Physics 115/242

Comparison of methods for integrating the simple harmonic oscillator.

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I. THE SIMPLE HARMONIC OSCILLATOR

The energy (sometimes called the "Hamiltonian") of the simple harmonic oscillator is

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{1}$$

where m is the mass, k is the spring constant, and $p = m\dot{x}$ is the momentum. In the numerical examples we will set $m = k = 1$ so the angular frequency, ω and period, T, are given by

$$\omega = \sqrt{\frac{k}{m}} = 1, \qquad T = \frac{2\pi}{\omega} = 2\pi. \tag{2}$$

We also have

$$2E = p^2 + x^2 \qquad \text{(a const.)} \tag{3}$$

Hence a "phase space" plot, i.e. the trajectory in the x-p plane, should be a circle of radius $\sqrt{2E}$.

Following standard practice, Newton's equation of motion

$$\ddot{x} = -x \tag{4}$$

will be written as two first order differential equations

$$\dot{x} = p, \tag{5}$$

$$\dot{p} = -x. \tag{6}$$

We will numerically integrate these equations for three methods that have been described in class:

- Euler method,
- second order Runge Kutta (RK2),
- fourth order Runge Kutta (RK4),

with initial conditions, $x = 1, p = 0$. Hence $2E = 1$ and the radius of the circle in the phase space plots is unity. We will use a time step $h = 0.02T$ so it takes 50 time steps to go perform one cycle of the oscillator.

The formulae for stepping forward in time the differential equations (5) and (6) are:

- Euler method

$$\begin{aligned} x_{n+1} &= x_n + hp_n, \\ p_{n+1} &= p_n - hx_n, \end{aligned}$$

- second order Runge Kutta (RK2)

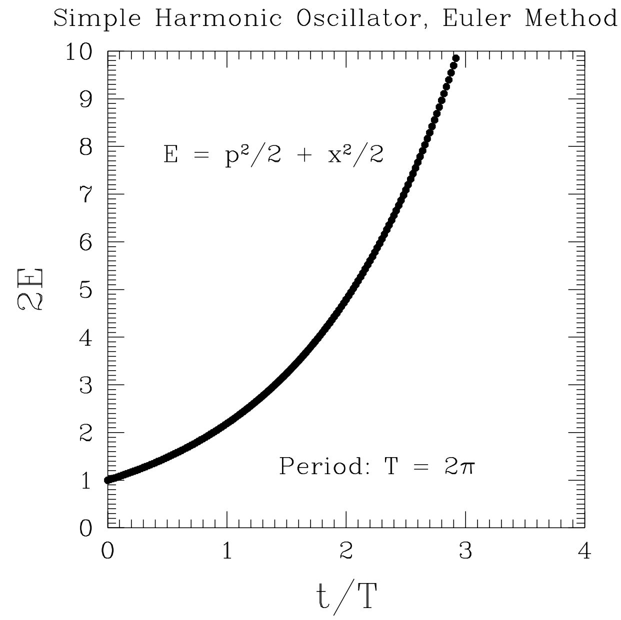
$$\begin{aligned} k_1^x &= p_n, \qquad\qquad k_1^p = -x_n, \\ k_2^x &= p_n + hk_1^p, \qquad k_2^p = -(x_n + hk_1^x), \\ x_{n+1} &= x_n + \frac{h}{2}(k_1^x + k_2^x), \\ p_{n+1} &= p_n + \frac{h}{2}(k_1^p + k_2^p), \end{aligned}$$

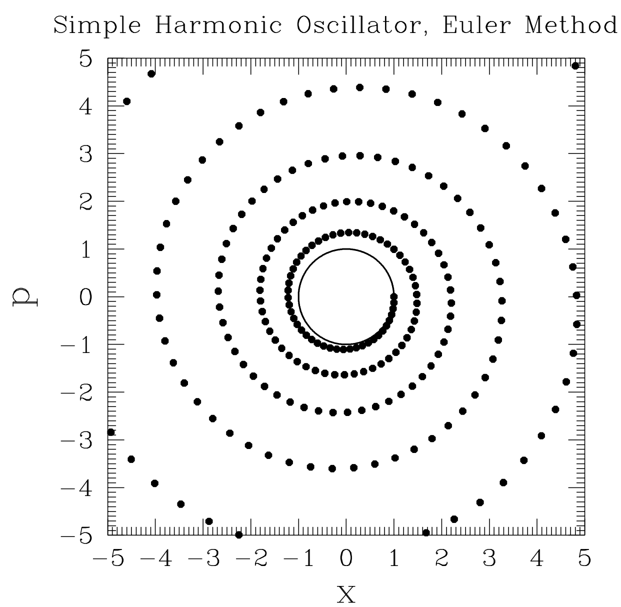
- fourth order Runge Kutta (RK4)

$$\begin{aligned} k_1^x &= p_n, \qquad\qquad k_1^p = -x_n, \\ k_2^x &= p_n + \frac{h}{2}k_1^p, \qquad k_2^p = -(x_n + \frac{h}{2}k_1^x), \\ k_3^x &= p_n + \frac{h}{2}k_2^p, \qquad k_3^p = -(x_n + \frac{h}{2}k_2^x), \\ k_4^x &= p_n + hk_3^p, \qquad k_4^p = -(x_n + hk_3^x), \\ x_{n+1} &= x_n + \frac{h}{6}(k_1^x + 2k_2^x + 2k_3^x + k_4^x), \\ p_{n+1} &= p_n + \frac{h}{6}(k_1^p + 2k_2^p + 2k_3^p + k_4^p). \end{aligned}$$

II. EULER METHOD

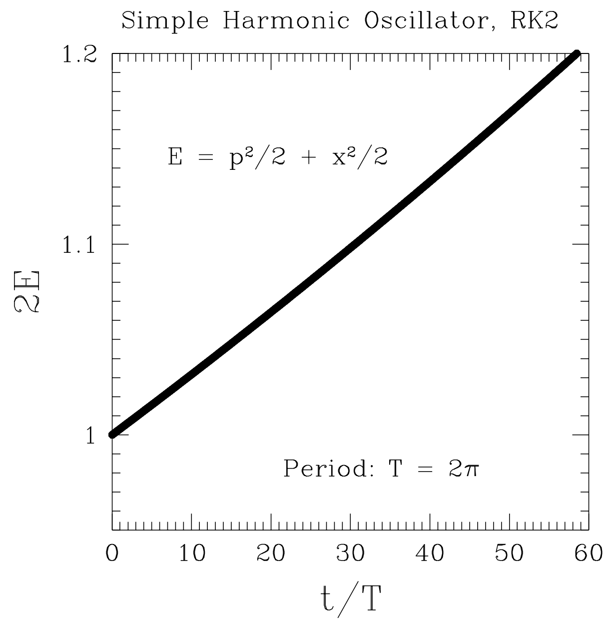
The figures show that the energy very quickly deviates from its correct value and grows without bound. The phase space plot is badly in error even after one cycle. This illustrates that the Euler method is terrible and so **I don't recommend its use.**

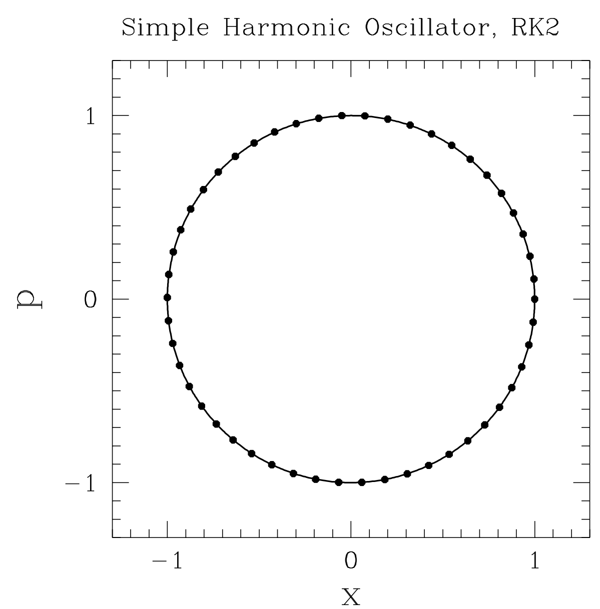




III. SECOND ORDER RUNGE-KUTTA (RK2)

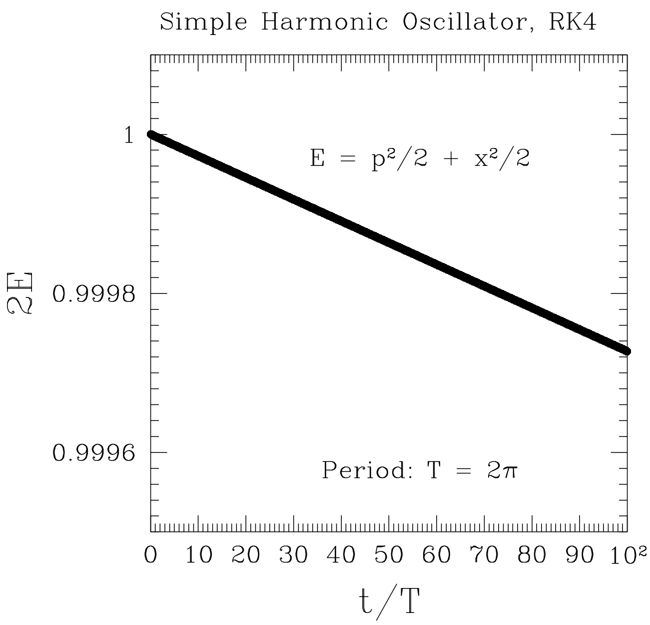
The first figure shows that the energy deviates from its exact value much more slowly than with the Euler method and the phase space plot shows that one cycle is tracked pretty accurately, within the thickness of the lines. (Remember the exact phase space plot is a circle of radius unity.) Hence, if you want a simple scheme, **use RK2 but not Euler**.





IV. FOURTH ORDER RUNGE-KUTTA (RK4)

This figure, which has a highly blown up scale on the vertical axis, shows that RK4 keeps the energy constant to very high precision. All in all, **RK4 is very accurate** but quite simple and so is the method of choice for many people. Combined with "adaptive stepsize control" (not necessary for the simple harmonic oscillator, and not covered in this course) it is very powerful. A phase space plot (not shown) looks essentially perfect.



V. ANHARMONIC OSCILLATOR

We conclude by showing some results for an anharmonic oscillator using the RK2 method. We take the potential energy to be

$$V(x) = \frac{x^6}{6}\,, \tag{7}$$

which is close to zero for $|x| < 1$ and then has very steep (almost vertical) walls at $x = \pm 1$. Hence the particle will travel with almost constant velocity for $|x| < 1$, and will rebound suddently when it gets to $x = \pm 1$. The figures below show that this expected behavior is well reproduced by RK2.