PHYSICS 115/242

Homework 5

Due in class, Monday, May 9.

1. *Central Limit Theorem*

   (a) With x_i a random number with a uniform distribution between 0 and 1 show analytically that the distribution of X, where

   $$X = \sqrt{\frac{12}{N}} \sum_{i=1}^{N} (x_i - 1/2),$$

   has zero mean and variance unity.

   (b) For a suitable choice of N (12 is convenient but not essential) verify numerically that X has (to a good approximation) a *Gaussian* distribution by computing the first 6 moments for a sufficiently large sample,
   i.e. Generate a large number of values for X and show that for your sample $\langle X \rangle \simeq \langle X^3 \rangle \simeq \langle X^5 \rangle \simeq 0, \langle X^2 \rangle \simeq 1, \langle X^4 \rangle \simeq 3, \langle X^6 \rangle \simeq 15$.
   Comment: If you use $N = 12$ you will find that $\langle x^6 \rangle$ is too low (because the tails of the distribution are not well represented). This can be rectified by using a larger value for N.

2. *Lorentzian distribution*

   (a) Explain how to generate random numbers with a Lorentzian distribution

   $$P(x) = \frac{1}{\pi} \frac{1}{1 + x^2}, \qquad (-\infty < x < \infty).$$

   (b) Generate a (fairly large) sample of numbers with this distribution (don't print them out!) and calculate what fraction lie in the range $|x| < 1$. Compare your numerical result with the exact answer.

   (c) In part (b) I didn't ask you to calculate moments. Why not?

3. *Monte Carlo Integration with error bars*

   (a) Evaluate the following integral

   $$\int_1^2 \ln x \, dx$$

   using Monte Carlo integration. You *must* give an error bar (obtained from the computation) for your answer. Do one run with a large number of points (N), and from this *one* set of data estimate the error bar in your answer.

   (b) The error bar is one standard deviation, so, assuming that you have enough points for the central limit theory to apply in which case the distribution of the sample mean is Gaussian, there is a probability of 68% that the exact result lies in within σ of the exact answer, 95.5% probability within $2\,\sigma$, 99.8% probaility within $3\,\sigma$, etc. (where here σ refers generically to the standard deviation of the Gaussian).

Evaluate the integral analytically, compare your result from Monte Carlo integration with the exact result, and comment.

4. *Multi-dimensional Monte Carlo Integral*
   Consider the following 10-dimensional integral

$$I = \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_{10}\, (x_1 + x_2 + \cdots + x_{10})^2 .$$

   (a) Show that the exact answer is 155/6.
   ** The hint is not necessary. By integrating each term it is easy to show that $I_N = N/3 + N(N-1)/2$. *Hint:* If we define

$$I_N = \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_N\, (x_1 + x_2 + \cdots + x_N)^2$$

   show that

$$I_N = \tfrac{1}{3} + \tfrac{1}{2}(N-1) + I_{N-1}.$$

   (b) Estimate the answer numerically using a Monte Carlo method. Obtain the error bar on your estimate and compare with the exact answer.

5. *Random Walk*
   Consider a random walker who starts at $x = 0$ and walks along a line along the x-axis. At each time step, $t = 1, 2, 3, \cdots$, the walker moves one step to the right or one step to the left with equal probability. By averaging over a sufficiently large number of walks show numerically that

$$\langle x(t)\rangle \simeq 0; \quad \langle x^2(t)\rangle \simeq t$$

   where the average $\langle \cdots \rangle$ is over your sample of walks. You should plot (or produce a neat table of) $\langle x(t)\rangle$ and $\langle x^2(t)\rangle$ against t for a range of t.