu q_\nu}{q^2}} \tag{35}$$

is the *longitudinal projection operator*. Similarly, from Eq. (32) and $\mathbf{g}(\mathbf{q}) = i\mathbf{q} \times \mathbf{E}(\mathbf{q})$, we get

$$\boxed{\mathbf{E}^T(\mathbf{q}) = -\mathbf{q} \times (\mathbf{q} \times \mathbf{E}(\mathbf{q})) = \mathbf{E}(\mathbf{q}) - \frac{\mathbf{q} \cdot \mathbf{E}(\mathbf{q})}{q^2}\, \mathbf{q}\,,} \tag{36}$$

or equivalently

$$E^T_\mu(\mathbf{q}) = \mathcal{P}^T_{\mu\nu}(\mathbf{q}) E_\nu(\mathbf{q}), \tag{37}$$

where

$$\boxed{\mathcal{P}^T_{\mu\nu}(\mathbf{q}) = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}} \tag{38}$$

is the *transverse projection operator*. Note that the sum of the longitudinal and transverse projection operators is unity:

$$\mathcal{P}^L(\mathbf{q}) + \mathcal{P}^T(\mathbf{q}) = 1\,. \tag{39}$$

We now see that to project out the longitudinal (or transverse) part of a vector field in real space, $\mathbf{E}(\mathbf{r})$ say, we first Fourier transform, then apply the longitudinal (or transverse) projection operator, and finally do the inverse transform back.

We give some examples of Fourier transforms of vector fields in Appendix A, indicating whether they are longitudinal or transverse. For example we see in Eq. (A3) that the electrostatic field from an electric dipole $\mathbf{p}$ is equal to

$$\mathbf{E}^{\text{dip}}(\mathbf{q}) = -4\pi\, \frac{\mathbf{p} \cdot \mathbf{q}}{q^2}\, \mathbf{q} \quad \text{or} \quad E^{\text{dip}}_\mu(\mathbf{q}) = -4\pi\, \mathcal{P}^L_{\mu\nu}(\mathbf{q}) p_\nu, \tag{40}$$

i.e. is just (-4π times) the longitudinal projection operator acting the (constant) vector field $\mathbf{p}$. Similarly the magnetic field due to a magnetic dipole $\mathbf{m}$ is given, according to Eq. (A6) by

$$\mathbf{B}^{\text{dip}}(\mathbf{q}) = 4\pi \left(\mathbf{m} - \frac{(\mathbf{m} \cdot \mathbf{q})}{q^2}\, \mathbf{q} \right) \quad \text{or} \quad B^{\text{dip}}_\mu(\mathbf{q}) = 4\pi\, \mathcal{P}^T_{\mu\nu}(\mathbf{q}) \mu_\nu, \tag{41}$$

and is just (4π times) the transverse projection operator acting on the constant vector $\mathbf{m}$.

Note that since the sum of the longitudinal and transverse projection operators is unity, Eq. (39), the difference between the expressions for the fields due to a magnetic dipole and an electric dipole is independent of $\mathbf{q}$, which, when Fourier transformed to real space, becomes a delta function at $\mathbf{r} = 0$. To be precise (using $\mathbf{p}$ here to denote the magnetic, as well as the electric, dipole moment)

$$\mathbf{B}^{\text{dip}}(\mathbf{r}) = \mathbf{E}^{\text{dip}}(\mathbf{r}) + 4\pi \mathbf{p} \delta(\mathbf{r})\,. \tag{42}$$

Hence the expressions for the fields to magnetic and electric dipoles are *the same* for $\mathbf{r} \neq 0$, as is well known[1, 2], but it is important not to forget about the extra delta function contribution at the origin in the magnetic case.

We now consider Maxwell's' equations when Fourier transformed to $\mathbf{q}$ and ω. The Maxwell equation for the longitudinal component of $\mathbf{E}$, Eq. (19), becomes

$$\boxed{iq\, E^L(\mathbf{q}, \omega) = 4\pi\rho(\mathbf{q}, \omega)\,,} \tag{43}$$

and Eqs. (21) and (22) which describe the transverse components become

$$\boxed{\mathbf{q} \times \mathbf{E}^T(\mathbf{q}, \omega) - \frac{\omega}{c}\mathbf{B}^T(\mathbf{q}, \omega) = 0,} \tag{44}$$

$$\boxed{i\mathbf{q} \times \mathbf{B}^T(\mathbf{q}, \omega) + i\frac{\omega}{c}\mathbf{E}^T(\mathbf{q}, \omega) = \frac{4\pi}{c}\mathbf{J}^T(\mathbf{q}, \omega)\,.} \tag{45}$$

We also note that the longitudinal component of $\mathbf{B}$ vanishes

$$\boxed{\mathbf{B}^L(\mathbf{q}, \omega) = 0\,.} \tag{46}$$

Since $\mathbf{B}$ is always transverse, we shall omit the superscript "T" on $\mathbf{B}$ from now on. The continuity equation, Eq. (5), is expressed in terms of Fourier transformed variables as

$$\boxed{q\, J^L(\mathbf{q}, \omega) - \omega\rho(\mathbf{q}, \omega) = 0\,.} \tag{47}$$

When Fourier transformed, the relation between the electric and magnetic fields and the scalar and vector potentials, Eq. (6), becomes

$$\boxed{\mathbf{E}^L(\mathbf{q}, \omega) = -i\mathbf{q}V(\mathbf{q}, \omega) + \frac{i\omega}{c}\mathbf{A}^L(\mathbf{q}, \omega), \qquad \mathbf{E}^T(\mathbf{q}, \omega) = \frac{i\omega}{c}\mathbf{A}^T(\mathbf{q}, \omega)} \tag{48}$$

$$\boxed{\mathbf{B}(\mathbf{q}, \omega) = i\mathbf{q} \times \mathbf{A}^T(\mathbf{q}, \omega)\,,} \tag{49}$$

and the gauge transformation, Eq. (23), becomes

$$\boxed{V'(\mathbf{q}, \omega) = V + \frac{i\omega}{c}\chi(\mathbf{q}, \omega), \qquad \mathbf{A}^{L\prime}(\mathbf{q}, \omega) = \mathbf{A}^L(\mathbf{q}, \omega) + \mathbf{q}\chi(\mathbf{q}, \omega),} \tag{50}$$

with $\mathbf{A}^T$ unchanged. The longitudinal electric field can be represented either entirely by a scalar potential, or entirely by a (time-dependent) vector potential, or by a combination of both. Because of gauge invariance, the same results must be obtained in each case. Later, we will use this result to obtain some important relations.

III. MAXWELL'S EQUATIONS IN MATTER

We shall be interested considering the response of the system to some "external" charge ρ_{ext} and current $\mathbf{J}_{\text{ext}}$. In the absence of the material these would produce electric and magnetic fields $\mathbf{E}_{\text{ext}}$ and $\mathbf{B}_{\text{ext}}$. However, the system responds and generates additional charges and currents, ρ_{int} and $\mathbf{J}_{\text{int}}$, and corresponding electric and magnetic fields $\mathbf{E}_{\text{int}}$ and $\mathbf{B}_{\text{int}}$. Maxwell's equations refer to the *total* charges, currents and fields, *i.e.*

$$\mathbf{E} = \mathbf{E}_{\text{int}} + \mathbf{E}_{\text{ext}}, \quad \mathbf{B} = \mathbf{B}_{\text{int}} + \mathbf{B}_{\text{ext}}, \quad \rho = \rho_{\text{int}} + \rho_{\text{ext}}, \quad \mathbf{J} = \mathbf{J}_{\text{int}} + \mathbf{J}_{\text{ext}} \, . \tag{51}$$

In this handout, we shall imagine the "external" charges and currents to be formed in a localized region inside the material. We can do this by combining different Fourier components just as one builds up a wavepacket out of plane waves of a range of wavevectors.

Of course, this is not experimentally realizable. In practice the external charges and currents really are outside the sample. In order to treat this situation one needs to relate the fields outside the sample to those inside it, which is not trivial because of currents and charges on the *surface* of the sample.

The conventional way to treat this difficulty[1, 2] in the electrostatic case is to introduce the polarization field $\mathbf{P}$, related to the induced charge density by

$$\boldsymbol{\nabla} \cdot \mathbf{P} = -\rho_{\text{int}}, \quad \text{so} \quad 4\pi P^L = -\mathbf{E}_{\text{int}} \, . \tag{52}$$

One also introduces the electric displacement vector $\mathbf{D}$ such that

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \tag{53}$$

and

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 4\pi\rho_{\text{ext}} \, . \tag{54}$$

We should note, though, that only the longitudinal part of $\mathbf{P}$ (and hence $\mathbf{D}$) is uniquely determined by Eq. (52). It is very convenient to define the polarization so that it vanishes outside the sample, which means that $\mathbf{P}$ (and hence $\mathbf{D}$) have a transverse component, whereas $\mathbf{E}$ does not (remember we are considering electrostatics here). The advantage of working with $\mathbf{D}$, and assuming some relationship (usually linear) between $\mathbf{P}$ and $\mathbf{E}$, is that one avoids having to consider explicitly the charges on the surface of the dielectric.

Similarly in the magnetostatic case, one introduces the magnetization, $\mathbf{M}$, related to the internal currents by

$$c\boldsymbol{\nabla}\times\mathbf{M}=\mathbf{J}_{\rm int}\quad\text{so}\quad 4\pi\mathbf{M}^T=\mathbf{B}_{\rm int}\,. \tag{55}$$

and the auxiliary field $\mathbf{H}$ where

$$\boldsymbol{\nabla}\times\mathbf{H}=\frac{4\pi}{c}\mathbf{J}_{\rm ext} \tag{56}$$

so

$$\mathbf{B}=\mathbf{H}+4\pi\mathbf{M}\,. \tag{57}$$

Analogously to the electrostatic case, $\mathbf{M}$ (and hence $\mathbf{H}$) have a longitudinal part, as well as a transverse part. whereas $\mathbf{B}$ is purely transverse. By introducing $\mathbf{H}$ and $\mathbf{M}$, and assuming some relationship between $\mathbf{M}$ and $\mathbf{H}$ (or $\mathbf{B}$), one avoids having to treat explicitly the currents on the surface of the material.

In this handout, we focus on the linear response of the medium to applied fields, without the additional complications of the (separate) problem of the relation between the fields inside and outside the sample. We therefore consider the "external" charges and currents to be *localized* in a region of the sample far from the surface. We decompose them into their different Fourier components, and compute how the system responds. The surface of the material is sent to infinity and does not enter the calculations. We will therefore not use $\mathbf{D}$ or $\mathbf{H}$ in the rest of this handout.

It will be useful to treat separately the response to longitudinal and transverse fields. For example, if we apply an electric field $\mathbf{E}(\mathbf{q},\omega)$ and determine the resulting induced current $\mathbf{J}_{\rm int}(\mathbf{q},\omega)$, Ohm's law tells us that

$$J_{{\rm int}\,\mu}=\sigma_{\mu\nu}(\mathbf{q},\omega)E_\nu\,, \tag{58}$$

where $\sigma_{\mu\nu}(\mathbf{q},\omega)$ is the conductivity tensor. For an isotropic medium the only independent components of $\sigma_{\mu\nu}(\mathbf{q},\omega)$ are the longitudinal and transverse components, *i.e.*

$$\begin{aligned}\sigma_{\mu\nu}(\mathbf{q},\omega) &= \mathcal{P}^L_{\mu\nu}(\mathbf{q})\sigma^L(\mathbf{q},\omega)+\mathcal{P}^T_{\mu\nu}(\mathbf{q})\sigma^T(\mathbf{q},\omega), && (59)\\ &= \boxed{\frac{q_\mu q_\nu}{q^2}\,\sigma^L(\mathbf{q},\omega)+\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\,\sigma^T(\mathbf{q},\omega)\,.} && (60)\end{aligned}$$

In this handout we will calculate the longitudinal and transverse conductivities, $\sigma^L(\mathbf{q},\omega)$ and $\sigma^T(\mathbf{q},\omega)$, of the electron gas within a certain approximation known as the random phase approximation.

A. The Longitudinal Maxwell's equations in matter

Let us first consider the longitudinal response of the system, which is determined by Eq. (43). We write this as

$$iq\, E^L(\mathbf{q},\omega) = 4\pi\left(\rho_{\rm int}(\mathbf{q},\omega) + \rho_{\rm ext}(\mathbf{q},\omega)\right) , \tag{61}$$

where we replaced $\mathbf{q}\cdot\mathbf{E}^L$ by qE^L since $\mathbf{E}^L$, being longitudinal, is in the direction of $\mathbf{q}$. We assume that the system responds *linearly* to the external perturbation. In particular,

$$\rho_{\rm int} \equiv (-e)\delta n \tag{62}$$

(where δn is the change in the electron *number density*) will be proportional to the electrostatic potential energy eV in the system. Here we will write the longitudinal electric field in terms of a scalar potential, see Eq. (48) and the discussion following it. Hence we have

$$\mathbf{E}^L(\mathbf{q},\omega) = -i\mathbf{q}V(\mathbf{q},\omega)\,, \tag{63}$$

and write

$$\langle\rho_{\rm int}(\mathbf{q},\omega)\rangle = (-e)\ \langle\delta n(\mathbf{q},\omega)\rangle = (-e)\ \chi_{\rm sc}(\mathbf{q},\omega)\ (-e)V(\mathbf{q},\omega) = -\frac{e^2}{iq^2}\,\chi_{\rm sc}(\mathbf{q},\omega)\, qE^L(\mathbf{q},\omega)\,, \tag{64}$$

where $\chi_{\rm sc}(\mathbf{q},\omega)$ is called the "screened" density response function because it describes the change in the number density in response to the *total* electric potential energy $(-e)V$, not the *external* potential. (The total potential includes the screening effects of the electrons which tends to reduce the potential from that produced by the external charges. Longitudinal screening effects are quantified in Eq. (69) below.) Substituting Eq. (64) into Eq. (61) gives

$$iqE^L(\mathbf{q},\omega) = 4\pi\rho_{\rm ext}(\mathbf{q},\omega) + \frac{4\pi i e^2}{q^2}\chi_{\rm sc}(\mathbf{q},\omega)\, qE^L(\mathbf{q},\omega)\,. \tag{65}$$

This can be written as

$$\boxed{i\epsilon^L(\mathbf{q},\omega)\, qE^L(\mathbf{q},\omega) = 4\pi\rho_{\rm ext}(\mathbf{q},\omega)\,,} \tag{66}$$

which only involves the external charge density, where the *longitudinal dielectric constant* $\epsilon(\mathbf{q},\omega)$ is given by

$$\boxed{\epsilon^L(\mathbf{q},\omega) = 1 - \frac{4\pi e^2}{q^2}\chi_{\rm sc}(\mathbf{q},\omega)\,.} \tag{67}$$

Note that $V(\mathbf{q}) = 4\pi e^2/q^2$, which appears in this expression, is the Fourier transform of the Coulomb potential e^2/r.

Since the external electric field satisfies

$$iq\mathbf{E}^L_{\rm ext}(\mathbf{q},\omega) = 4\pi\rho_{\rm ext}(\mathbf{q},\omega)\,, \tag{68}$$

we see from Eq. (66) that $\epsilon(\mathbf{q},\omega)$ describes the extent to which the external field is reduced (*i.e.* screened) by the charges of the system:

$$\boxed{\frac{E^L(\mathbf{q},\omega)}{E^L_{\rm ext}(\mathbf{q},\omega)} = \frac{1}{\epsilon^L(\mathbf{q},\omega)}\,.} \tag{69}$$

Alternatively, because of Eq. (63), we can write this in terms of the potential rather than the field:

$$\frac{V(\mathbf{q},\omega)}{V_{\rm ext}(\mathbf{q},\omega)} = \frac{1}{\epsilon^L(\mathbf{q},\omega)}\,. \tag{70}$$

From Eq. (69) we see that there can be fluctuations in the longitudinal field without an external field when

$$\boxed{\epsilon^L(\mathbf{q},\omega) = 0\,,} \tag{71}$$

which gives the condition for longitudinal excitations in the system to be sustained.

Incidentally, if we consider the change in density in response to the *external*, rather than the total, potential, *i.e.*

$$\langle\rho_{\rm int}(\mathbf{q},\omega)\rangle = (-e)\ \chi(\mathbf{q},\omega)\ (-e)V_{\rm ext}(\mathbf{q},\omega) \tag{72}$$

where χ is the density response function, we find

$$\frac{1}{\epsilon^L(\mathbf{q},\omega)} = 1 + \frac{4\pi e^2}{q^2}\chi(\mathbf{q},\omega)\,, \tag{73}$$

where

$$\chi(\mathbf{q},\omega) = \frac{\chi_{\rm sc}(\mathbf{q},\omega)}{1 - \dfrac{4\pi e^2}{q^2}\chi_{\rm sc}(\mathbf{q},\omega)} = \frac{\chi_{\rm sc}(\mathbf{q},\omega)}{\epsilon^L(\mathbf{q},\omega)}\,. \tag{74}$$

Next we consider the case of *dielectrics*, which do not conduct and have a finite dielectric constant for $\mathbf{q}$ and $\omega \to 0$. From Eqs. (52)–(54) we see that, in the $(\mathbf{q},\omega) \to 0$ limit, we can write

$$\mathbf{P} = \chi_{\rm diel}\mathbf{E}, \quad \text{or} \quad \mathbf{E}_{\rm int} = -4\pi\chi_{\rm diel}\mathbf{E}\,. \tag{75}$$

where, according to Eq. (64), the dielectric susceptibility $\chi_{\rm diel}$ is related to the screened density response by

$$\boxed{\chi_{\rm diel} = -e^2 \lim_{q\to 0}\frac{\chi_{\rm sc}(\mathbf{q},0)}{q^2}\,.} \tag{76}$$

Note that the static dielectric constant of a dielectric is related to $\chi_{\rm diel}$ by

$$\epsilon(0,0) = 1 + 4\pi\chi_{\rm diel}\,. \tag{77}$$

Whereas $\epsilon(0,0)$ is finite for a dielectric (insulator), we shall see that for a metal $\epsilon(\mathbf{q},\omega)$ diverges for $(\mathbf{q},\omega) \to 0$. Noting that $\chi_{\rm diel} > 0$, and also noting Eq. (134) below, we see that $\chi_{\rm sc}(\mathbf{q},0)$ is negative. (This is the standard definition. In my view it is unfortunate that $\chi_{\rm sc}$ is not defined such that $\chi_{\rm sc}(\mathbf{q},0) > 0$.)

Up to now we have described the longitudinal response of the system in terms of the change in the density. It is often convenient, instead, to describe it in terms of the resulting current, and we define the longitudinal conductivity, $\sigma^L(\mathbf{q},\omega)$, by Ohm's law:

$$\boxed{\langle \mathbf{J}^L_{\rm int}(\mathbf{q},\omega)\rangle = \sigma^L(\mathbf{q},\omega)\mathbf{E}^L(\mathbf{q},\omega)\,.} \tag{78}$$

Inserting the continuity equation, Eq. (47), into Eq. (64) we find

$$\langle \mathbf{J}^L_{\rm int}(\mathbf{q},\omega)\rangle = ie^2\frac{\omega}{q^2}\chi_{\rm sc}(\mathbf{q},\omega)\mathbf{E}^L(\mathbf{q},\omega)\,, \tag{79}$$

which gives

$$\sigma^L(\mathbf{q},\omega) = ie^2\frac{\omega}{q^2}\chi_{\rm sc}(\mathbf{q},\omega)\,. \tag{80}$$

Comparing with Eq. (67) we find the relationship between ϵ^L and σ^L:

$$\boxed{\epsilon^L(\mathbf{q},\omega) = 1 + \frac{4\pi i}{\omega}\sigma^L(\mathbf{q},\omega)\,.} \tag{81}$$

In a conductor $\sigma^L(\mathbf{q},\omega)$ is finite for $(\mathbf{q},\omega) \to 0$, so $\epsilon(\mathbf{q},\omega)$ diverges in this limit, By contrast, for an insulator, $\epsilon(0,0)$ is finite, see also Eqs. (75)–(77), so $\sigma(0,\omega)$ is imaginary (non-dissipative) and vanishes like $i\omega$ as $\omega \to 0$.

It will also turn out to be useful to consider the gauge in which $\mathbf{E}^L$ is given by a vector potential $\mathbf{A}^L$, rather than a scalar potential V, see Eq. (50) and the discussion below it. From Eq. (48) we see that in this gauge

$$\mathbf{E}^L(\mathbf{q},\omega) = \frac{i\omega}{c}\mathbf{A}^L(\mathbf{q},\omega)\,. \tag{82}$$

We define $\chi^L_{\rm sc,\,j}(\mathbf{q},\omega)$ as the (screened) response of the (number) current to (e/c times) the vector potential, *i.e.*

$$\langle \mathbf{J}^L_{\rm int}(\mathbf{q},\omega)\rangle = (-e)\langle \mathbf{j}^L(\mathbf{q},\omega)\rangle = -\frac{e^2}{c}\chi^L_{\rm sc,\,j}(\mathbf{q},\omega)\mathbf{A}^L(\mathbf{q},\omega), \tag{83}$$

where the number current $\mathbf{j}$ is related to the induced electrical current $\mathbf{J}_{\rm int}$ by

$$\mathbf{J}_{\rm int} = (-e)\mathbf{j}\,, \tag{84}$$

and has continuity equation analogous to Eq. (47),

$$q\, j^L(\mathbf{q},\omega) = \omega\, \delta n(\mathbf{q},\omega)\,, \tag{85}$$

where δn is the change in the electron number density. From Eq. (82) the conductivity is related to $\chi^L_{\rm sc,\,j}(\mathbf{q},\omega)$ by

$$\boxed{\sigma^L(\mathbf{q},\omega) = -\frac{e^2}{i\omega}\chi^L_{\rm sc,\,j}(\mathbf{q},\omega),} \tag{86}$$

and so, from Eq. (81) the dielectric constant can be expressed in terms of $\chi^L_{\rm sc,\,j}(\mathbf{q},\omega)$ by

$$\boxed{\epsilon^L(\mathbf{q},\omega) = 1 - \frac{4\pi e^2}{\omega^2}\,\chi^L_{\rm sc,\,j}(\mathbf{q},\omega)\,.} \tag{87}$$

Comparing with Eq. (67), we see that

$$\chi^L_{\rm sc,\,j}(\mathbf{q},\omega) = \frac{\omega^2}{q^2}\chi_{\rm sc}(\mathbf{q},\omega)\,, \tag{88}$$

which is just a consequence of the continuity equation, Eq. (47). Physically, Eq. (88) reflects that a longitudinal electric field can be represented either by a scalar potential, the response to which is controlled by $\chi_{\rm sc}(\mathbf{q},\omega)$, or by a longitudinal vector potential, the response of which is controlled by $\chi^L_{\rm sc,\,j}(\mathbf{q},\omega)$, *i.e.* it reflects *gauge invariance.*

An important consequence of Eq. (88) is that $\chi^L_{\rm sc,\,j}$ vanishes at $\omega = 0$,

$$\boxed{\chi^L_{\rm sc,\,j}(\mathbf{q},0) = 0} \tag{89}$$

for *all* $\mathbf{q}$. This is because a static longitudinal vector potential does not give rise to an electric or magnetic field, and so has no observal effect.

To conclude this section, the longitudinal electrical response of the system to an external perturbation is given by Eq. (66) and is characterized by the dielectric constant $\epsilon^L(\mathbf{q},\omega)$ (or equivalently by $\sigma^L(\mathbf{q},\omega)$ or $\chi^L_{\rm sc,\,j}(\mathbf{q},\omega)$). Relationships between the different linear response functions are summarized in Appendix G.

B. The Transverse Maxwell's equations in matter

Now we consider the Maxwell equations for transverse fields, Eqs. (44) and (45). We multiply Eq. (44) by $\mathbf{q}\times$ and substitute for $\mathbf{B}$ from Eq. (45) to get

$$i(\omega^2 - c^2q^2)\mathbf{E}^T(\mathbf{q},\omega) = 4\pi\omega\mathbf{J}^T(\mathbf{q},\omega)\,, \tag{90}$$

where we used $\mathbf{q} \times (\mathbf{q} \times \mathbf{E}^T) = -q^2 \mathbf{E}^T$ (since $\mathbf{q} \cdot \mathbf{E}^T = 0$).

Writing $\mathbf{J}^T = \mathbf{J}^T_{\text{int}} + \mathbf{J}^T_{\text{ext}}$ and assuming that $\mathbf{J}^T_{\text{int}}$ is proportional to the (total) electric field (Ohm's law),

$$\boxed{\langle \mathbf{J}^T_{\text{int}}(\mathbf{q},\omega) \rangle = \sigma^T(\mathbf{q},\omega) \mathbf{E}^T(\mathbf{q},\omega),} \tag{91}$$

Eq. (90) becomes

$$\boxed{i \left[\omega^2 \left(1 + \frac{4\pi i \sigma^T(\mathbf{q},\omega)}{\omega} \right) - c^2 q^2 \right] \mathbf{E}^T(\mathbf{q},\omega) = 4\pi\omega \mathbf{J}^T_{\text{ext}}(\mathbf{q},\omega)\,.} \tag{92}$$

We can combine the two terms involving ω^2 by defining a transverse dielectric constant $\epsilon^T(\mathbf{q},\omega)$ by

$$\boxed{\epsilon^T(\mathbf{q},\omega) = 1 + \frac{4\pi i \sigma^T(\mathbf{q},\omega)}{\omega},} \tag{93}$$

which has exactly the same form as the corresponding expression for the longitudinal functions, Eq. (81).

If we define a field $\widetilde{\mathbf{D}}(\mathbf{q},\omega)$ by

$$\widetilde{\mathbf{D}}(\mathbf{q},\omega) = \epsilon^T(\mathbf{q},\omega) \mathbf{E}^T(\mathbf{q},\omega) \tag{94}$$

then

$$\widetilde{\mathbf{D}}(\mathbf{q},\omega) = \mathbf{E}^T(\mathbf{q},\omega) + \frac{4\pi i}{\omega}\, \mathbf{J}^T_{\text{int}}(\mathbf{q},\omega), \tag{95}$$

which can be instructively written in the time domain as

$$\frac{\partial \widetilde{\mathbf{D}}(\mathbf{r},t)}{\partial t} = \frac{\partial \mathbf{E}^T(\mathbf{r},t)}{\partial t} + 4\pi \mathbf{J}^T_{\text{int}}(\mathbf{r},t)\,. \tag{96}$$

In the absence of the material, $\mathbf{E} = \mathbf{E}_{\text{ext}}$ and $\sigma^T = 0$, so Eq. (92) becomes

$$i(\omega^2 - c^2 q^2) \mathbf{E}^T_{\text{ext}}(\mathbf{q},\omega) = 4\pi\omega \mathbf{J}^T_{\text{ext}}(\mathbf{q},\omega)\,, \tag{97}$$

and dividing this into Eq. (92) gives

$$\boxed{\frac{\mathbf{B}^T(\mathbf{q},\omega)}{\mathbf{B}^T_{\text{ext}}(\mathbf{q},\omega)} = \frac{\mathbf{E}^T(\mathbf{q},\omega)}{\mathbf{E}^T_{\text{ext}}(\mathbf{q},\omega)} = \frac{\omega^2 - c^2 q^2}{\omega^2 \epsilon^T(\mathbf{q},\omega) - c^2 q^2} = \frac{1}{1 + 4\pi i \omega \sigma^T(\mathbf{q},\omega)/(\omega^2 - c^2 q^2)}\,,} \tag{98}$$

where we used that $\mathbf{B} \propto \mathbf{E}$, see Eq. (44). We see that in the longitudinal case, Eq. (69), the ratio of the total to external electric fields is the inverse of the dielectric constant. The same is not true in the transverse case because of the factors of ω^2 and $c^2 q^2$. Physically the reason is that transverse fluctuations couple to electromagnetic waves which propagate with speed c (in vacuum). Note however, that Eqs. (69) and (98) *are* equivalent for $q = 0$. We shall see, quite generally, that the difference between the longitudinal and transverse responses vanishes at $q = 0$.

From Eq. (98) we see that the condition for self-sustaining transverse fluctuations (*i.e.* that exist without an external current) is

$$\boxed{\omega^2 = \frac{c^2 q^2}{\epsilon^T(\mathbf{q},\omega)}\,.} \tag{99}$$

We can also relate the transverse conductivity to the response to current response to a vector potential, as we did for the longitudinal case. Following the same steps as in Eqs. (82) to (87), we have

$$\langle \mathbf{J}^T_{\rm int}(\mathbf{q},\omega)\rangle = -\frac{e^2}{c}\chi^T_{\rm sc,\,j}(\mathbf{q},\omega)\mathbf{A}^T(\mathbf{q},\omega), \tag{100}$$

$$\boxed{\sigma^T(\mathbf{q},\omega) = -\frac{e^2}{i\omega}\chi^T_{\rm sc,\,j}(\mathbf{q},\omega),} \tag{101}$$

and

$$\boxed{\epsilon^T(\mathbf{q},\omega) = 1 - \frac{4\pi e^2}{\omega^2}\,\chi^T_{\rm sc,\,j}(\mathbf{q},\omega)\,.} \tag{102}$$

It is useful to use Eq. (101) to rewrite Eq. (98), which describes transverse screening, in terms of $\chi^T_{\rm sc,\,j}(\mathbf{q},\omega)$ as follows

$$\boxed{\frac{\mathbf{B}^T(\mathbf{q},\omega)}{\mathbf{B}^T_{\rm ext}(\mathbf{q},\omega)} = \frac{\mathbf{E}^T(\mathbf{q},\omega)}{\mathbf{E}^T_{\rm ext}(\mathbf{q},\omega)} = \frac{1}{1 + 4\pi e^2\chi^T_{\rm sc,\,j}(\mathbf{q},\omega)/(c^2q^2-\omega^2)}\,,} \tag{103}$$

For the longitudinal case, we have Eq. (88), which leads to Eq. (89), as a result of charge conservation. However, there is no analogue of Eq. (88) for the transverse case. Nonetheless, unless there are long-range current correlations, we expect the longitudinal and transverse current response functions to be equal at $\mathbf{q} = 0$. Since $\chi^L_{\rm sc,\,j}(0,0) = 0$ according to Eq. (89), we then have

$$\chi^T_{\rm sc,\,j}(0,0) = 0\,. \tag{104}$$

We should emphasize, however, that the assumption of no long-range current correlations breaks down for a superconductor, and Eq. (104) is not true for these materials. Furthermore, there is no reason to suppose that $\chi^T_{\rm sc,\,j}(\mathbf{q},0)$ vanishes for $\mathbf{q} \neq 0$, and we shall now see that there is a term proportional to q^2 which gives the magnetostatic response of the material.

From Eqs. (55) to (57), we see that the low frequency and long wavelength magnetic response can be conventionally written as

$$\mathbf{M} = \chi_{\text{mag}}\mathbf{B}_{\text{ext}}, \quad \text{or} \quad \mathbf{B}_{\text{int}} = 4\pi\chi_{\text{mag}}\mathbf{B}_{\text{ext}}, \tag{105}$$

where χ_{mag} is the magnetic susceptibility. In fact, we will find it more convenient to define the magnetic susceptibility as the response to the total field $\mathbf{B}$, rather than the external field $\mathbf{B}_{\text{ext}}$. We therefore define

$$\mathbf{B}_{\text{int}} = 4\pi\chi'_{\text{mag}}\mathbf{B}, \tag{106}$$

where

$$\boxed{\mu = 1 + 4\pi\chi_{\text{mag}} = (1 - 4\pi\chi'_{\text{mag}})^{-1}\,,} \tag{107}$$

in which

$$\boxed{\mu = \mathbf{B}/\mathbf{B}_{\text{ext}}} \tag{108}$$

is known as the magnetic permeability.

Comparing with Eqs. (103) and noting that $cq \gg \omega$ in this limit, one has

$$\boxed{\chi'_{\text{mag}} = -\frac{e^2}{c^2}\lim_{q\to 0}\frac{\chi^T_{\text{sc, j}}(\mathbf{q}, 0)}{q^2}\,,} \tag{109}$$

which is the magnetic analogue of Eq. (76). Eq. (109) implies that the transverse current response vanishes for $\mathbf{q} \to 0$ like q^2, as stated above. The coefficient of q^2 then gives the low frequency magnetic susceptibility χ'_{mag}. This is generally very small because of the factor of c^2 in the denominator of Eq. (109).

It is important to note that in a superconductor $\lim_{\mathbf{q}\to 0}\chi^T_{\text{sc, j}}(\mathbf{q}, 0)$ tends to a (positive) *constant*, rather than vanishing like q^2, which means that the (wave-vector dependent) magnetic susceptibility $\chi'_{\text{mag}}(\mathbf{q})$ *diverges* like $-1/q^2$ according to Eq. (109).[1] The minus sign means that the response is diamagnetic. Eqs. (107) and (108) then show that $\mu = 0$ in this limit and $\mathbf{B} = 0$, *i.e.* there is flux expulsion, which is known as the Meissner effect and is a fundamental property of a superconductor.

To conclude this section, we have introduced several linear response functions, for both the longitudinal and transverse cases. The reason is that, depending on the circumstances, one or the other of them may be more convenient. They are all simply related and the relationships between them are summarized in Appendix G.

[1] From Eq. (107), we see that the conventionally defined magnetic susceptibility χ_{mag} tends to the uninspiring value of $-1/(4\pi)$.

IV. LONGITUDINAL RESPONSE OF THE ELECTRON GAS

In this section we consider microscopically the response of the electron gas to a longitudinal perturbation. We will mainly consider a simple approximation, known as the Random Phase Approximation (RPA). The next section will be a similar treatment of the transverse response.

A. Formalism

As discussed in the previous section, if we apply a time dependent electric field $\mathbf{E}_{\text{ext}}(\mathbf{r}, t)$ to an electron gas, the electrons move to screen the field. The effects of the screening are described by a time and space dependent longitudinal dielectric constant $\epsilon^L(\mathbf{q}, \omega)$, see Eq. (69). As we showed, see also Refs. [3, 4], the dielectric constant can be related to the screened density response of the system by Eq. (67), and to the unscreened density response $\chi(\mathbf{q}, \omega)$ by Eq. (73).

To compute the density response of the system to an external potential V_{ext} we first note that potential enters the Hamiltonian in the form

$$\int (-e)V_{\text{ext}}(\mathbf{r}, t)\hat{n}(\mathbf{r})\, d\mathbf{r} = \int (-e)V_{\text{ext}}(\mathbf{q}, t)\hat{n}^\dagger(\mathbf{q})\, d\mathbf{q}, \tag{110}$$

where $\hat{n}(\mathbf{r})$ is the number operator for particles at $\mathbf{r}$, see Appendix B. We see that the (number) density (at wave vector $\mathbf{q}$) couples to $(-e)V_{\text{ext}}(\mathbf{q})$. We then want the response of the charge density $\rho^\dagger(\mathbf{q}) = (-e)\hat{n}^\dagger(\mathbf{q})$ to this perturbation. According the theory of linear response, see *e.g.* Ref.[4, 5] and Eq. (C10) of Appendix C, we have $\langle \rho_{\text{int}}(\mathbf{q}, \omega)\rangle = (-e)\ \chi(\mathbf{q}, \omega)\ (-e)V_{\text{ext}}(\mathbf{q}, \omega)$, Eq. (72), with

$$\chi(\mathbf{q}, \omega) = \sum_{n,m} P_n |\langle m|\hat{n}^\dagger_{\mathbf{q}}|n\rangle|^2 \left[\frac{1}{E_n - E_m + \omega + i\eta} - \frac{1}{E_m - E_n + \omega + i\eta}\right], \tag{111}$$

where

$$\hat{n}^\dagger_{\mathbf{q}} = \sum_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}+\mathbf{q}\,\sigma} c_{\mathbf{k}\sigma} \tag{112}$$

is the (number) density creation operator at wavevector $\mathbf{q}$, and η is a small positive number representing a small imaginary part of the frequency. When the real part of the denominator of Eq. (111) vanishes, $\chi(\mathbf{q}, \omega)$, and hence $\epsilon^L(\mathbf{q}, \omega)$, picks up an imaginary part giving rise to damping and the the sign of η is necessary to get the sign of the damping term correct (*i.e.* to make sure that one has damping rather than exponential growth of fluctuations). We write the real and imaginary parts of $\chi(\mathbf{q}, \omega)$ as $\chi'(\mathbf{q}, \omega)$ and $\chi''(\mathbf{q}, \omega)$ respectively, *i.e.*

$$\chi(\mathbf{q}, \omega) = \chi'(\mathbf{q}, \omega) + i\chi''(\mathbf{q}, \omega)\,. \tag{113}$$

In Eq. (111) the states $|n\rangle$ are exact eigenstates of the many-body system, and these are occupied with Boltzmann probabilities $P_n = \exp(-\beta E_n)/Z$ with $Z = \sum_m \exp(-\beta E_m)$.

Physically, $\chi(\mathbf{q}, \omega)$ represents the change in density in response to an addition time and space-dependent potential. In systems with Coulomb interactions, the effects of *screening*, in which the *effective* interaction between two charges is reduced by the motion of the other charges, is very important. For this reason it is often useful to relate the dielectric constant to change in density in response to a change in the screened (*i.e.* total) potential (comprising the external potential and the change in the potential due to the other electrons). In this case, as discussed in the previous section and Refs. [3, 4], Eq. (73) is replaced by Eq. (67), where the screened density response $\chi_{\rm sc}(\mathbf{q}, \omega)$ is related to the unscreened response by Eq. (74). One can write a general expression for $\chi_{\rm sc}(\mathbf{q}, \omega)$, similar to Eq. (111), except that the effects of screening are *not* to be included when evaluating the matrix elements[4] otherwise these effects would be double counted.[2]

The difference between the screened and unscreened responses can be conveniently visualized by diagrammatic perturbation theory[4, 6]. This is not the place to go into this big subject in detail, but if you have already had some exposure to it, the following comments may be useful. We represent the propagation of electrons by solid lines with arrows, and the Coulomb potential $V(\mathbf{q}) = 4\pi e^2/q^2$ by a dashed line. Diagrams for the density-density correlation function $\chi(\mathbf{q}, \omega)$ have two "sources" where an electron-hole pair (density fluctuation) is created or disappears. These are represented by the small circles. An important theorem is that all diagrams must be connected. Some examples are shown in Fig. 1.

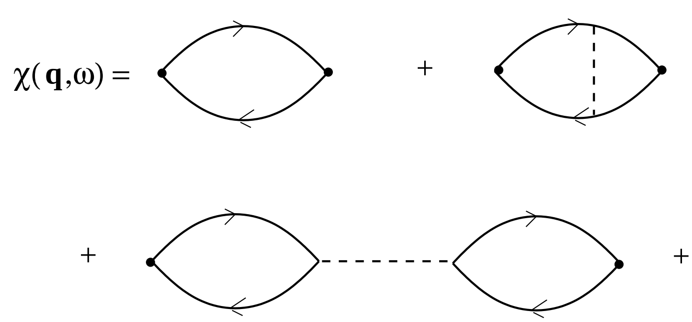

FIG. 1: Some diagrams for $\chi(\mathbf{q}, \omega)$. The small circles indicate where the density fluctuation (particle-hole pair) is created and destroyed. The solid lines represent propagation of electrons and the dashed line is the Coulomb interaction $V(q) = 4\pi e^2/q^2$.

[2] To be precise[4], the matrix element $\langle m|\hat{n}(\mathbf{q})|m\rangle$ is replaced by $\langle m|\hat{n}(\mathbf{q})|m\rangle^{(1)}$ where $\langle m|\hat{n}(\mathbf{q})|m\rangle^{(1)} = \langle m|\hat{n}(\mathbf{q})|m\rangle \epsilon(\mathbf{q}, \omega_{mn})$, where $\omega_{mn} = E_m - E_n$.

The uncsreened density response is the sum of all such digrams (according to some rules which we are not going to describe here). A nice feature of the diagrammatic method is that the screened response turns out to be just the sum of all diagrams which *cannot be split into separate pieces by cutting one interaction (dashed) line.* If we denote the sum of all such diagrams by a shaded "blob", denoted by $\chi_b(\mathbf{q},\omega)$, then the diagrams for $\chi(\mathbf{q},\omega)$ are given by those in Fig. 2.

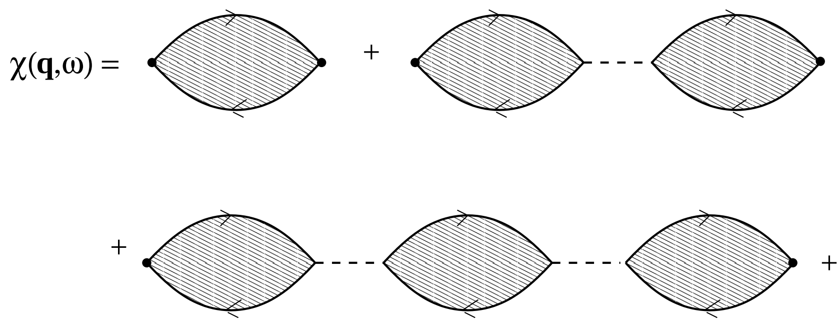

FIG. 2: Representing the sum of all diagrams in terms of the the sum of all diagrams which cannot be split in two by cutting an interaction line. The latter is represented by the shaded "blob". The dashed line is $V(\mathbf{q}) = 4\pi e^2/q^2$, so the contributions of all diagrams form the geometric series in Eq. (114). This shows that the "blob" is the screened density response $\chi_{\rm sc}(\mathbf{q},\omega)$.

$$\begin{aligned}\chi(\mathbf{q},\omega) &= \chi_{\rm b}(\mathbf{q},\omega) + \chi_{\rm b}(\mathbf{q},\omega)V(q)\chi_{\rm b}(\mathbf{q},\omega) + \chi_{\rm b}(\mathbf{q},\omega)V(q)\chi_{\rm b}(\mathbf{q},\omega)V(q)\chi_{\rm b}(\mathbf{q},\omega) + \cdots \\ &= \frac{\chi_{\rm b}(\mathbf{q},\omega)}{1 - V(q)\chi_{\rm b}(\mathbf{q},\omega)}\,. \end{aligned} \tag{114}$$

Comparing with Eq. (74) we see that $\chi_{\rm b}(\mathbf{q},\omega) = \chi_{\rm sc}(\mathbf{q},\omega)$.

Some diagrams for $\chi_{\rm sc}(\mathbf{q},\omega)$ are shown in Fig. 3.

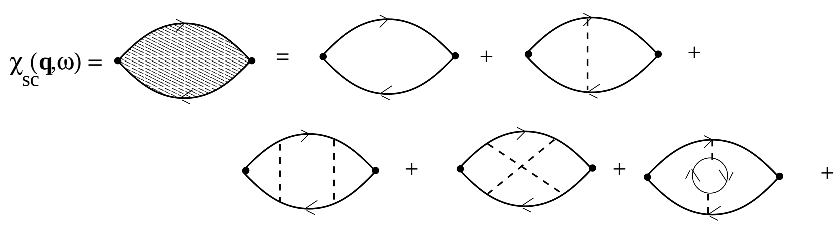

FIG. 3: Some diagrams for $\chi_{\rm sc}(\mathbf{q},\omega)$.

To leading order, one just takes the first diagram, which corresponds to non-interacting electrons,*i.e.* $\chi_{\rm sc}(\mathbf{q},\omega) = \chi_0(\mathbf{q},\omega)$. This approximation, known as the "Random Phase Approximation" (RPA) is quite successful, and is all that we shall consider here. Physically, the RPA assumes that the most important effect of the interactions is screening. Once we have included that, we can represent the system by non-interacting electrons.

From Eq. (111), evaluated for free electrons, and Eq. (112), we find

$$\boxed{\chi_0(\mathbf{q},\omega) = 2\sum_{\mathbf{k}} f_{\mathbf{k}}(1-f_{\mathbf{k+q}}) \left[\frac{1}{\omega - \epsilon_{\mathbf{k+q}} + \epsilon_{\mathbf{k}} + i\eta} - \frac{1}{\omega + \epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k}} + i\eta}\right]} \tag{115}$$

where $\epsilon_{\mathbf{k}} = k^2/(2m)$ is the energy of an electron in state $\mathbf{k}$, and

$$f_{\mathbf{k}} = \frac{1}{\exp(\beta[\epsilon - \mu]) + 1} \tag{116}$$

is the occupancy of this single-particle state, *i.e.* the Fermi-Dirac distribution. At $T = 0$, $f_{\mathbf{k}}$ is 1 for $k < k_F$ and 0 for $k > k_F$, where k_F is the Fermi wavevector. The overall factor of 2 in Eq. (115) comes from a sum over spin. Eq. (115) describes processes in which a particle in state $\mathbf{k}$ (which is occupied with probability $f_{\mathbf{k}}$) is scattered into state $\mathbf{k}+\mathbf{q}$ (which is empty with probability $1 - f_{\mathbf{k+q}}$).

Eq. (115) can be written in different ways which turn out to be convenient. For example, if we make the replacement $\mathbf{k} \to -(\mathbf{k}+\mathbf{q})$ (so $\mathbf{k}+\mathbf{q} \to -\mathbf{k}$) in the second term in Eq. (115) we have

$$\begin{aligned} \chi_0(\mathbf{q},\omega) &= 2\sum_{\mathbf{k}} \{f_{\mathbf{k}}(1-f_{\mathbf{k+q}}) - f_{\mathbf{k+q}}(1-f_{\mathbf{k}})\} \frac{1}{\omega - \epsilon_{\mathbf{k+q}} + \epsilon_{\mathbf{k}} + i\eta} \\ &= \boxed{2\sum_{\mathbf{k}} \frac{f_{\mathbf{k}} - f_{\mathbf{k+q}}}{\omega - \epsilon_{\mathbf{k+q}} + \epsilon_{\mathbf{k}} + i\eta}\,.} \end{aligned} \tag{117}$$

In the last expression if we write separately the $f_{\mathbf{k}}$ and $f_{\mathbf{k+q}}$ terms and make the substitution $\mathbf{k} \to -(\mathbf{k}+\mathbf{q})$ in the latter, we have

$$\chi_0(\mathbf{q},\omega) = 2\sum_{\mathbf{k}} f_{\mathbf{k}} \left\{\frac{1}{\omega - \epsilon_{\mathbf{k+q}} + \epsilon_{\mathbf{k}} + i\eta} - \frac{1}{\omega + \epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k}} + i\eta}\right\}. \tag{118}$$

Interestingly this last expression is just the same as Eq. (115) without the factor of $1 - f_{\mathbf{k+q}}$ (which incorporates the fact that an electron cannot scatter into a state if there is one already there, *i.e.* the exclusion principle). It is curious that this factor does not affect the final answer.

Eq. (118) has the advantage that, at $T = 0$, the only situation considered from now on, the region of integration is simply $k < k_F$, *i.e.*

$$\boxed{\chi_0(\mathbf{q},\omega) = 2\sum_{k<k_F} \left\{\frac{1}{\omega - \epsilon_{\mathbf{k+q}} + \epsilon_{\mathbf{k}} + i\eta} - \frac{1}{\omega + \epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k}} + i\eta}\right\}.} \tag{119}$$

We will now evaluate $\chi_0(\mathbf{q},\omega)$ in several special limits and then comment qualitatively on the general case.

B. The limit $\omega \ll v_F q$

We note that if $\mathbf{q}$ is small

$$\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} \simeq v_F \, \hat{\mathbf{k}} \cdot \mathbf{q} \,, \tag{120}$$

where $v_F = k_F/m$ is the Fermi velocity. Hence, in the limit $\omega \ll v_F q$ we can neglect ω in Eq. (119), and get

$$\begin{aligned} \chi_0(\mathbf{q}, 0) &= 2 \sum_{k<k_F} \left\{ \frac{1}{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}} + i\eta} + \frac{1}{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}} - i\eta} \right\} \\ &= 4\mathcal{P} \sum_{k<k_F} \frac{1}{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}} \,, \end{aligned} \tag{121}$$

where $\mathcal{P}$ denotes the principal part. We see that the result is real.

We will first evaluate $\chi_0(\mathbf{q}, 0)$ for $\mathbf{q} \to 0$. In fact this is easiest starting from Eq. (117) since

$$\chi_0(\mathbf{q} \to 0, 0) = 2 \sum_{\mathbf{k}} \frac{f_{\mathbf{k}} - f_{\mathbf{k}+\mathbf{q}}}{\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}} = \int_0^\infty \rho(\epsilon) \left\{ \frac{\partial n(\epsilon)}{\partial \epsilon} \delta\epsilon_{\mathbf{k},\mathbf{q}} \right\} \frac{1}{\delta\epsilon_{\mathbf{k},\mathbf{q}}} \, d\epsilon = \boxed{-\rho(\epsilon_F)} \,, \tag{122}$$

where $\delta\epsilon_{\mathbf{k},\mathbf{q}} = \epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}$. Here we have replaced the sum over $\mathbf{k}$ by an integral over energy with $2 \sum_{\mathbf{k}} \to \int \rho(\epsilon) \, d\epsilon$, where $\rho(\epsilon)$ is the density of one particle levels. We have also used that for $T \to 0$, $\partial n(\epsilon)/\partial(\epsilon) = -\delta(\epsilon - \epsilon_F)$. (Strictly speaking this derivation takes the limit $\mathbf{q} \to 0$ before $T \to 0$. However, the order of these limits does not matter, as we shall see below by direct evaluation of Eq. (121)).

From Eqs. (67) and (121) we see that for $\mathbf{q} \to 0$

$$\boxed{\epsilon(\mathbf{q}, 0) = 1 + \frac{\kappa^2}{q^2} \,,} \tag{123}$$

where

$$\boxed{\kappa^2 = 4\pi e^2 \rho(\epsilon_F) = \frac{6\pi n e^2}{\epsilon_F} \,,} \tag{124}$$

in which we used that the density of states at the Fermi energy is given by

$$\rho(\epsilon_F) = \frac{3n}{2\epsilon_F} \,, \tag{125}$$

where n is the particle density. Eq. (124) is due to Thomas and Fermi, and κ is the Thomas-Fermi inverse screening length. If we apply a static external potential due to a point charge of strength Ze, *i.e.* $V_{\text{ext}}(\mathbf{q}) = 4\pi Ze/q^2$, it follows from Eqs. (70) and (123) that the screened potential is given by

$$V_{\text{sc}}(\mathbf{q}) = \frac{4\pi Ze}{\kappa^2 + q^2} \,. \tag{126}$$

Fourier transforming gives

$$V_{\rm sc}(\mathbf{r}) = \frac{Ze}{r} e^{-\kappa r} \,, \tag{127}$$

showing that the Coulomb interaction is screened at distances greater than κ^{-1}.

Next we evaluate $\chi_0(\mathbf{q}, 0)$ for arbitrary $\mathbf{q}$ from Eq. (121). Transforming to spherical polars gives

$$\begin{aligned} \chi_0(\mathbf{q}, 0) &= -4 \frac{2m}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{k_F} k^2 dk \, \mathcal{P} \int_0^{\pi} \sin\theta \, d\theta \frac{1}{q^2 + 2qk\cos\theta} & (128) \\ &= -\frac{2m}{\pi^2} \int_0^{k_F} k^2 dk \, \mathcal{P} \int_{-1}^{1} \frac{du}{q^2 + 2kqu} & (129) \\ &= -\frac{m}{\pi^2 q} \int_0^{k_F} k \log \left[\left| \frac{q+2k}{q-2k} \right| \right] dk \,. & (130) \end{aligned}$$

Evaluating the integral over k gives

$$\chi_0(\mathbf{q}, 0) = -\frac{m}{8\pi^2 q} \left\{ 4k_F q + \left[(2k_F)^2 - q^2 \right] \log \left[\left| \frac{q+2k_F}{q-2k_F} \right| \right] \right\} . \tag{131}$$

Now the density of states at the Fermi surface can be expressed as

$$\rho(\epsilon_F) = \frac{m}{\pi^2} k_F \,, \tag{132}$$

as shown in any treatment of free electrons. Hence we can write Eq. (131) as

$$\chi_0(\mathbf{q}, 0) = -\frac{\rho(\epsilon_F)}{2} \frac{1}{4qk_F} \left\{ 4k_F q + \left[(2k_F)^2 - q^2 \right] \log \left[\left| \frac{q+2k_F}{q-2k_F} \right| \right] \right\} \tag{133}$$

i.e.

$$\boxed{\chi_0(\mathbf{q}, 0) = -\rho(\epsilon_F) F(x)} \tag{134}$$

where

$$\boxed{F(x) = \frac{1}{2} + \frac{1}{4x} \left(1 - x^2 \right) \log \left[\left| \frac{1+x}{1-x} \right| \right] \,,} \tag{135}$$

with

$$x = \frac{q}{2k_F} \,. \tag{136}$$

For $x \to 0, F(x)$ has the expansion

$$F(x) = 1 - \frac{x^2}{3} + O(x^4) \,, \tag{137}$$

and at $x = 1, F(x)$ has an infinite slope due to the singularity in the log. In the limit $x \to \infty$ one finds that $F(x) \to 0$. A plot of $F(x)$ is shown in Fig. 4

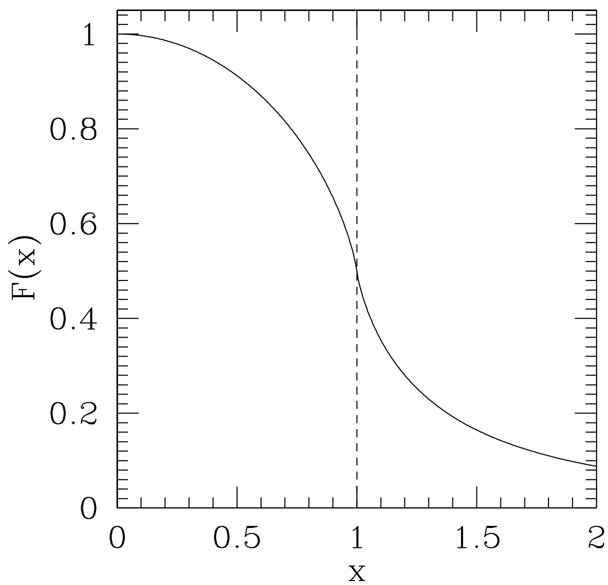

FIG. 4: A sketch of the function $F(x)$, given by Eq. (137), which determines the static limit of the dielectric constant in the RPA. The curve has vertical slope at $x = 1$ because of the logarithmic singularity in Eq. (137).

From Eqs. (67), (124) and (134) the longitudinal static dielectric constant is given, in the RPA, by

$$\epsilon^L(\mathbf{q}, 0) = 1 + \frac{\kappa^2}{q^2} F\left(\frac{q}{2k_F}\right) . \tag{138}$$

Using this equation, and Eq. (70) with $\omega = 0$, to determine the static screened Coulomb interaction, the logarithmic singularity in $F(x)$ turns out to control the long distance part of $V_{\rm sc}(\mathbf{r})$. This does not actually tend to zero exponentially as predicted by Eq. (127) (which only considered the small-$\mathbf{q}$ part of $\epsilon^L(\mathbf{q}, \omega)$) but rather decays with a power of r and oscillates: $V_{\rm sc}(\mathbf{r}) \propto \cos(2k_F r)/r^3$ for $r \to \infty$. These oscillations are known Friedel oscillations.

It is often convenient to express the response of the electron gas in terms of the conductivity $\sigma^L(\mathbf{q}, \omega)$ rather than or $\epsilon^L(\mathbf{q}, \omega)$. The connection between $\sigma^L(\mathbf{q}, \omega)$ and $\epsilon^L(\mathbf{q}, \omega)$ is given in Eq. (81). If we take $\omega \to 0$ and then $\mathbf{q} \to 0$ (the limits we have been taking in this section) we find from Eqs. (122) and (80) that, in the RPA,

$$\boxed{\lim_{\mathbf{q}\to 0} \lim_{\omega^L \to 0} \sigma(\mathbf{q}, \omega) = -\frac{i\kappa^2\omega}{4\pi q^2}} \, , \tag{139}$$

where κ, given by Eq. (124), is the Thomas-Fermi inverse screening length. Hence the conductivity is imaginary (*i.e.* non-dissipative) and vanishes for $\omega \to 0$.

C. The limit $\omega \gg v_F q$

To evaluate $\chi_0(\mathbf{q}, \omega)$ in this limit we start with Eq. (119) which we write as

$$\chi_0(\mathbf{q}, \omega) = 4 \sum_{k<k_F} \frac{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}}{(\omega + i\eta)^2 - (\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}})^2} \,. \tag{140}$$

For $\omega \gg v_F q$ we can neglect the $(\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}})^2$ factor in the denominator, to obtain

$$\chi_0(\mathbf{q} \to 0, \omega) = \frac{4}{\omega^2} \frac{1}{2m} \sum_{k<k_F} \left(q^2 + 2\mathbf{q} \cdot \mathbf{k}\right) \,. \tag{141}$$

The term involving $\mathbf{q} \cdot \mathbf{k}$ gives zero when averaged over the direction of $\mathbf{k}$, and since q^2 is just a constant it can be taken outside the sum. This just leaves $2 \sum_{k<k_F}$ which simply counts the states (in a unit volume) and so gives n, the particle density including both spin species. Hence we have

$$\chi_0(\mathbf{q} \to 0, \omega) = \frac{q^2 n}{m\omega^2} \,. \tag{142}$$

Substituting into Eq. (67) gives

$$\boxed{\epsilon^L(0, \omega) = 1 - \frac{\omega_p^2}{\omega^2} \,,} \tag{143}$$

where ω_p, called the plasmon frequency, is given by

$$\boxed{\omega_p^2 = \frac{4\pi n e^2}{m} \,.} \tag{144}$$

We discussed earlier that longitudinal excitations occur when $\epsilon^L(\mathbf{q}, \omega) = 0$, and Eq. (143) shows that this happens for $\omega = \omega_p$ at $\mathbf{q} = 0$. Since we are considering the density response of the system, this "plasmon mode" must be a longitudinal density fluctuation. Normally, density fluctuations give longitudinal sound waves whose frequency is proportional to q. However, here the long-range Coulomb interaction gives these modes a finite frequency as $q \to 0$.[3]

The same result for ω_p can be obtained classically by considering the $\mathbf{q} = 0$ oscillations of the negative charge density of the electron gas relative to the (assumed) uniform positive background, see Fig. 5. The electric field is 4π times the surface charge density nx where x is the displacement. Hence the force on an electron is $-4\pi n e^2 x$ (the minus sign because the force is opposite to the direction of x). Hence we obtain simple harmonic motion at frequency ω_p.

[3] The mechanism by which long-range interactions give excitations whose frequency normally vanishes as $q \to 0$ (so-called "Goldstone modes") a *finite* value for $q \to 0$ is called, by our particle-physics colleagues, the "Higgs mechanism".

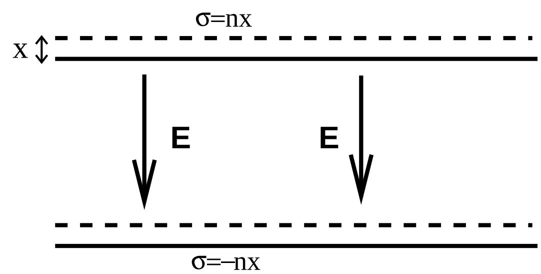

FIG. 5: The displacement of the negative charges relative to the positive charges, gives an electric field which sets up simple harmonic motion at the plasmon frequency. Hence the same plasmon frequency, Eq. (144), is obtained classically.

We have derived Eq. (143) only within the RPA, but we show in Appendix F, see Eq. (F2), that, due to a "sum-rule", $\epsilon^L(\mathbf{q},\omega)$ is given *precisely* by $1-\omega_p^2/\omega^2$ for $\omega \to \infty$. Furthermore, as discussed in Pines and Noziéres[4], Eq. (143) is exactly true at $\mathbf{q}=0$ for *all* ω, and hence the plasmon frequency is given *exactly* by Eq. (144) at $\mathbf{q}=0$.

If we convert Eq. (143) into an expression for the conductivity, using Eq. (81), we find the conductivity to be entirely imaginary (but see the discussion below). Denoting the real and imaginary parts of the conductivity by σ_1 and σ_2 respectively[4], *i.e.*

$$\sigma(\mathbf{q},\omega) = \sigma_1(\mathbf{q},\omega) + i\sigma_2(\mathbf{q},\omega), \tag{145}$$

we have

$$\sigma_2^L(0,\omega) = \frac{ne^2}{m\omega}\,, \tag{146}$$

In fact, the conductivity cannot be entirely imaginary for all ω because Kramers-Kronig relations connect the real and imaginary parts, see Eqs. (D3) and (D4). for example, according to Eq. (D4), the imaginary part is given in terms of the real part by

$$\sigma_2^L(0,\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\sigma_1^L(0,\omega')}{\omega-\omega'}\,d\omega'\,. \tag{147}$$

Since $\sigma_2^L(0,\omega)$ is given by Eq. (146), we must have

$$\sigma_1^L(0,\omega) = \frac{ne^2\pi}{m}\,\delta(\omega)\,. \tag{148}$$

The delta function at $\omega = 0$ means that we have a *perfect conductor*. This unphysical result occurs because the electrons do not scatter in the RPA. In practice, collisions between electrons at finite-T, and scattering off impurities even at $T=0$, would give a finite dc conductivity.

---

[4] It is conventional to use one prime to denote the real part of a linear response function and two primes to denote the imaginary part, see *e.g.* Eq. (113), but, for some reason, to use subscripts "1" and "2" for the same purpose when dealing with the conductivity. We follow standard usage here.

We can put in scattering phenomenologically by introducing a relaxation time τ into the conductivity as follows:

$$\boxed{\sigma^L(0,\omega) = \frac{ne^2\tau}{m}\frac{1}{1-i\omega\tau}\,,} \tag{149}$$

which reproduces Eqs. (146) and (148) for $\tau \to \infty$. Note that

$$\lim_{\tau\to\infty}\frac{1}{\tau^{-1}-i\omega} = \lim_{\tau\to\infty}\frac{i}{\omega+i\tau^{-1}} = \mathcal{P}\left(\frac{i}{\omega}\right) + \pi\delta(\omega)\,. \tag{150}$$

Eq. (149) is the result of the Drude theory of electrical conductivity. The real part of σ^L has a peak centered at $\omega = 0$ and of width τ^{-1}, known as the "Drude peak".

From Eq. (80) the corresponding expression for $\epsilon^L(0,\omega)$ is

$$\epsilon^L(0,\omega) = 1 - \frac{\omega_p^2}{\omega(\omega+i/\tau)}\,. \tag{151}$$

Note that from Eq. (148) or (149) we have

$$\boxed{\int_0^\infty \sigma_1^L(0,\omega)\,d\omega = \frac{ne^2}{2m} = \frac{\omega_p^2}{8}\,,} \tag{152}$$

where ω_p is given by Eq. (144). (Using Eq. (148) we only get half the contribution from the delta function because the integral starts at 0.) Eq. (152) is true in general, not just in the RPA, as shown in Appendices E, and F. It is an important "sum-rule" which is very helpful in analyzing experimental data for the conductivity obtained, typically, from reflectivity measurements. Actually, reflectivity involves the transverse, not the longitudinal, response but these are equal at $\mathbf{q} = 0$ since there is no way to distinguish between longitudinal and transverse in this limit. (Our explicit calculations confirm this.) In optical measurements, $\mathbf{q}$ is not exactly zero, but the speed of light is so large that $\mathbf{q}$ is very close to zero and so the difference between the longitudinal and transverse responses is negligible.

We emphasize that we have found very different results for $\sigma^L(\mathbf{q},\omega)$ in the long-wavelength, low-frequency region, depending on the order in which the limits $\omega \to 0, \mathbf{q} \to 0$ are taken. If we let $\omega \to 0$ first, we see in Eq. (139) that the conductivity is imaginary (*i.e.* non-dissipative) and vanishes for $\omega \to 0$. This is because we set up a long-wavelength static potential, in which the electrons come to a new equilibrium with no current flow. By contrast, if we take $\mathbf{q} \to 0$ first we set up a potential which is uniform in space and oscillates slowly with frequency, which gives rise to a current. In other words, to get the dc conductivity, we have to let $\mathbf{q} \to 0$ first:

$$\boxed{\sigma_{\rm dc} = \lim_{\omega\to 0}\lim_{\mathbf{q}\to 0}\sigma^L(\mathbf{q},\omega)\,.} \tag{153}$$

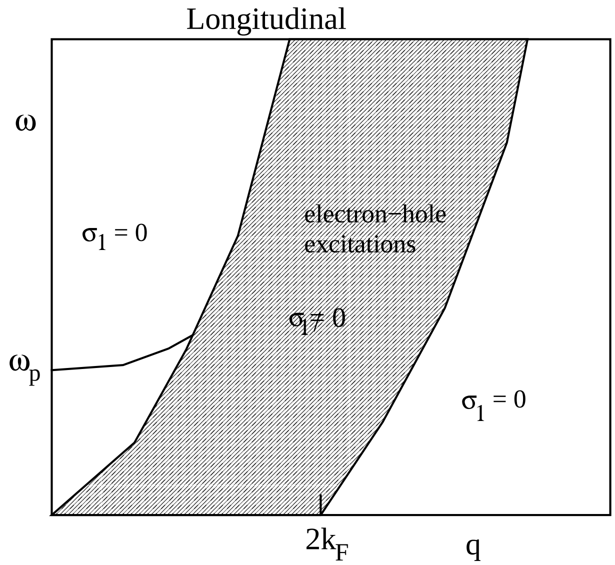

FIG. 6: The longitudinal excitation spectrum of the electron gas in the RPA. Excitations occur where $\epsilon^L(\mathbf{q}, \omega) = 0$. The shaded region is where electron-hole excitations can be created. The upper boundary of it is given by $\omega = (1/2m)(q^2 + 2qk_F)$ and the lower boundary by $\omega = (1/2m)(q^2 - 2qk_F)$ or 0, whichever is greater. There are also collective excitations known as plasmons. The plasmon dispersion relation is also sketched. In the shaded region $\epsilon_2(\mathbf{q}, \omega)$, and hence, because of Eq. (81), $\sigma_1(\mathbf{q}, \omega)$, are non zero.

D. The general case

Results for $\chi_0(\mathbf{q}, \omega)$ and the corresponding expressions for $\epsilon^L(\mathbf{q}, \omega)$ and $\sigma^L(\mathbf{q}, \omega)$ for arbitrary $\mathbf{q}$ and ω are given in Refs. [3] and [4]. We will not give these rather complicated expressions here. The main new feature, beyond what we have seen so far, is the appearance of an imaginary part in $\epsilon^L(\mathbf{q}, \omega)$ (real part of $\sigma^L(\mathbf{q}, \omega)$) when the denominator of Eq. (115) vanishes, *i.e.* when

$$\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} = \pm\omega \, . \tag{154}$$

When this happens a density fluctuation can decay into a particle-hole pair. The region where this can occur are

$$\omega \; < \; \frac{1}{2m}(q^2 + 2qk_F) \tag{155}$$

$$\omega \; > \; \frac{1}{2m}\begin{cases} 0 & (q < 2k_F) \\ (q^2 - 2qk_F) & (q > 2k_F) \end{cases} \tag{156}$$

This region is shown in Fig. 6.

In addition there is an elementary excitation at the the plasmon frequency, $\omega_p(q)$. The condition for this is $\epsilon^L(\mathbf{q}, \omega) = 0$, see Eq. (71). A plasmon involves a collective excitation of the whole electron gas. For $q = 0$, ω_p is given by Eq. (144) and at small q this is modified to

$$\omega_p(\mathbf{q}) \simeq \omega_p \left\{ 1 + \frac{3}{10} \left(\frac{q v_F}{\omega_p} \right)^2 + \cdots \right\} . \tag{157}$$

The plasmon dispersion relation is also sketched in Fig. 6.

E. Relation to the current response and gauge invariance

In the previous section we have described the longitudinal response of the electron gas in terms of the response of the *density* to an external *scalar* potential. Alternatively, we could have looked at the response of the longitudinal *current* to a *vector* potential. Since the density and longitudinal currents are related by the continuity equation, Eq. (47), the two formulations are equivalent, and response functions for density and longitudinal current are related by the simple expression in Eq. (88). However, we shall find it useful to do the current formulation here for the longitudinal case, because when, in the next section, we do the transverse response (which does not couple to the density and so there one can *only* consider the current response) it will be helpful to compare with analogous expressions for the longitudinal case.

If we replace the external scalar potential in Eq. (110) by an external longitudinal vector potential, the Hamiltonian is

$$\mathcal{H}_{\mathbf{A}} = \frac{1}{2m} \sum_i \Big(\mathbf{p}_i + \frac{e}{c} \mathbf{A}_{\text{ext}}(\mathbf{r}_i, t) \Big)^2 + \sum_{i<j} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|} . \tag{158}$$

We can write this as

$$\mathcal{H}_{\mathbf{A}} = \mathcal{H} + \frac{e}{c} \int \mathbf{j}_p(\mathbf{r}) \cdot \mathbf{A}_{\text{ext}}(\mathbf{r}, t)\, d\mathbf{r} + \frac{e^2}{2mc^2} \sum_i \mathbf{A}_{\text{ext}}(\mathbf{r}_i, t)^2 , \tag{159}$$

where

$$\mathbf{j}_p(\mathbf{r}) = \frac{1}{2m} \sum_i \left[\mathbf{p}_i \delta(\mathbf{r} - \mathbf{r}_i) + \delta(\mathbf{r} - \mathbf{r}_i) \mathbf{p}_i \right] , \tag{160}$$

is the *paramagnetic* current density, see Appendix B. The *total* current is given, according to Appendix B, by

$$\mathbf{j}(\mathbf{r}) = \mathbf{j}_p(\mathbf{r}) + \frac{e}{mc} \hat{n}(\mathbf{r}) \mathbf{A}_{\text{ext}}(\mathbf{r}, t), \tag{161}$$

where the second term on the RHS is called the *diamagnetic* current. The paramagnetic current $\mathbf{j}_p$ is useful because it is *independent of* $\mathbf{A}_{\text{ext}}$. However it is the *total* current $\mathbf{j}$ (paramagnetic plus diamagnetic) which enters in the continuity equation, Eq. (85), and which is gauge invariant.[5]

[5] Gauge invariance means that one performs the following transformations which leave $\mathbf{E}, \mathbf{B}$, and the energy levels unchanged: $V \to V - c^{-1} \partial \chi / \partial t, \mathbf{A} \to= \mathbf{A} + \boldsymbol{\nabla} \chi$ and $|n\rangle \to \exp[-ie\chi(\mathbf{r}, t)/c]\, |n\rangle$.

We will be interested in the linear response of the total current in response to a magnetic vector potential, Fourier transformed to $\mathbf{q}$ and ω:

$$\langle \mathbf{j}_p^\mu(\mathbf{q},\omega)\rangle = \frac{e}{c}\chi_{j_p}^{\mu\nu}(\mathbf{q},\omega)\mathbf{A}_{\rm ext}^\nu(\mathbf{q},\omega)\,, \tag{162}$$

where μ and ν are cartesian indices. For an isotropic medium (assumed here) $\chi_{j_p}^{\mu\nu}$ has just two distinct elements, the longitudinal and transverse parts, *i.e.*

$$\boxed{\langle \mathbf{j}_p^L(\mathbf{q},\omega)\rangle = \frac{e}{c}\chi_{j_p}^{L}(\mathbf{q},\omega)\mathbf{A}_{\rm ext}^L(\mathbf{q},\omega),} \tag{163}$$

$$\boxed{\langle \mathbf{j}_p^T(\mathbf{q},\omega)\rangle = \frac{e}{c}\chi_{j_p}^{T}(\mathbf{q},\omega)\mathbf{A}_{\rm ext}^T(\mathbf{q},\omega)\,.} \tag{164}$$

From Eqs. (159) and (163), and the discussion of linear response in Appendix C, we find that the longitudinal response function of the paramagnetic current is given by

$$\boxed{\chi_{j_p}^L(\mathbf{q},\omega) = \sum_n P_n \sum_m |\langle m|j_p^L(\mathbf{q})|n\rangle|^2\left[\frac{1}{E_n-E_m+\omega+i\eta} - \frac{1}{E_m-E_n+\omega+i\eta}\right]\,,} \tag{165}$$

where $j_p^L(\mathbf{q})$ is the component of $\mathbf{j}_p(\mathbf{q})$ along the direction of $\mathbf{q}$.

In addition we need to include diamagnetic current, the second term on the RHS of Eq. (161). To linear order in $\mathbf{A}_{\rm ext}(\mathbf{q},\omega)$ we just take the average of the density operator $\langle\hat{n}(\mathbf{r})\rangle$, which is equal to n, the mean density. Hence, the response of the total (longitudinal) current to a longitudinal vector potential is

$$\langle \mathbf{j}^L(\mathbf{q},\omega)\rangle = \frac{e}{c}\chi_{j}^{L}(\mathbf{q},\omega)\mathbf{A}_{\rm ext}^L(\mathbf{q},\omega)\,, \tag{166}$$

where

$$\boxed{\chi_j^L(\mathbf{q},\omega) = \frac{n}{m} + \chi_{j_p}^L(\mathbf{q},\omega)\,.} \tag{167}$$

The $\omega = 0$ part of $\chi_{j_p}^L(\mathbf{q},\omega)$ is equal to

$$\chi_{j_p}^L(\mathbf{q},0) = 2\sum_n P_n \sum_m \frac{|\langle m|j_p^L(\mathbf{q})|n\rangle|^2}{E_n-E_m}\,, \tag{168}$$

which, from Eq. (B22), can be written as

$$\boxed{\chi_{j_p}^L(\mathbf{q},0) = -\frac{n}{m}\,.} \tag{169}$$

where n is the electron density, so

$$\chi_j^L(\mathbf{q},0) = 0 \tag{170}$$

for *all* $\mathbf{q}$, as noted earlier in Eq. (89).

Hence, from Eqs. (167) and (169),

$$
\begin{aligned}
\chi_j^L(\mathbf{q},\omega) &= \chi_{j_p}^L(\mathbf{q},\omega) - \chi_{j_p}^L(\mathbf{q},0) && (171)\\
&= \sum_n P_n \sum_m |\langle m|j_p^L(\mathbf{q})|n\rangle|^2 \left[\frac{1}{E_n - E_m + \omega + i\eta} - \frac{1}{E_n - E_m} - \frac{1}{E_m - E_n + \omega + i\eta} + \frac{1}{E_m - E_n}\right] && (172)\\
&= \sum_n P_n \sum_m |\langle m|j_p^L(\mathbf{q})|n\rangle|^2 \frac{\omega^2}{(E_m - E_n)^2}\, \frac{2(E_m - E_n)}{(\omega + i\eta)^2 - (E_m - E_n)^2}\,, && (173)
\end{aligned}
$$

From Eq. (B17) this becomes

$$
\begin{aligned}
\chi_j^L(\mathbf{q},\omega) &= \frac{\omega}{q^2} \sum_n P_n \sum_m |\langle m|\hat{n}(\mathbf{q})|n\rangle|^2 \frac{2(E_m - E_n)}{(\omega + i\eta)^2 - (E_m - E_n)^2} && (174)\\
&= \frac{\omega^2}{q^2} \sum_n P_n \sum_m |\langle m|\hat{n}(\mathbf{q})|n\rangle|^2 \left[\frac{1}{E_n - E_m + \omega + i\eta} - \frac{1}{E_m - E_n + \omega + i\eta}\right], && (175)\\
&= \boxed{\frac{\omega^2}{q^2}\chi(\mathbf{q},\omega)\,,} && (176)
\end{aligned}
$$

where the last line follow from Eq. (111). This can also be obtained from the continuity equation, see Eq. (88).

The paramagnetic current response can be calculated diagrammatically, as for the density response. We can take over the diagrams in Fig. 1 except that the small circle, which denoted there matrix elements of the density operator $\hat{n}(\mathbf{q})$, now represent matrix elements of the longitudinal paramagnetic current $\mathbf{j}_p^J(\mathbf{q},\omega)$. If the electron lines meeting at a circle have wavevectors $\mathbf{k}$ and $\mathbf{k}+\mathbf{q}$, then, using the matrix elements for the current operator in Eq. (B13), it follows that the small circles have a factor[6] $k\cos\theta + q/2$, where θ is the angle between $\mathbf{q}$ and $\mathbf{k}$.

As we recall from the earlier parts of this section, it is more convenient to consider the *screened* response, *i.e.* the response to the total field (in this case vector potential) including that produced by the electrons. It is shown by Pines and Nozières[4] p. 256–260, that the screened response functions have very similar properties to the unscreened ones. The screened longitudinal current response is given by

$$
\boxed{\langle \mathbf{j}_p^L(\mathbf{q},\omega)\rangle = \frac{e}{c}\chi_{\mathrm{sc,\,j_p}}^L(\mathbf{q},\omega)\mathbf{A}^L(\mathbf{q},\omega).} \qquad (177)
$$

As for the case of the screened *density* response, one can represent the perturbation expansion for $\mathbf{j}_p^L(\mathbf{q},\omega)$ by a sum of diagrams which cannot be broken by cutting a single interaction line, see

[6] Because the matrix elements of the density operator are those given in Eq. (B4), this factor is unity when calculating the density response.

Fig. 3. The only differences are (i) the small circles have a "matrix element" factor of $k\cos\theta + q/2$ (as for the unscreened response) and (ii) there is an additional diamagnetic contribution of n/m (again as for the unscreened response)[4], so

$$\boxed{\chi^L_{\mathrm{sc,\,j}}(\mathbf{q},\omega) = \frac{n}{m} + \chi^L_{\mathrm{sc,\,j_p}}(\mathbf{q},\omega)\,.} \tag{178}$$

Furthermore[4],

$$\boxed{\chi^L_{\mathrm{sc,\,j_p}}(\mathbf{q},0) = -\frac{n}{m}\,.} \tag{179}$$

and

$$\boxed{\chi^L_{\mathrm{sc,\,j}} = \frac{\omega^2}{q^2}\chi_{\mathrm{sc}}(\mathbf{q},\omega)\,.} \tag{180}$$

The last three equations correspond to Eqs. (167), (169) and (176) for the unscreened case. In the last equation $\chi_{\mathrm{sc}}(\mathbf{q},\omega)$ is the screened *density* response.

As a consequence of these last two equations and Eq. (67), the longitudinal dielectric constant can be written as

$$\boxed{\epsilon^L(\mathbf{q},\omega) = 1 - \frac{4\pi e^2}{\omega^2}\chi^L_{\mathrm{sc,\,j}}(\mathbf{q},\omega) = 1 - \frac{4\pi e^2}{\omega^2}\left[\frac{n}{m} + \chi^L_{\mathrm{sc,\,j_p}}(\mathbf{q},\omega)\right]\,.} \tag{181}$$

Using Eq. (81) one can also relate the conductivity to the current responses:

$$\boxed{\sigma^L(\mathbf{q},\omega) = e^2\,\frac{i}{\omega}\,\chi^L_{\mathrm{sc,\,j}}(\mathbf{q},\omega) = e^2\,\frac{i}{\omega}\left[\frac{n}{m} + \chi^L_{\mathrm{sc,\,j_p}}(\mathbf{q},\omega)\right]\,.} \tag{182}$$

It is instructive to evaluate $\chi^L_{\mathrm{sc,\,j}}(\mathbf{q},\omega)$ in the RPA to check that it it reproduces Eq. (180). In the RPA, $\chi^L_{\mathrm{sc,\,j_p}}(\mathbf{q},\omega)$ is given by $\chi_{0,\,\mathrm{j_p}}(\mathbf{q},\omega)$, the longitudinal current response for non-interacting electrons (just as we evaluated $\chi_{\mathrm{sc}}(\mathbf{q},\omega)$ in the RPA from the density response for non-interacting electrons). The matrix element of the component of $\mathbf{j}_p$ along $\mathbf{q}$ connecting states in which an electron in state $\mathbf{k}$ is destroyed and one in state $\mathbf{k}+\mathbf{q}$ is created, is $m^{-1}(k\cos\theta + q/2)$ according to Eq. (B13), where θ is the angle between $\mathbf{q}$ and $\mathbf{k}$. Using Eq. (165) we get, by comparison with Eq. (119),

$$\chi^L_{0,\,\mathrm{j_p}}(\mathbf{q},\omega) = \frac{2}{m^2}\sum_{k<k_F}\left(k\cos\theta + \frac{q}{2}\right)^2\left\{\frac{1}{\omega - \epsilon_{\mathbf{k}+\mathbf{q}} + \epsilon_{\mathbf{k}} + i\eta} - \frac{1}{\omega + \epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} + i\eta}\right\}\,, \tag{183}$$

where θ is the angle between $\mathbf{q}$ (and $\mathbf{j}$) and $\mathbf{k}$. For $\omega = 0$ this simplifies too

$$\begin{aligned}
\chi^L_{0,\,\mathrm{j_p}}(\mathbf{q}, 0) &= -\frac{4}{m} \sum_{k<k_F} \left(k\cos\theta + \frac{q}{2}\right)^2 \frac{1}{\frac{q^2}{2} + qk\cos\theta}, \\
&= -\frac{4}{mq^2} \sum_{k<k_F} \left(\frac{q^2}{2} + qk\cos\theta\right), \\
&= -\frac{2}{m} \sum_{k<k_F} 1, \\
&= \boxed{-\frac{n}{m}}, \qquad (184)
\end{aligned}$$

in agreement with Eq. (179).

Using this result, a bit more algebra leads to

$$\chi^L_{0,\,\mathrm{j}}(\mathbf{q}, \omega) = \chi^L_{0,\,\mathrm{j_p}}(\mathbf{q}, \omega) + \frac{n}{m} = \frac{\omega^2}{q^2} \chi_0(\mathbf{q}, \omega)\,, \qquad (185)$$

in agreement with Eq. (180), where the LHS is the screened current response in the RPA, and the RHS is the screened density response function, $\chi_0(\mathbf{q}, \omega)$, in the RPA, see Eq. (119),

V. TRANSVERSE RESPONSE OF THE ELECTRON GAS

A. Formalism

In the previous section we considered the longitudinal response of the electron gas both in terms of the density response and the current response. The transverse response, by contrast, does not involve the density, and so we can *only* consider the current response. The formalism has been worked out above for the longitudinal response and it turns out that we can simply transcribe the results to get the transverse case.[7] In particular, the screened current response is the sum of a diamagnetic and paramagnetic part

$$\chi^T_{\mathrm{sc},\,\mathrm{j}}(\mathbf{q}, \omega) = \frac{n}{m} + \chi^T_{\mathrm{sc},\,\mathrm{j_p}}(\mathbf{q}, \omega)\,, \qquad (186)$$

which is analagous to Eq. (178), where $\chi^T_{\mathrm{sc},\,\mathrm{j_p}}(\mathbf{q}, \omega)$ is defined by

$$\langle \mathbf{j}^T_p(\mathbf{q}, \omega)\rangle = \frac{e}{c} \chi^T_{\mathrm{sc},\,\mathrm{j_p}}(\mathbf{q}, \omega) \mathbf{A}^T(\mathbf{q}, \omega)\,. \qquad (187)$$

Furthermore, from Eq. (186), the transverse dielectric constant $\epsilon^T(\mathbf{q}, \omega)$, and conductivity, $\sigma^T(\mathbf{q}, \omega)$, which govern the transverse screening according to Eq. (98), are related to the screened

[7] This is not fully obvious, and is glossed over in most of the texts. There is some discussion in Pines and Nozières[4].

transverse paramagnetic current response by

$$\boxed{\epsilon^T(\mathbf{q},\omega) = 1 - \frac{4\pi e^2}{\omega^2}\chi^T_{\mathrm{sc,j}}(\mathbf{q},\omega) = 1 - \frac{4\pi e^2}{\omega^2}\left[\frac{n}{m} + \chi^T_{\mathrm{sc,j_p}}(\mathbf{q},\omega)\right] = 1 - \frac{\omega_p^2}{\omega^2} - \frac{4\pi e^2}{\omega^2}\chi^T_{\mathrm{sc,j_p}}(\mathbf{q},\omega)\,,} \tag{188}$$

$$\boxed{\sigma^T(\mathbf{q},\omega) = e^2\,\frac{i}{\omega}\,\chi^T_{\mathrm{sc,j}}(\mathbf{q},\omega) = e^2\,\frac{i}{\omega}\left[\frac{n}{m} + \chi^T_{\mathrm{sc,j_p}}(\mathbf{q},\omega)\right]\,.} \tag{189}$$

Furthermore, the ratio of the transverse electric (or magnetic) field to the "external" field, given by Eq. (103), in terms of the total current response. We have now seen it is useful to separate this into the sum of the diamagnetic response, n/m, and the paramagnetic response, $\chi^T_{\mathrm{sc,j_p}}(\mathbf{q},\omega)$ according to Eq. (186). With this separation, Eq. (103) becomes

$$\boxed{\frac{\mathbf{B}^T(\mathbf{q},\omega)}{\mathbf{B}^T_{\mathrm{ext}}(\mathbf{q},\omega)} = \frac{\mathbf{E}^T(\mathbf{q},\omega)}{\mathbf{E}^T_{\mathrm{ext}}(\mathbf{q},\omega)} = \frac{c^2q^2-\omega^2}{\omega_p^2 + c^2q^2 - \omega^2 + 4\pi e^2\chi^T_{\mathrm{sc,j_p}}(\mathbf{q},\omega)}\,,} \tag{190}$$

where in this, and Eq. (188), we note the appearance of the plasmon frequency, ω_p.

For the longitudinal case, the $\omega = 0$ limit of $\chi_{\mathrm{sc,j_p}}(\mathbf{q},\omega)$ is just $-n/m$ for *all* $\mathbf{q}$, see Eq. (169). We expect that the longitudinal and transverse cases to be equal at $\mathbf{q} = 0$ (unless there are long-range current-current correlations, which happens in a superconductor), but there is no reason for them to be equal at $\mathbf{q} \neq 0$. Hence we anticipate that

$$\chi^T_{\mathrm{sc,j_p}}(\mathbf{q},0) = -\frac{n}{m} + O(q^2)\,. \tag{191}$$

Diagramatically, $\chi^T_{\mathrm{sc,j_p}}(\mathbf{q},\omega)$ is represented by the sum of diagrams which can not by divided into two by cutting a single interaction (*i.e.* dashed) line[8], as in Fig. 3. The only difference compared with the calculation of the density response is that the circles each have a factor of the matrix element of (one of the two components of) the transverse current. From Eq. (B13) this factor is $k\sin\theta\cos\phi$ or $k\sin\theta\sin\phi$, where the electron lines meeting at the circle have wavevectors $\mathbf{k}$ and $\mathbf{k}+\mathbf{q}$, we take a coordinate system with the polar axis in the direction of $\mathbf{q}$, and θ and ϕ are the polar and azimuthal angles of $\mathbf{k}$.

In the RPA, $\chi^T_{\mathrm{sc,j_p}}(\mathbf{q},\omega)$ is replaced by its value for non-interacting electrons, just the first diagram in Fig. 3. We can evaluate this by considering Eq. (165), neglecting all interactions, and replacing the longitudinal current by one of the transverse components. According to Eq. (B13), the matrix element connecting states where an electron in state $\mathbf{k}$ is destroyed and one in state $\mathbf{k}+\mathbf{q}$

[8] The dashed line now represents the photon propagator $4\pi e^2/(c^2q^2-\omega^2)$, which also appears in Eq. (103), rather than the Coulomb interaction.

is created is $m^{-1}k\sin\theta\cos\phi$ for j_x, and $m^{-1}k\sin\theta\sin\phi$ for j_y (assuming $\mathbf{q}$ is in the z-direction). Since the average of $\sin^2\phi$ and $\cos^2\phi$ are both equal to 1/2 we get

$$\boxed{\chi^T_{0,\,\mathrm{j_p}}(\mathbf{q},\omega) = \frac{1}{m^2}\sum_{k<k_F}(k\sin\theta)^2\left\{\frac{1}{\omega-\epsilon_{\mathbf{k+q}}+\epsilon_{\mathbf{k}}+i\eta}-\frac{1}{\omega+\epsilon_{\mathbf{k+q}}-\epsilon_{\mathbf{k}}+i\eta}\right\}.} \tag{192}$$

Eq. (192) is to be compared with the longitudinal result in Eq. (183).

B. The limit $\omega \ll v_F q$

To consider the limit $\omega \ll v_F q$ we set $\omega = 0$ (as we did for the longitudinal case) and get

$$\chi^T_{0,\,\mathrm{j_p}}(\mathbf{q},0) = -\frac{2}{m^2}\mathcal{P}\sum_{k<k_F}\frac{k^2\sin^2\theta}{\epsilon_{\mathbf{k+q}}-\epsilon_{\mathbf{k}}} = -\frac{4}{m}\mathcal{P}\sum_{k<k_F}\frac{k^2\sin^2\theta}{q^2+2qk\cos\theta}\,. \tag{193}$$

Writing

$$-\frac{\sin^2\theta}{q^2+2qk\cos\theta} = \frac{-1+\cos^2\theta}{q^2+2qk\cos\theta} = \frac{\cos\theta}{2qk}-\frac{1}{4k^2}+\left(-1+\frac{q^2}{4k^2}\right)\frac{1}{q^2+2qk\cos\theta}\,. \tag{194}$$

and substituting into Eq. (193) gives

$$\begin{aligned}\chi^T_{0,\,\mathrm{j_p}}(\mathbf{q},0) &= -\frac{4}{m}\frac{1}{(2\pi)^3}2\pi\int_0^{k_F}k^2\,dk\left[\frac{2}{4}+\left(k^2-\frac{q^2}{4}\right)\mathcal{P}\int_{-1}^{1}\frac{du}{q^2+2qku}\right],\\ &= -\frac{4}{m}\frac{1}{(2\pi)^2}\int_0^{k_F}k^2\,dk\left\{\frac{2}{4}+\left(k^2-\frac{q^2}{4}\right)\frac{1}{2kq}\log\left[\left|\frac{q+2k}{q-2k}\right|\right]\right\},\\ &= -\frac{k_F^3}{6\pi^2 m}-\frac{1}{2\pi^2 qm}\int_0^{k_F}k\left(k^2-\frac{q^2}{4}\right)\log\left[\left|\frac{q+2k}{q-2k}\right|\right]\,dk\,.\end{aligned} \tag{195}$$

Using

$$n = 2\,\frac{1}{(2\pi)^3}\frac{4\pi k_F^3}{3} = \frac{1}{3\pi^2}k_F^3, \tag{196}$$

and performing the k integral gives

$$\chi^T_{0,\,\mathrm{j_p}}(\mathbf{q},0) = -\frac{n}{2m}+\frac{n}{m}\left\{-\frac{1}{2}+\frac{3}{8}(1+x^2)-\frac{3(1-x^2)^2}{16x}\log\left[\left|\frac{1+x}{1-x}\right|\right]\right\} \tag{197}$$

where

$$x = \frac{q}{2k_F}\,. \tag{198}$$

Hence the full static transverse current response, given by Eq. (186), is equal to

$$\boxed{\chi^T_{0,\,\mathrm{j}}(\mathbf{q},0) = \frac{n}{m}\left\{\frac{3}{8}(1+x^2)-\frac{3(1-x^2)^2}{16x}\log\left[\left|\frac{1+x}{1-x}\right|\right]\right\}\,.} \tag{199}$$

We are particularly interested in the low $\mathbf{q}$ limit (since this gives the static diamagnetic susceptibility), and expanding Eq. (199) in powers of x gives

$$\chi^T_{0,\mathrm{j}}(\mathbf{q}, 0) = \frac{n}{m} x^2 = \frac{n}{m} \frac{q^2}{4k_F^2} \, . \tag{200}$$

We could have got this result more easily by expanding the integral in Eq. (195) in powers of q before evaluating it.

From Eq. (109) it follows that the static magnetic susceptibility of the electron gas (ignoring the *spin* susceptibility) is

$$\chi'_{\mathrm{mag}} = -\frac{ne^2}{4mc^2k_F^2} = -\frac{1}{3}\left(\frac{e}{2mc}\right)^2 \rho(\epsilon_F) = \boxed{-\frac{1}{3}\chi_0^{\mathrm{Pauli}} \, ,} \tag{201}$$

where $\rho(\epsilon_F)$, the density of states at the Fermi energy is given by Eq. (125) and χ_0^{Pauli} is the Pauli spin susceptibility of the free electron gas. Note that, if we put in the factors of $\hbar$, the factor in brackets is $e\hbar/(2mc) \equiv \mu_B$, the Bohr magneton. We emphasize that Eq. (201) is the contribution to the magnetic susceptibility of the electron gas from its orbital motion. The minus sign indicates that this is a *diamagnetic* effect. Since χ'_{mag} is very small (of order 10^{-5}), the difference between χ'_{mag}, the response to the total magnetic field, and χ_{mag}, the response to the external field, is negligible and so Eq. (201) also gives the (more conventional) magnetic susceptibility χ_{mag}. Eq. (201) was first found by Landau from a calculation of the ground state energy.

We emphasize the results found in this section are very different from those found in Sec. IV B for the longitudinal response, in the same regime, $\omega \ll v_F q$.

It is also interesting to compute the leading imaginary part to $\chi_{0,\mathrm{j}}(\mathbf{q}, \omega)$ at low ω, and we quote the result in Sec. V D.

C. The limit $\omega \gg v_F q$

In this limit we can write Eq. (192) as

$$\chi^T_{0,\mathrm{jp}}(\mathbf{q}, \omega) = \frac{2}{m\omega^2} \sum_{k<k_F} (k \sin\theta)^2 \left(q^2 + 2qk\cos\theta\right) \, , \tag{202}$$

which tends to zero for $\omega \to \infty$. Hence only the first (diamagnetic) part of the current response in Eq. (186) contributes in this limit, and so Eq. (188) becomes

$$\boxed{\epsilon^T(\mathbf{q} \to 0, \omega) = 1 - \frac{\omega_p^2}{\omega^2} \, ,} \tag{203}$$

where ω_p is the plasmon frequency defined in Eq. (144). Eq. (203) is the same expression as for the longitudinal case, Eq. (143). One can add a relaxation time phenomenologically as in the longitudinal case, Eq. (151)

$$\epsilon^T(0,\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i/\tau)}\,. \tag{204}$$

We will use this expression in class to discuss the optical reflectivity of simple metals.

The condition for a transverse excitation to occur is given by $\omega^2 = c^2q^2/\epsilon^T(\mathbf{q},\omega)$, Eq. (99). Using Eq. (203) for $\epsilon^T(\mathbf{q},\omega)$ gives

$$\omega^2 = \omega_p^2 + c^2q^2 \tag{205}$$

for $q \to 0$, which shows that the plasmon rapidly mixes with electromagnetic waves as q increases. The dispersion of this mode is indicated in Fig. (7) below.

It is expected, *in general* that the longitudinal and transverse responses agree in the limit $\omega \gg v_F q$ since the information about the perturbation cannot propagate across one wavelength during the period of one oscillation (so how can the current know whether it is longitudinal or transverse?).

D. The general case

General expressions for $\chi^T_{0,\mathrm{j}}(\mathbf{q},\omega)$ according to Eq. (192) are given by Dressel and Grüner[3]. Naturally there is an imaginary part in the range where particle-hole excitations can be created. This is the same as for the longitudinal case.

There are differences in the collective excitations, though, which are now given by the solutions of Eq. (99) rather than Eq. (71). There is still a plasmon with frequency ω_p, see Eq. (144), at $q = 0$ but this quickly merges into the branch of electromagnetic radiation, $\omega = cq$ as q increases, see Fig, 7. In a more precise theory of the electron gas, and for a certain range of parameters, one can also have a collective transverse branch emerging out of the particle-hole continuum, see Fig. 3.3 of Pines and Nozières[4]. However, this does not occur in the RPA, for which the putative transverse branch lies inside the particle-hole continuum where it is very heavily damped.

It is also of interest to consider the leading imaginary (dissipative) part of the response at small ω. This is given by[3, 4]

$$\chi^T_{0,\mathrm{j}}(\mathbf{q},\omega) = -i\omega\frac{3\pi}{4}\frac{n}{qk_f}, \qquad (\text{small }\omega)\,, \tag{206}$$

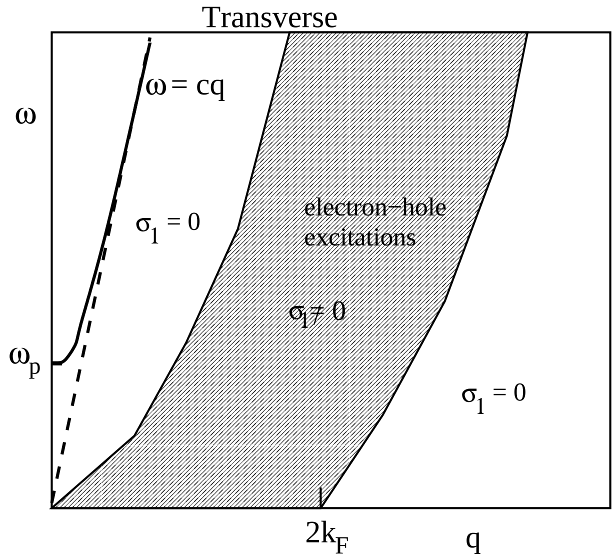

FIG. 7: The transverse excitation spectrum of the electron gas in the RPA, given by the solutions of Eq. (99). The shaded region is where electron-hole excitations can be created and is the same as for the longitudinal case in Fig. 6. At $q = 0$ there is a plasmon at $\omega = \omega_p$, but, unlike the longitudinal case, this quickly mixes strongly with electromagnetic waves as q is increased.

which, from Eq. (189), can be more conveniently written as

$$\boxed{\sigma^T(\mathbf{q}, 0) = \frac{3\pi}{4}\frac{ne^2}{qk_F}\,.} \tag{207}$$

Experimentally, $\sigma^T(\mathbf{q}, 0)$ enters in the "anomalous" skin effect. The reason is that an electromagnetic wave decays rapidly on entering a metal, and, as a result, q, which is complex, has very large real and imaginary parts. The imaginary part, q_i is the inverse of the "skin depth" δ_0, the distance the the wave propagates into the metal. For a clean metal at low temperatures, the conductivity can be very large and it turns out, as you will show in a homework problem, that the system can then be in a regime where $\delta_0 < \ell\,(= v_F\tau)$, where ℓ is the mean free path, the distance traveled by an electron between collisions. In this situation one expects that σ will be independent of τ, since the electrons don't have time to scatter in the region (near the surface) where the electric field is non-zero. Hence, Eq. (207) (which does involve τ since it is derived in the RPA) should be a good description provided, in addition, we have $\omega \ll v_F|q|$ (where we replace $|q|$ by δ^{-1}). The homework question asks you to evaluate the skin depth in this anomalous (*i.e.* $\delta_0 < \ell$) region.

VI. SUMMARY

We have considered the density and current response of the electron gas with a view to understanding optical reflectivity measurements of conductors. We found it convenient to consider separately the longitudinal and transverse response, and to Fourier transform with respect to time and space (where the definitions of longitudinal and transverse are particularly transparent). The absorption of electromagnetic radiation involves, of course, the transverse response. Because the interactions are of long range the analysis is simplified by considering the response to a screened perturbation, which includes the fields set up by the system in addition to the external fields.

Because the speed of light is so much higher than the Fermi velocity, the response of the electron system to absorption of light is effectively at very small $\mathbf{q}$, more precisely at $\omega \gg v_F q$, in which case the distinction between longitudinal and transverse response disappears, and, in most cases, we can set $\mathbf{q} = 0$ in calculating response functions.

We have introduced several linear response functions, and the relations between them are summarized in Appendix G.

We consider the response of the electron gas in the RPA, finding, in the $\mathbf{q} \to 0$ limit, (for both the longitudinal and transverse cases)

$$\epsilon(0, \omega) = 1 - \frac{\omega_p^2}{\omega^2}\,, \tag{208}$$

where the plasmon frequency ω_p is given by Eq. (144). This neglects damping of the single-particle excitations, which can be put in phenomenologically by using the expression

$$\epsilon(0, \omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i/\tau)}\,, \tag{209}$$

where τ is the relaxation time. From Eq. (93), this corresponds to the familiar Drude form for the conductivity

$$\sigma(0, \omega) = \frac{ne^2\tau}{m}\,\frac{1}{1 - i\omega\tau}. \tag{210}$$

The real part σ_1, given by

$$\sigma_1(0, \omega) = \frac{ne^2\tau}{m}\,\frac{1}{1 + \omega^2\tau^2}, \tag{211}$$

has a peak at $\omega = 0$, of width τ^{-1}, which is known as the "Drude peak".

For most metals $\omega_p\tau \gg 1$ and, using this information, we will investigate in class, the reflectivity of simple metals predicted by the Drude expression.

In more complicated metals, with strong correlations, the shape of the curve of $\sigma_1(0,\omega)$ will be different from that predicted by the Drude theory. However, the total area under the curve will the unchanged because of the important sum rule

$$\int_0^\infty \sigma_1(0,\omega) = \frac{\omega_p^2}{8} = \frac{\pi n e^2}{2m}\,. \tag{212}$$

Hence reflectivity data are often converted to data for σ_1 and interpreted as "shift of spectral weight" from one region of ω to another, since the total spectral weight is fixed by the sum rule.

Transverse propagating excitations exist when the condition

$$\omega^2 = \frac{c^2 q^2}{\epsilon^T(\mathbf{q},\omega)}\,. \tag{213}$$

Using Eq. (209) we see that for $\omega \ll \tau^{-1} \ll \omega_p$, $\epsilon^T(\mathbf{q},\omega)$ is real and negative, so that the wave is heavily damped with the real and imaginary parts of q equal. However, for $\omega \gg \omega_p$, $\epsilon^T(\mathbf{q},\omega)$ is real and positive and electromagnetic waves can propagate with little damping, see Fig. (7).

There can, of course, be other sources of damping of electromagnetic radiation in a metal in addition to exciting electrons in the conduction band. These include "inter-band"transitions, where, for example, an electron is excited from the valence band to the conduction band by absorption of a photon, and lattice vibrations.

In insulators, electromagnetic waves can be absorbed by lattice vibrations, or inter-band electronic transitions (or impurities). If these do not occur, *e.g.* in window glass, then light propagates without damping, $\epsilon(0,\omega)$ is real, and the speed of light is

$$v = \frac{c}{n(\omega)}\,, \tag{214}$$

where $n(\omega) = \epsilon(0,\omega)^{1/2}$ is the refractive index.

APPENDIX A: EXAMPLES OF LONGITUDINAL AND TRANSVERSE FIELDS

In this section we derive the Fourier transform of some familiar vector fields, showing whether they are longitudinal or transverse.

First of all we know that the electrostatic potential due to a point dipole of strength $\mathbf{p}$ is[1, 2]

$$V(\mathbf{r}) = \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = -\mathbf{p}\cdot\boldsymbol{\nabla}\left(\frac{1}{r}\right) . \tag{A1}$$

The Fourier transform of $1/r$ is $4\pi/q^2$ and so

$$V(\mathbf{q}) = -4\pi i\,\frac{\mathbf{p}\cdot\mathbf{q}}{q^2} . \tag{A2}$$

Converting this to the electric field using Eq. (63) gives

$$\boxed{\mathbf{E}(\mathbf{q}) = -4\pi\,\frac{\mathbf{p}\cdot\mathbf{q}}{q^2}\,\mathbf{q}\,,} \tag{A3}$$

which is longitudinal, *i.e.* in the direction of $\mathbf{q}$, as expected since static electric fields are always longitudinal.

Similarly the vector potential from a magnetic dipole of strength $\mathbf{m}$ is given by[1, 2]

$$\mathbf{A}(\mathbf{r}) = \frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} = -\mathbf{m}\times\boldsymbol{\nabla}\left(\frac{1}{r}\right) . \tag{A4}$$

The Fourier transform is

$$\mathbf{A}(\mathbf{q}) = 4\pi i\,\frac{\mathbf{q}\times\mathbf{m}}{q^2} . \tag{A5}$$

The corresponding magnetic field is $\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r})$, *i.e.* $\mathbf{B}(\mathbf{q}) = i\mathbf{q}\times\mathbf{A}(\mathbf{q})$, so

$$\boxed{\mathbf{B}(\mathbf{q}) = -4\pi\,\frac{\mathbf{q}\times(\mathbf{q}\times\mathbf{m})}{q^2} = 4\pi\left(\mathbf{m} - \frac{(\mathbf{m}\cdot\mathbf{q})}{q^2}\,\mathbf{q}\right) ,} \tag{A6}$$

which is transverse, *i.e.* perpendicular to $\mathbf{q}$, as expected since $\mathbf{B}$ is always transverse.

Finally, in this section, we compute the Fourier transform of a current I flowing around a circular loop of radius a in the x-y plane, *i.e.*

$$J_x(\mathbf{r}) = I(-\sin\theta)\delta(r-a)\delta(z) \tag{A7}$$

$$J_y(\mathbf{r}) = I(\cos\theta)\delta(r-a)\delta(z) \tag{A8}$$

$$J_z(\mathbf{r}) = 0 . \tag{A9}$$

where we have used cylindrical polar coordinates, see Fig. 8.

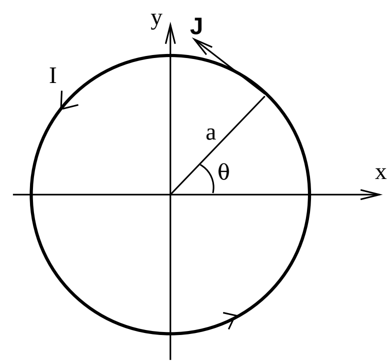

FIG. 8: Axis convention for the current loop.

Since the current is only at $z = 0$, the Fourier transform is independent of q_z. We therefore consider only the components of $\mathbf{q}$ in the x-y plane, and assume that $\mathbf{q}$ lies along the x direction. Hence the Fourier transform of J_x is

$$J_x(\mathbf{q}) = I \int_0^{2\pi} (-\sin\theta) e^{-iqr\cos\theta}\, d\theta \int_0^{\infty} r\,\delta(r-a) dr = 0\,, \tag{A10}$$

(see the definition of the spatial Fourier transform in Eq. (26)), because the θ integral vanishes (the contributions from θ and $(2\pi - \theta)$ vanish). We also have

$$\begin{aligned} J_y(\mathbf{q}) &= I \int_0^{2\pi} (\cos\theta) e^{-iqr\cos\theta}\, d\theta \int_0^{\infty} r\,\delta(r-a)\, dr && \text{(A11)} \\ &= -2\pi i\, I \int_0^{\infty} r J_1(qr) \delta(r-a)\, dr && \text{(A12)} \\ &= -2\pi i I a J_1(qa)\,, && \text{(A13)} \end{aligned}$$

where $J_1(x)$ is the Bessel function of order 1. Since $J_1(x) = x/2 + \cdots$ for $x \to 0$ we see that $\mathbf{J}(\mathbf{q}) \to 0$ for $\mathbf{q} \to 0$ (as expected since the current flows in a closed loop and so the current density integrated over all space vanishes).

We can write the expression for $\mathbf{J}(\mathbf{q})$, applicable for $\mathbf{q}$ in any direction, as

$$\boxed{\mathbf{J}(\mathbf{q}) = 2\pi I a J_1(qa)\, i\, \frac{\mathbf{q} \times \hat{z}}{q}\,.} \tag{A14}$$

This is transverse, *i.e.* perpendicular to $\mathbf{q}$. In fact, *any* current which does not cause a time-dependent charge density must be transverse, because a longitudinal current is related to the charge density by Eq. (47). For $a \to 0$ we get

$$\mathbf{J}(\mathbf{q}) = \pi i a^2 I \mathbf{q} \times \hat{z} = i\, c\, \mathbf{q} \times \mathbf{m}\,, \tag{A15}$$

where $\mathbf{m} = c^{-1} \pi a^2 I \hat{z}$ is the *magnetic moment* of the current loop.

To check this result, we compare it with Eq. (A5), which gives

$$q^2\mathbf{A}(\mathbf{q}) = \frac{4\pi}{c}\mathbf{J}(\mathbf{q}). \tag{A16}$$

This can be usefully written in real space as

$$\nabla^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}\,, \tag{A17}$$

which is correct, and is a well known result in magnetostatics[1, 2]

APPENDIX B: THE DENSITY AND CURRENT OPERATORS

The (number) density is given by

$$\hat{n}(\mathbf{r}) = \sum_i \delta(\mathbf{r} - \mathbf{r}_i), \tag{B1}$$

where $\mathbf{r}_i$ is the position of the i-th electron. The spatial Fourier transform is given by

$$\hat{n}(\mathbf{q}) = \int \hat{n}(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}\,d\mathbf{r} = \sum_i e^{-i\mathbf{q}\cdot\mathbf{r}_i}\,. \tag{B2}$$

It is convenient to go to the second quantized representation, working in a plane wave set of basis states $|\mathbf{k}\rangle = V^{-1/2}e^{i\mathbf{k}\cdot\mathbf{r}}$. As shown in standard texts in quantum mechanics, a 1-body operator $\hat{\mathcal{U}} = \sum_i \hat{U}(\mathbf{r}_i)$ can be represented in second-quantized notation by

$$\hat{\mathcal{U}} = \sum_{\mathbf{k}\sigma,\,\mathbf{k}'\sigma'} \langle\mathbf{k}\sigma|\hat{U}(\mathbf{r})|\mathbf{k}'\sigma'\rangle c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}'\sigma'}\,, \tag{B3}$$

where $c^\dagger$ and c are creation and destruction operators. Here $\hat{U}(\mathbf{r}) = e^{-i\mathbf{q}\cdot\mathbf{r}}$ and so the matrix element is zero unless $\sigma' = \sigma$ and $\mathbf{k}+\mathbf{q} = \mathbf{k}'$, *i.e.*

$$\boxed{\hat{n}(\mathbf{q}) = \sum_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}-\mathbf{q}\,\sigma}c_{\mathbf{k}\sigma}\,.} \tag{B4}$$

The Hermitian conjugate operator is

$$\hat{n}^\dagger(\mathbf{q}) = \sum_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}-\mathbf{q}\,\sigma} = \sum_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}+\mathbf{q}\,\sigma}c_{\mathbf{k}\sigma}\,. \tag{B5}$$

Next we discuss the (number) current[4, 5]

$$\mathbf{j}(\mathbf{r}) = \frac{1}{2}\sum_i \left[\mathbf{v}_i\delta(\mathbf{r}-\mathbf{r}_i) + \delta(\mathbf{r}-\mathbf{r}_i)\mathbf{v}_i\right]\,, \tag{B6}$$

where $\mathbf{v}_i$ is the velocity operator. The velocity is related to the canonical momentum $\mathbf{p}_i$ by

$$\mathbf{v}_i = \frac{1}{m}\left[\mathbf{p}_i + \frac{e}{c}\mathbf{A}(\mathbf{r}_i)\right] , \tag{B7}$$

where $\mathbf{A}(\mathbf{r})$ is the vector potential. If follows that the current can be written as the sum of two parts:

$$\boxed{\mathbf{j}(\mathbf{r}) = \mathbf{j}_p(\mathbf{r}) + \frac{e}{mc}\hat{n}(\mathbf{r})\mathbf{A}(\mathbf{r}),} \tag{B8}$$

where the *paramagnetic current* operator is given by

$$\mathbf{j}_p(\mathbf{r}) = \frac{1}{2m}\sum_i \left[\mathbf{p}_i\delta(\mathbf{r}-\mathbf{r}_i) + \delta(\mathbf{r}-\mathbf{r}_i)\mathbf{p}_i\right] , \tag{B9}$$

and the second term in Eq. (B8) is called the *diamagnetic current* operator. The Fourier components of the paramagnetic current are given by

$$\mathbf{j}_p(\mathbf{q}) = \int \mathbf{j}(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}\, d\mathbf{r} = \frac{1}{2m}\sum_i \left[\mathbf{p}_i e^{-i\mathbf{q}\cdot\mathbf{r}_i} + e^{-i\mathbf{q}\cdot\mathbf{r}_i}\mathbf{p}_i\right] = \frac{1}{m}\sum_i \left(\mathbf{p}_i + \frac{\mathbf{q}}{2}\right) e^{-i\mathbf{q}\cdot\mathbf{r}_i} , \tag{B10}$$

where to get the last expression we used

$$[f(\mathbf{r}), \mathbf{p}] = i\boldsymbol{\nabla} f(\mathbf{r}) . \tag{B11}$$

To obtain the second quantized form of $\mathbf{j}(\mathbf{q})$ we need the matrix element $\langle \mathbf{k}\sigma|\mathbf{j}(\mathbf{q})|\mathbf{k}'\sigma'\rangle$. Since $\mathbf{p}|k\rangle = \mathbf{k}|k\rangle$ we have

$$\langle \mathbf{k}\sigma|\mathbf{j}_p(\mathbf{q})|\mathbf{k}'\sigma'\rangle = \frac{1}{m}\left(\mathbf{k} + \frac{\mathbf{q}}{2}\right) \delta_{\mathbf{k},\mathbf{k}'+\mathbf{q}}\, \delta_{\sigma\sigma'} . \tag{B12}$$

It follows that

$$\boxed{\mathbf{j}_p(\mathbf{q}) = \frac{1}{m}\sum_{\mathbf{k}\,\sigma} \left(\mathbf{k} + \frac{\mathbf{q}}{2}\right) c^\dagger_{\mathbf{k}-\mathbf{q}\,\sigma} c_{\mathbf{k}\sigma} ,} \tag{B13}$$

see also Ref. [5].

The density and current operators are related by a continuity equation[4, 5], since the particle number is conserved. In $\mathbf{q}$-space this is

$$\frac{\partial}{\partial t}\,\hat{n}(\mathbf{q}) = -i\mathbf{q}\cdot\mathbf{j}(\mathbf{q}) = -iq\, j^L(\mathbf{q}) , \tag{B14}$$

(see also Eq. (47)). Note that this only involves the longitudinal component of $\mathbf{J}$. However the time derivative of $\hat{n}(\mathbf{q})$ is also given by the equation of motion

$$\frac{\partial}{\partial t}\,\hat{n}(\mathbf{q}) = -i\left[\hat{n}(\mathbf{q}), \mathcal{H}\right] , \tag{B15}$$

and so

$$[\hat{n}(\mathbf{q}), \mathcal{H}] = q\, j^L(\mathbf{q}) \tag{B16}$$

If we sandwich this last expression between exact eigenstates of $\mathcal{H}$ we get

$$\boxed{(E_m - E_n)\, \langle n|\hat{n}(\mathbf{q})|m\rangle = q\, \langle n|j^L(\mathbf{q})|m\rangle\, ,} \tag{B17}$$

which will turn out to be useful.

It will also be useful to evaluate the thermal average of the commutator of Eq. (B16) with $\hat{n}^\dagger(\mathbf{q})$. The left hand side is given by

$$\sum_n P_n \langle n| \left[[\hat{n}(\mathbf{q}), \mathcal{H}]\, , \hat{n}^\dagger(\mathbf{q}) \right] |n\rangle = \sum_n P_n \sum_m (E_m - E_n)\, \left[|\langle n|\hat{n}(\mathbf{q})|m\rangle|^2 + |\langle n|\hat{n}^\dagger(\mathbf{q})|m\rangle|^2 \right]\, . \tag{B18}$$

Now $\hat{n}^\dagger(\mathbf{q}) = \hat{n}(-\mathbf{q})$, and, because of time reversal invariance, for each pair of states n and m with momentum difference $\mathbf{q}$, there will be another pair, n' and m', degenerate with the first pair, which are separated by momentum $-\mathbf{q}$. Hence the two terms on the RHS of Eq. (B18) are equal so

$$2 \sum_n P_n \sum_m (E_m - E_n)\, |\langle n|\hat{n}(\mathbf{q})|m\rangle|^2\, . \tag{B19}$$

This is the the thermal average of the commutator of the LHS of Eq. (B16) with $\hat{n}^\dagger(\mathbf{q})$. We now consider the RHS:

$$\begin{aligned} q \sum_n P_n \langle n| \left[J^L(\mathbf{q}), \hat{n}^\dagger(\mathbf{q}) \right] |n\rangle &= \frac{q}{m} \sum_n P_n \langle n| \left[j^L(\mathbf{q}), \hat{n}^\dagger(\mathbf{q}) \right] |n\rangle \\ &= q \sum_i \sum_n P_n \langle n| \left[\mathbf{p}_i^L e^{-i\mathbf{q}\cdot\mathbf{r}_i}, e^{i\mathbf{q}\cdot\mathbf{r}_i} \right] |n\rangle \\ &= q \sum_n P_n \langle n| \frac{qn}{m} |n\rangle \\ &= \frac{q^2 n}{m}\, , \end{aligned} \tag{B20}$$

where $n = \sum_i (1)$ is the number of electrons (per unit volume, which is being assumed throughout). To get the second line in Eq. (B20) we used Eq. (B13), and to get the third line we evaluated the commutator of $\mathbf{p}_i$ with $e^{i\mathbf{q}\cdot\mathbf{r}}$ using Eq. (B11). Equating Eq. (B19) to Eq. (B20) gives

$$\boxed{\sum_n P_n \sum_m (E_m - E_n)\, |\langle n|\hat{n}(\mathbf{q})|m\rangle|^2 = \frac{q^2 n}{2m}\, ,} \tag{B21}$$

which is known as the $\boxed{f\text{-sum rule.}}$ It is related to the sum rule of the same name in atomic physics.

If we use Eq. (B17) we can reexpress the f-sum rule in terms of matrix elements of the current. We will always assume that such expressions are to be evaluated in the absence of any fields, and so the total current is equal to the paramagnetic current. Hence we have

$$\boxed{\sum_n P_n \sum_m \frac{|\langle n|j_p^L(\mathbf{q})|m\rangle|^2}{E_m - E_n} = \frac{n}{2m}\,.} \tag{B22}$$

Usually we evaluate expressions like this in the absence of any electromagnetic fields, in which case the full current is equal to the paramagnetic current.

APPENDIX C: LINEAR RESPONSE THEORY

Suppose we have a system described by a Hamiltonian $\mathcal{H}$ and add to it time-dependent perturbation

$$\mathcal{H}_I = \int d\mathbf{r}\, f(\mathbf{r},t)A(\mathbf{r}), \tag{C1}$$

where $f(\mathbf{r},t)$ is the amplitude of the perturbation, switched on gradually at large negative times, and $A(\mathbf{r})$ is the operator of the system to which the perturbation couples. We want to know how the system responds to the perturbation. For this handout, all we will need to know is the expectation value of $\mathbf{A}$ itself.

It is convenient to Fourier transform with respect to space and time. Since the perturbation is switched on "adiabatically" from $t = -\infty$, the time dependence of $f(t)$ is $e^{-i(\omega+i\eta)t}$, where η is a small positive quantity, which ensures that the perturbation vanished at $t = -\infty$. We therefore write

$$f(\mathbf{q},t) = f(\mathbf{q},\omega)e^{-i(\omega+i\eta)t}\,. \tag{C2}$$

In adiabatic switching on, one neglects real transitions between states induced by the perturbation, so the probability of the system being in a given state is unchanged. Expectation values change, therefore, only because of the change in the wavefunctions. We will now evaluate this change to first order in the perturbation.

We write an exact unperturbed, time-independent, state as $|n\rangle$ and the time dependent state as $|\psi_n(t)\rangle = |n\rangle e^{-iE_nt}$. According to time-dependent perturbation theory[7] the perturbed state $|\psi_n(t)\rangle'$ is given, to first order by

$$|\psi_n(t)\rangle' = |n\rangle e^{-iE_nt} + \sum_m c_m(t)\,|m\rangle e^{-iE_mt}\,, \tag{C3}$$

where

$$i\dot{c}_m(t) = f(\mathbf{q},t)\langle m|A(\mathbf{q})|n\rangle e^{i(E_m-E_n)t} = f(\mathbf{q},\omega)\langle m|A(\mathbf{q})|n\rangle e^{i(E_m-E_n-\omega-i\eta)t} . \tag{C4}$$

Integrating from $t = -\infty$ gives

$$c_m(t) = f(\mathbf{q},\omega)e^{-i(\omega+i\eta)t}\,\frac{\langle m|A(\mathbf{q})|n\rangle}{E_n - E_m + \omega + i\eta} . \tag{C5}$$

Similarly, the complex conjugate wavefunction

$$\langle\psi_n(t)|' = \langle n|e^{iE_nt} + \sum_m \bar{c}_m(t)\langle m|e^{iE_mt} , \tag{C6}$$

is given to first order by (note that the time dependence is not to be complex-conjugated)

$$\bar{c}_m(t) = f(\mathbf{q},\omega)e^{-i(\omega+i\eta)t}\,\frac{\langle n|A(\mathbf{q})|m\rangle}{E_n - E_m - \omega - i\eta} . \tag{C7}$$

Hence the expectation value of A in state $|n\rangle$ is given by

$$\begin{aligned}\langle\psi_n(t)|A(\mathbf{q})|\psi_n(t)\rangle \;=\; & \langle n|A(\mathbf{q})|n\rangle + \\ & f(\mathbf{q},\omega)e^{-i(\omega+i\eta)t}|\langle n|A(\mathbf{q})|m\rangle|^2\left[\frac{1}{E_n - E_m + \omega + i\eta} + \frac{1}{E_n - E_m - \omega - i\eta}\right]\end{aligned} \tag{C8}$$

We assume that the expectation value of $A(\mathbf{q})$ is zero in the absence of the perturbation. The system is in initial state $|n\rangle$ with Boltzmann probability P_n and so, summing over n, one finds[4, 5] that the linear response function $\chi_A(\mathbf{q},\omega)$, defined by

$$\langle A(\mathbf{q},\omega)\rangle = \chi_A(\mathbf{q},\omega)\, f(\mathbf{q},\omega), \tag{C9}$$

is given by

$$\boxed{\chi_A(\mathbf{q},\omega) = \sum_{n,m} P_n|\langle n|A(\mathbf{q})|m\rangle|^2\left[\frac{1}{E_n - E_m + \omega + i\eta} - \frac{1}{E_m - E_n + \omega + i\eta}\right] .} \tag{C10}$$

In the time domain, Eqs. (C9) and (C10) can be written as

$$\langle A(\mathbf{q},t)\rangle = \int_{-\infty}^{\infty} \chi_A(\mathbf{q},t-t')f(\mathbf{q},t')\,dt' , \tag{C11}$$

where

$$\boxed{\chi_A(\mathbf{q},t-t') = -i\theta(t-t')\langle[A(\mathbf{q},t), A(\mathbf{q},t')]\rangle} \tag{C12}$$

where $\theta(t)$ is the theta function (equal to 0 for $t < 0$ and 1 for $t > 0$), $[A, B] = AB - BA$ is the commutator, and $A(\mathbf{q},t) = \exp(i\mathcal{H}t)A(\mathbf{q})\exp(-i\mathcal{H}t)$ is the time-dependent Heisenberg operator.

The theta function reflects *causality*, *i.e.* the effects of the perturbation at time t' are only felt at later times t. Because of the θ function the upper limit of the integral in Eq. (C11) could be replaced by t. In addition, substituting $f(\mathbf{q},t) = f(\mathbf{q},\omega)e^{-i(\omega+i\eta)t}$, Eq. (C11) becomes

$$\langle A(\mathbf{q},t)\rangle = f(\mathbf{q},\omega)e^{-i(\omega+i\eta)t}\int_{-\infty}^{t} \chi_A(\mathbf{q},t-t')e^{-i(\omega+i\eta)(t'-t)}\,dt'\,, \tag{C13}$$

or

$$\langle A(\mathbf{q},t)\rangle = \langle A(\mathbf{q},\omega)\rangle e^{-i(\omega+i\eta)t}\,, \tag{C14}$$

where

$$\langle A(\mathbf{q},\omega)\rangle = \chi_A(\mathbf{q},\omega)f(\mathbf{q},\omega), \tag{C15}$$

with

$$\boxed{\chi_A(\mathbf{q},\omega) = \int_{0}^{\infty} \chi_A(\mathbf{q},t)e^{i(\omega+i\eta)t}\,dt\,,} \tag{C16}$$

which shows the relation between the time and frequency forms of the linear response function χ_A.

The response function $\chi_A(\mathbf{q},\omega)$ has a real part, obtained by the principal part of the sum (converted to an integral in practice) in Eq. (C10) and an imaginary part arising when the real part of the denominator vanishes. Using

$$\frac{1}{x+i\eta} = \mathcal{P}\left(\frac{1}{x}\right) - i\pi\delta(x), \tag{C17}$$

we have $\chi_A(\mathbf{q},\omega) = \chi'_A(\mathbf{q},\omega) + i\chi''_A(\mathbf{q},\omega)$, where

$$\chi''_A(\mathbf{q},\omega) = -\pi\sum_{n,m} P_n|\langle n|A(\mathbf{q})|m\rangle|^2\left[\delta(\omega-(E_m-E_n)) - \delta(\omega-(E_n-E_m))\right] \tag{C18}$$

It will be useful to consider the "first moment" of $\chi''_A(\mathbf{q},\omega)$, *i.e.*

$$\boxed{\int_0^{\infty} \omega\chi''_A(\mathbf{q},\omega)\,d\omega = \sum_n P_n\left\{\pi\sum_m (E_n-E_m)|\langle n|A(\mathbf{q})|m\rangle|^2\right\}\,,} \tag{C19}$$

where we used that $\chi''_A(\mathbf{q},\omega)$ is an odd function of ω and so the integral from 0 to ∞ of $\omega\chi''_A(\mathbf{q},\omega)$ is half the integral from $-\infty$ to ∞.

It is also useful to consider the linear response at complex frequency z, so we generalize Eq. (C16) to

$$\chi_A(\mathbf{q},z) = \int_0^{\infty} \chi_A(\mathbf{q},t)e^{izt}\,dt \quad (\mathrm{Im}(z) > 0). \tag{C20}$$

This is well defined for $\mathrm{Im}(z) > 0$, *i.e.* the upper half complex plane, since $\exp(-\mathrm{Im}(z)t)$ tends to zero as $t \to \infty$. We conclude that $\chi_A(\mathbf{q}, z)$ is *analytic* in the upper half plane. We will use this fact in Appendix D to derive important relationships connecting the real and imaginary parts of $\chi_A(\mathbf{q}, \omega)$, known as the Kramers-Kronig relations. Physically these relations (and the analyticity of $\chi_A(\mathbf{q}, \omega)$ in the upper half-plane) are due to causality.

We note that the representation of $\chi_A(\mathbf{q}, \omega)$ in terms of exact unperturbed eigenstates of the Hamiltonian in Eq. (C10) can be trivially generalized to complex z:

$$\boxed{\chi_A(\mathbf{q}, z) = \sum_{n,m} P_n |\langle n|A(\mathbf{q})|m\rangle|^2 \left[\frac{1}{E_n - E_m + z} - \frac{1}{E_m - E_n + z}\right] .} \tag{C21}$$

Furthermore this definition also makes sense for $\mathrm{Im}(z) < 0$. In that case one can show that

$$\chi_A(\mathbf{q}, z) = \int_{-\infty}^{0} \chi_A^{\mathrm{adv}}(\mathbf{q}, t) e^{-izt}\, dt \quad (\mathrm{Im}(z) < 0), \tag{C22}$$

where the "advanced" response function $\chi_A^{\mathrm{adv}}(\mathbf{q}, t)$ is given by

$$\chi_A^{\mathrm{adv}}(\mathbf{q}, t - t') = i\theta(t' - t)\langle [A(\mathbf{q}, t), A(\mathbf{q}, t')]\rangle \, . \tag{C23}$$

This is to be compared with $\chi_A(\mathbf{q}, t - t')$ in Eq. (C12), which is called the "retarded" response function.

Equation (C21) shows that $\chi_A(\mathbf{q}, z)$ is an analytic function *everywhere* in the complex plane except on the real axis. However, it is only equal to the physical "retarded" response function (which respects causality) for $\mathrm{Im}(z) > 0$.

To investigate what happens on the real axis we compare Eq. (C21) with Eq. (C18), and use Eq. (C17). The result is that that $\chi_A(\mathbf{q}, z)$ can be written in a "spectral representation"

$$\boxed{\chi_A(\mathbf{q}, z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\chi_A''(\mathbf{q}, \omega')}{\omega' - z}\, d\omega' \, .} \tag{C24}$$

For $z = \omega + i\eta$ this correctly gives our basic definition of the real an imaginary parts,

$$\chi_A(\mathbf{q}, \omega + i\eta) = \chi_A'(\mathbf{q}, \omega) + i\chi_A''(\mathbf{q}, \omega) \, , \tag{C25}$$

(see Eqs. (C17) and (D3)), while for $z = \omega - i\eta$ we get

$$\chi_A(\mathbf{q}, \omega - i\eta) = \chi_A'(\mathbf{q}, \omega) - i\chi_A''(\mathbf{q}, \omega) \, . \tag{C26}$$

We therefore now understand the analytic properties of $\chi_A(\mathbf{q}, z)$. It is analytic everywhere in the complex plane, except along the part of the real axis where $\chi_A''(\mathbf{q}, \omega) \neq 0$, where there is a branch cut with a discontinuity of size $2\chi_A''(\mathbf{q}, \omega)$.

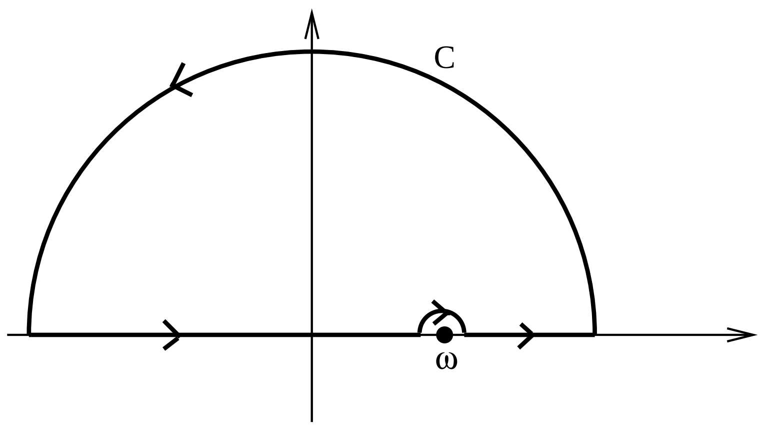

FIG. 9: Contour in the complex frequency (z) plane used to derive the Kramers-Kronig relations. The radius of the outer semicircle tends to infinity. The radius of the small semicircle about $z = \omega$ tends to zero.

APPENDIX D: KRAMERS-KRONIG RELATIONS

In Appendix C we showed that the linear response function $\chi(z)$ is an analytic function of complex frequency z in the upper-half plane because of causality. We now use this property to derive important relationships between the real and imaginary parts of $\chi(\omega + i\eta) = \chi'(\omega) + i\chi''(\omega)$, due to Kramers and Kronig.

We evaluate

$$\oint_C \frac{\chi(z)}{z - \omega} \, dz \, , \tag{D1}$$

over the contour shown in Fig. (9).

Because $\chi(z)$ is analytic inside the region of integration and the pole at $z = \omega$ is also excluded, the integral is 0. The contribution from the "semicircle at infinity" vanishes if $\chi(z) \to 0$ for $z \to \infty$.[9] The integral along the real axis is a principal value integral, and the contribution from the semicircle is $-i\pi$ times the residue at $z = \omega$, *i.e.* $-i\pi\chi(\omega + i\eta) = -i\pi[\chi'(\omega) + i\chi''(\omega)]$. Hence we have

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{\chi'(\omega') + i\chi''(\omega')}{\omega' - \omega} \, d\omega' - i\pi \left[\chi'(\omega) + i\chi''(\omega) \right] = 0 \, . \tag{D2}$$

Equation real and imaginary parts gives the desired Kramers-Kronig relations:

$$\boxed{\chi'(\omega) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega} \, d\omega' \, ,} \tag{D3}$$

[9] If $\chi(z) \to$ const. for $z \to \infty$ then $\chi(z)$ is replaced by $\chi(z) -$ const. in the integrand in Eq. (D1) and in the subsequent results.

$$\boxed{\chi''(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi'(\omega')}{\omega' - \omega}\, d\omega' \,.} \tag{D4}$$

APPENDIX E: SUM RULE FOR THE CONDUCTIVITY

Let us consider Eq. (C19) for the density response function, $\chi(\mathbf{q}, \omega)$, see Eq. (72). We have

$$\int_0^\infty \omega\chi''(\mathbf{q}, \omega)\, d\omega = \sum_n P_n \left\{ \pi \sum_m (E_n - E_m) |\langle n|\hat{n}(\mathbf{q})|m\rangle|^2 \,. \right\} \tag{E1}$$

From the f-sum rule in Eq. (B21) we can simplify this to

$$\int_0^\infty \omega\chi''(\mathbf{q}, \omega)\, d\omega = -\frac{\pi q^2 n}{2m} \,. \tag{E2}$$

It is shown in Ref. [4] and Appendix F that the same sum rule holds for the screened response $\chi_{\rm sc}(\mathbf{q}, \omega)$, *i.e.*

$$\int_0^\infty \omega\chi''_{\rm sc}(\mathbf{q}, \omega)\, d\omega = -\frac{\pi q^2 n}{2m} \,. \tag{E3}$$

Using the relation between $\chi_{\rm sc}(\mathbf{q}, \omega)$ and the conductivity in Eq. (80) we can write Eq. (E3) as a sum rule for the real part of the longitudinal conductivity:

$$\int_0^\infty \sigma_1^L(\mathbf{q}, \omega)\, d\omega = \frac{\pi q^2 n}{2m}\,\frac{e^2}{q^2} = \frac{\pi n e^2}{2m} \,, \tag{E4}$$

or equivalently

$$\boxed{\int_0^\infty \sigma_1^L(\mathbf{q}, \omega)\, d\omega = \frac{\omega_p^2}{8} \,,} \tag{E5}$$

where the plasmon frequency, ω_p is given by Eq. (144).

For the longitudinal conductivity, the sum rule in Eq. (E5) holds for all $\mathbf{q}$. In fact, we are more often interested in the frequency dependent *transverse* conductivity, because this is what governs experimental measurements of the reflectivity. However, at $q = 0$ there is no distinction between longitudinal and transverse, and so the sum rule must also hold for the $q = 0$ transverse conductivity:

$$\boxed{\int_0^\infty \sigma_1^T(0, \omega)\, d\omega = \frac{\omega_p^2}{8} \,.} \tag{E6}$$

This is very useful in analyzing reflectivity data.

Next we consider the zero-frequency limit of the longitudinal paramagnetic current response function, evaluated in zero field. Eq. (C10) gives

$$\chi^L_{j_p}(\mathbf{q}, 0) = 2\sum_n P_n \sum_m \frac{|\langle n|j_p^L(\mathbf{q})|m\rangle|^2}{E_n - E_m} \,. \tag{E7}$$

Using the f-sum rule in the form in Eq. (B22), we find the surprisingly simple result

$$\boxed{\chi^L_{j_p}(\mathbf{q},0) = -\frac{n}{m}\,.} \tag{E8}$$

We will use this result in Sec. IV E.

APPENDIX F: SUM RULE FOR THE SCREENED RESPONSE

In Appendix E we derived a simple expression, Eq. (E2), for the first moment of the imaginary part of the unscreened density response function $\chi(\mathbf{q},\omega)$. Here we show that the same sum rule, Eq. (E3), holds for the unscreened density response function, $\chi_{\rm sc}(\mathbf{q},\omega)$. We used Eq. (E3) in the derivation of the important sum rule for the conductivity, Eq. (E5).

Now $\chi(\mathbf{q},\omega)$ (or equivalently $1/\epsilon(\mathbf{q},\omega)$) being the response to an external perturbation must satisfy *causality*, and hence, as shown in the books, must be an analytic function of complex ω in the upper half plane. This analyticity leads to Kramers-Kronig relations relating the real an imaginary parts of the response, see Eqs. (D3) and (D4). However, $\chi_{\rm sc}(\mathbf{q},\omega)$ (or equivalently $\epsilon(\mathbf{q},\omega)$) is a *construct*; it does not measure the response to an external field but to a *screened* field including fields generated by the system. Hence one cannot assume causality in determining the analytic its properties. Nonetheless more complicated arguments, e.g. p. 206–209 in Pines and Nozières[4], lead to the same conclusion as for $\chi(\mathbf{q},\omega)$, namely $\chi_{\rm sc}(\mathbf{q},\omega)$ is an analytic function of complex ω in the upper half plane. Hence $\chi_{\rm sc}(\mathbf{q},\omega)$ also satisfies Kramers-Kronig relations.

If we take Eq. (D3) at large ω and use the fact that $\chi''(\mathbf{q},\omega')$ is an odd function of ω' we get

$$\chi'(\mathbf{q},\omega) = \frac{1}{\pi}\int_0^\infty \chi''(\mathbf{q},\omega')\left(\frac{1}{\omega'-\omega} - \frac{1}{-\omega'-\omega}\right) d\omega' \approx -\frac{2}{\pi\omega^2}\int_0^\infty \omega'\chi''(\mathbf{q},\omega')d\omega' = \frac{q^2 n}{m\omega^2}\,, \tag{F1}$$

for $\omega \to \infty$, where we used the sum rule, Eq. (E2), to obtain the last equality. Thus we see that

this sum rule is actually just a consequence of the behavior of the *real* part of $\chi(\mathbf{q},\omega)$ at large ω.

In the large-ω limit, $\chi(\mathbf{q},\omega)$ is very small and so, from Eq. (73), it follows that the dielectric constant is given by

$$\epsilon(\mathbf{q},\omega) \approx 1 - \frac{\omega_p^2}{\omega^2}\,. \tag{F2}$$

We derived this expression in the RPA in Sec. IV C, Eq. (143), for $\mathbf{q} = 0$, and now we see that it is *exactly* true for $\omega \to \infty$.

By comparing Eq. (73) with Eq. (67) we see that, in the same large ω limit where $\epsilon(\mathbf{q},\omega)$ is close to unity, $\chi'_{\rm sc}(\mathbf{q},\omega) = \chi'(\mathbf{q},\omega)$, *i.e.*

$$\chi'_{\rm sc}(\mathbf{q},\omega) \approx \frac{q^2 n}{m\omega^2}\,. \tag{F3}$$

Since, as discussed above, $\chi_{\rm sc}(\mathbf{q},\omega)$ also satisfies Kramers-Kronig relations,

$$\chi'_{\rm sc}(\mathbf{q},\omega) \;=\; \mathcal{P}\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\chi''_{\rm sc}(\mathbf{q},\omega')}{\omega'-\omega}\,d\omega', \tag{F4}$$

$$\chi''_{\rm sc}(\mathbf{q},\omega) \;=\; -\mathcal{P}\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\chi'_{\rm sc}(\mathbf{q},\omega')}{\omega'-\omega}\,d\omega', \tag{F5}$$

we can reverse the steps in Eq. (F1) to get (for large ω)

$$\frac{q^2 n}{m\omega^2}\approx\chi'_{\rm sc}(\mathbf{q},\omega)=\frac{1}{\pi}\int_0^{\infty}\chi''_{\rm sc}(\mathbf{q},\omega')\left(\frac{1}{\omega'-\omega}-\frac{1}{-\omega'-\omega}\right)d\omega'=-\frac{2}{\pi\omega^2}\int_0^{\infty}\omega'\chi''_{\rm sc}(\mathbf{q},\omega')d\omega'\,, \tag{F6}$$

so we obtain Eq. (E3).

APPENDIX G: SUMMARY OF RELATIONSHIPS BETWEEN THE LINEAR RESPONSE FUNCTIONS

This appendix summarizes the relationships between the different linear response functions.

The (screened) density response is defined by

$$\langle\rho_{\rm int}(\mathbf{q},\omega) = e^2\,\chi_{\rm sc}(\mathbf{q},\omega)\,V(\mathbf{q},\omega)\,. \tag{G1}$$

The conductivity is defined by Ohm's law:

$$\langle J_{{\rm int}\,\mu}\rangle = \sigma_{\mu\nu}(\mathbf{q},\omega)E_\nu\,, \tag{G2}$$

where $\sigma_{\mu\nu}(\mathbf{q},\omega)$ is the conductivity tensor. For an isotropic medium the only independent components of $\sigma_{\mu\nu}(\mathbf{q},\omega)$ are the longitudinal and transverse components, *i.e.*

$$\sigma_{\mu\nu}(\mathbf{q},\omega) \;=\; \mathcal{P}^L_{\mu\nu}(\mathbf{q})\sigma^L(\mathbf{q},\omega)+\mathcal{P}^T_{\mu\nu}(\mathbf{q})\sigma^T(\mathbf{q},\omega), \tag{G3}$$

$$\;=\; \frac{q_\mu q_\nu}{q^2}\,\sigma^L(\mathbf{q},\omega)+\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\sigma^T(\mathbf{q},\omega)\,. \tag{G4}$$

Writing the electric field in terms of its longitudinal and transverse components we see that the longitudinal and transverse responses don't mix (still assuming that the system is isotropic):

$$\langle\mathbf{J}^{(L,T)}_{\rm int}(\mathbf{q},\omega)\rangle = \sigma^{(L,T)}(\mathbf{q},\omega)\,\mathbf{E}^{(L,T)}(\mathbf{q},\omega)\,, \tag{G5}$$

The (screened) current response also has longitudinal and transverse components, like the conductivity, which are defined by

$$\langle\mathbf{J}^{(L,T)}_{\rm int}(\mathbf{q},\omega)\rangle = -\frac{e^2}{c}\,\chi^{(L,T)}_{\rm sc,\,j}(\mathbf{q},\omega)\,\mathbf{A}^{(L,T)}(\mathbf{q},\omega)\,, \tag{G6}$$

We have shown that the dielectric constant, conductivity, and (screened) current response, are related (for both the longitudinal and transverse cases) by

$$\epsilon^{(L,T)}(\mathbf{q},\omega) \;=\; 1 + \frac{4\pi i}{\omega}\,\sigma^{(L,T)}(\mathbf{q},\omega), \tag{G7}$$

$$\epsilon^{(L,T)}(\mathbf{q},\omega) \;=\; 1 - \frac{4\pi e^2}{\omega^2}\,\chi^{(L,T)}_{\mathrm{sc,\,j}}(\mathbf{q},\omega). \tag{G8}$$

where the total current response $\chi_{\mathrm{sc,\,j}}(\mathbf{q},\omega)$ is related to the paramagnetic current response (the part of the current that is independent of $\mathbf{A}$) by

$$\chi^{(L,T)}_{\mathrm{sc,\,j}}(\mathbf{q},\omega) = \frac{n}{m} + \chi^{(L,T)}_{\mathrm{sc,\,j_p}}(\mathbf{q},\omega)\,, \tag{G9}$$

and so Eq. (G8) can be written as

$$\epsilon^{(L,T)}(\mathbf{q},\omega) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{4\pi e^2}{\omega^2}\,\chi^{(L,T)}_{\mathrm{sc,\,j_p}}(\mathbf{q},\omega). \tag{G10}$$

In addition, for the longitudinal case only, there is a relationship between $\epsilon(\mathbf{q},\omega)$ and the (screened) density response given by

$$\epsilon^{L}(\mathbf{q},\omega) = 1 - \frac{4\pi e^2}{q^2}\,\chi_{\mathrm{sc}}(\mathbf{q},\omega). \tag{G11}$$

In these expressions, $\epsilon^L(\mathbf{q},\omega)$ is the ratio of the total longitudinal electric field to the external field, Eq. (69), while for the transverse case the ratio of total field to the external field (either electric or magnetic) is only equal to $\epsilon^T(\mathbf{q},\omega)$ at $\mathbf{q}=0$. For $\mathbf{q}\neq 0$ the ratio is more complicated because the system couples to electromagnetic waves and the result is given by Eq. (98) or equivalently (103).

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[1] *Classical Electrodynamics* by D. J. Jackson, Wiley (1975).

[2] *Introduction to Electrodynamics* by David G. Griffiths, Prentice Hall, 1989.

[3] *Electrodynamics of Solids* by Martin Dressel and George Grüner, Cambridge University Press (2002).

[4] *The Theory of Quantum Liquids: Volume 1, Normal Fermi Liquids* by David Pines and Philippe Nozières, Addison Wesley (1988).

[5] *Quantum Theory of the Electron Liquid* by Gabriele Giuliani and Giovanni Vignale, Cambridge University Press (2005).

[6] *Quantum Theoretical Methods in Statistical Physics* by A. A. Abrikosov, L. P. Gorkov, and I. Y. Dzyaloshinskii.

[7] *Quantum Mechanics* by L. D. Lifshitz and E. M. Lifshitz.