PHYSICS 250
Homework 8
Due in the last class, Friday December 7.

1. *Bookwork*
   For a differential equation of the Sturm-Liouville type show that
   (a) The eigenvalues are real.
   (b) The eigenfunctions are orthogonal.

2. *Variational Problem*
   You are familiar with the solution of the simple harmonic oscillator problem in quatum mechanics with Hamiltonian
   $$\mathcal{H} = -\frac{1}{2}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}x^2\psi(x) = E\psi(x)\,.$$
   Note that we have set $\hbar = m = \omega = 1$ where m is the mass and ω the classical frequency of oscillation.

   The ground state wavefunction $\psi_0(x)$ is actually a Gaussian, but let's suppose you don't know that. Suppose instead you guess that the groundstate wavefunction is actually an exponential
   $$\psi(x) = Ae^{-\kappa|x|}\,,$$
   where κ is a parameter and A is a normalization constant, which you should determine,

   Estimate the ground state energy by the variational approach, in which you vary κ.

   Compare your answer with the exact result.

3. Kelly, Qu. 8, Ch. 7.

4. Kelly, Qu. 13, Ch. 7.
   Note: For part (a) a precise discuss of the meaning of "self adjoint" is given in Sec. 7.3.1.

5. Consider the probability distribution
   $$P_1(x) = \begin{cases} 2(1-x), & (0 < x < 1), \\ 0, & (\text{otherwise})\,. \end{cases}$$
   (a) Determine the mean and standard deviation of P_1.
   (b) The skewness, s, of a distribution is defined by
   $$s = \langle (x - \langle x \rangle)^3 \rangle / \sigma^3\,,$$
   where σ^2 is the variance. Find the skewness for $P_1(x)$ and also for the distribution
   $$P_2(x) = \begin{cases} 1, & (0 < x < 1), \\ 0, & (\text{otherwise})\,. \end{cases}$$
   Note that the skewness vanishes for a distribution, like P_2, which is symmetric about its mean.

6. A radioactive object has, *on average* 5 decays per second. What is the probability of detecting
   (a) 0

(b) 1

(c) 5

(d) 20

decays in a second.

7. For a Gaussian distribution show that the probability that a result is obtained within $k\sigma$ from the mean is $\mathrm{erf}(k/\sqrt{2})$ where $\mathrm{erf}(x)$ is the error function.

8. A casino claims that there is a 50/50 chance of winning at a particular game. I play 100 times and lose 70 times.

   (a) Show that this corresponds to 4 times the standard deviation if the probability of winning is really 50%.

   (b) We showed in class that the distribution for this problem (the binomial distribution) goes over to a Gaussian distribution for a large number of trials. Since 100 is quite a large number we assume that this is a good approximation here. From the previous question, we see that the probabililty of a $\pm 4\sigma$ deviation, or greater, is $\mathrm{erfc}(4/\sqrt{2})$. You are given that $\mathrm{erfc}(4/\sqrt{2}) = 0.00006334$. Do you think that the casino was being honest in claiming a 50/50 chance of winning?

9. *An example where the central limit theorem doesn't hold*
   Consider the Lorentzian distribution

$$P(x) = \frac{1}{\pi} \frac{1}{1 + x^2} \, .$$

   Note that the standard devation does not exist (and hence the central limit theorem does not apply.)

   (a) By computing the Fourier transform, as discussed in class, obtain the distribution of the sum of N random variables with a Lorentzian distribution

   (b) Hence show that the distribution of the mean of N variables is *precisely the original Lorentzian distribution.*

   Hence, in contrast to cases where the central limit theorem applies, the distribution of the sum does *not* become sharper than the distribution of an individual variable.

10. I carry out the following series of measurements for a particular quantity, x, which is subject to random noise:

$$1.1, \; 0.9, \; 0.95, \; 1.05, \; 1.0 \, .$$

   What is my best estimate for $\langle x \rangle$ and what is the error bar on this estimate?