Physics 115/242
The Kepler Problem

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I. INTRODUCTION

We consider motion of a planet around the sun, the Kepler problem, see e.g. Garcia, "Numerical Methods for Physics", Sec. 3.1.

Let the planet have mass m and velocity $\vec{v}$, and the sun have mass M. The total energy is given by

$$E = \frac{1}{2} m v^2 - \frac{GMm}{r}, \tag{1}$$

where $\vec{r}$ is the distance from the sun to the planet. The energy is conserved, as is the angular momentum:

$$\vec{L} = m \vec{r} \times \vec{v}. \tag{2}$$

The motion is in a plane so the only non-zero compoment is

$$L_z \equiv L = m(x v_y - y v_x)\,. \tag{3}$$

The force on the planet is given by

$$\vec{F}(\vec{r}) = -\frac{GMm}{r^2}\,\hat{r}\,. \tag{4}$$

The negative sign indicates that this is an attractive (inwards) force.

II. CIRCULAR MOTION

For the special case of circular motion the sum of the gravitational force in Eq. (4) and the centripetal force $m v_{\rm circ}^2 / r_{\rm circ}$ is zero, i.e.

$$\frac{m v_{\rm circ}^2}{r_{\rm circ}} = \frac{GMm}{r_{\rm circ}^2}, \tag{5}$$

so

$$v_{\rm circ} = \sqrt{\frac{GM}{r_{\rm circ}}}\,. \tag{6}$$

It is easy to check that the potential energy is minus twice the kinetic energy so the total energy is

$$E_{\mathrm{circ}} = -\frac{1}{2}\frac{GMm}{r_{\mathrm{circ}}}. \tag{7}$$

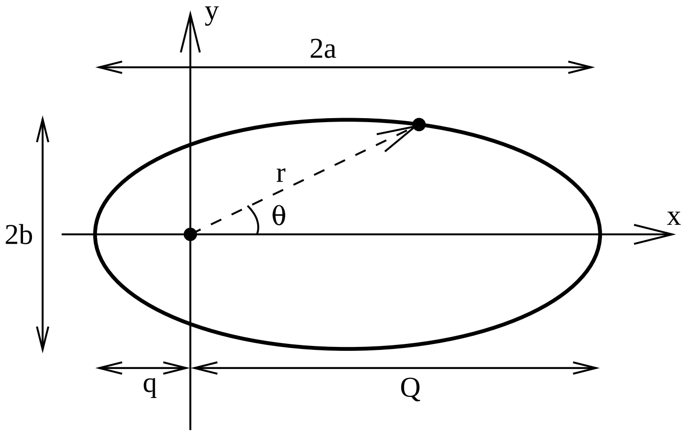
The angular momentum, L_{circ}, and period of the circular orbit, T_{circ}, are given by

$$L_{\mathrm{circ}} = m v_{\mathrm{circ}} r = m\sqrt{GM r_{\mathrm{circ}}} \tag{8}$$

$$T_{\mathrm{circ}} = 2\pi \frac{r_{\mathrm{circ}}}{v_{\mathrm{circ}}} = \frac{2\pi}{\sqrt{GM}}\, r_{\mathrm{circ}}^{3/2}\,. \tag{9}$$

III. THE GENERAL CASE: ELLIPTICAL MOTION

In general the motion is not circular but an ellipse with the sun at one focus, see the figure below.



If a and b are the semi-major and semi-minor axes ($b \le a$) then the eccentricity, ϵ, is defined to be

$$\epsilon = \sqrt{1 - \frac{b^2}{a^2}}. \tag{10}$$

We now quote without proof that the formula for the ellipse is

$$r(\theta) = \frac{a(1-\epsilon^2)}{1 - \epsilon\cos\theta}. \tag{11}$$

Hence, the perihelion (distance at closest approach) is given by

$$q = (1-\epsilon)a, \tag{12}$$

while the aphelion (distance when furthest away) is given by

$$Q = (1+\epsilon)a. \tag{13}$$

We also quote without proof that the eccentricity, angular momentum and energy are related by

$$\epsilon = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}. \tag{14}$$

From Eqs. (7) and (8) we see that Eq. (14) correctly gives $\epsilon = 0$ for a circular orbit.

One can also show that the period is related to the semi-major axis a by

$$T = 2\pi \frac{a^{3/2}}{\sqrt{GM}} \qquad \left[= \frac{2\pi}{\sqrt{GM}} \left(\frac{Q}{1+\epsilon} \right)^{3/2} \right], \tag{15}$$

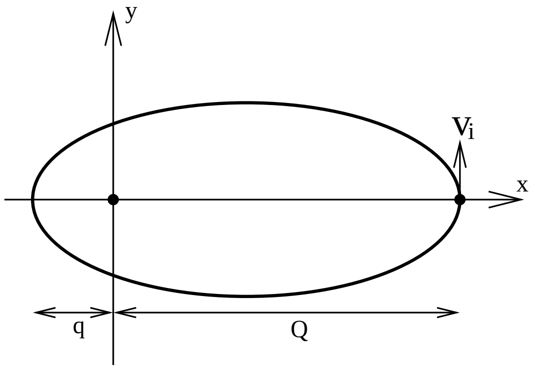
which is known as Kepler's third law. Furthermore, the energy can also be expressed in terms of a by

$$E = -\frac{1}{2}\frac{GMm}{a}\,. \tag{16}$$

Comparing the general expressions for the energy and period in Eqs. (16) and (15) with those for a circular orbit in Eqs. (7) and (9), we see that they are the same except that $r_{\rm circ}$ is replaced by a in the general case.

IV. RELATION TO SPEED AT APHELION

In the numerical problems in the homework, we start the planet at the aphelion with speed v_i, see the figure below.



Hence we have

$$L = mQv_i, \qquad E = \frac{1}{2}mv_i^2 - \frac{GMm}{Q}\,. \tag{17}$$

The expression for the eccentricity, Eq. (14), can then be written in the following simple form

$$\epsilon = 1 - \frac{v_i^2}{v_{\rm circ}^2}, \tag{18}$$

where v_{circ} is given by Eq. (6) with $r = Q$, and we have assumed $v_i \le v_{\mathrm{circ}}$ (so that the planet starts at the aphelion rather than the perihelion). Similarly, from Eqs. (11) and (13) we can write

$$\frac{r(\theta)}{Q} = \frac{1-\epsilon}{1-\epsilon\cos\theta}, \tag{19}$$

and, from Eqs. (9) and (15),

$$\frac{T}{T_{\mathrm{circ}}} = \frac{1}{(1+\epsilon)^{3/2}}\,. \tag{20}$$

V. NUMERICS

In the numerical calculations we set $m = M = G = Q = 1$. From the above results it follows that

$$\begin{aligned} v_{\mathrm{circ}} &= 1, & (21)\\ T_{\mathrm{circ}} &= 2\pi, & (22)\\ \epsilon &= 1 - v_i^2, & (23)\\ T &= \frac{2\pi}{(2-v_i^2)^{3/2}} = \frac{2\pi}{(1+\epsilon)^{3/2}}, & (24)\\ r &= \frac{1-\epsilon}{1-\epsilon\cos\theta}\,. & (25) \end{aligned}$$

Orbits of different eccentricity are produced by different values of v_i in the region $0 < v_i \le 1$. For $v_i = 1$ the orbit is circular ($\epsilon = 0$). If we were to set set $v_i = 0$ ($\epsilon = 1$) the planet would fall into the sun and crash.

For $\epsilon \ll 1$, the orbit is close to circular and the speed is nearly constant. However, for $\epsilon \to 1$, the ellipse becomes more and more “squashed”, and the magnitude and direction of the velocity change rapidly when the planet approaches the sun. This requires a smaller time step to preserve accuracy. In fact, for $\epsilon \to 1$, it would be better to use adaptive step-size control so the time-step would be automatically reduced when the planet gets close to the sun and increased again when the planets moves away. (However, adaptive step-size control will not be needed for the homework.)