PHYSICS 250
Homework 7

Due in class, Monday November 26.

1. Consider the following first order differential equation:

$$\frac{dx}{dt} + \gamma x(t) = F(t)\,,$$

with $\gamma > 0$. In the absence of the "forcing term", $F(t)$, the (homogeneous) equation, $x'(t)+\gamma x(t) = 0$, has solution

$$x(t) = Ae^{-\gamma t}\,,$$

where A is a constant. As discussed in class, the solution of the inhomogeneous equation (i.e. including the forcing term) can be written as

$$x(t) = \int_{-\infty}^{\infty} G(t-t')F(t')\,dt'\,,$$

where the Green function $G(t)$ satisfies

$$\frac{dG}{dt} + \gamma\, G(t) = \delta(t)\,.$$

We assume the boundary condition that x is zero before the "force" $F(t)$ starts to act and so $G(t) = 0$ for $t < 0$.

(a) Determine $G(t)$ *either* by working in the time domain, and determining the value of the constant A above needed to get the correct discontinuity at $t = 0$ due to the delta function, *or* use Fourier transforms (which automatically puts in the boundary condition that the solution vanishes at $t = \pm\infty$).

(b) Denoting the Fourier transform of $G(t)$ by $\chi(\omega)$, determine $\chi'(\omega)$ and $\chi''(\omega)$, the real and imaginary parts, and verify, by doing the integrals, that they are related by Kramers-Kronig relations

$$\begin{aligned}\chi''(\omega) &= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi'(\omega')}{\omega'-\omega}\,d\omega'\,,\\ \chi'(\omega) &= -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi''(\omega')}{\omega'-\omega}\,d\omega'\,.\end{aligned}$$

Note: Remember that the Kramers-Kronig relations are simply the result of causality, $G(t) = 0$ for $t < 0$) (and the fact that $\chi(\omega) \to 0$ for $|\omega| \to 0$).

2. Kelly, Qu. 7, Ch. 6.
Note: For part (c), you should find that

$$\begin{aligned}\overline{x}(t) &= \frac{\hbar k_0}{m}\,t\,,\\ \sigma^2(t) &= \sigma_0^2 + \left(\frac{\hbar t}{m\sigma_0}\right)^2\,.\end{aligned}$$

Note that at long times $\sigma(t) \propto t$. Physically, the reason for this is that different pieces of the wavefunction propagate with different velocities.

3. Kelly, Qu. 8, Ch. 6.

4. Kelly, Qu. 11, Ch. 6.
   Note: Certain materials have regions of frequency where the refractive index, $n(\omega)$, is less than unity, implying a phase velocity, $v_p = c/n(\omega)$, greater than c, the speed of light in vacuum. The point of this question is for you to show that, nonetheless, no signal can propagate faster than c, in agreement with special relativity.

5. Kelly, Qu. 12, Ch. 6.
   Note: For part (d) you are asked to *verify* that the given solution satisfies the equation provided certain relations between the parmeters are satisfied. You may used *Mathematica* to help you with this.

6. Kelly, Qu. 13, Ch. 6.
   Note: For part (c), you are only asked to *verify* the given solution, not to derive it "from first principles". You may use *Mathemtica* to help you, if you wish.