The Sine Gordon Equation

In[1]:= Clear["Global`*"]

■ The Equation

The equation is

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \sin u
\]

In[2]:= sinegordon\_eq[u_] := D[u[x, t], {x, 2}] - D[u[x, t], {t, 2}] == Sin[u[x, t]] // FullSimplify

Without the right hand side we would have the wave equation with the wave speed, \(c\), set to unity.

■ One Soliton Solutions

We display a soliton solution; first of all for the special case of no time dependence. We verify that the equation is satisfied by giving the command sinegordon\_eq and seeing that the result is True.

In[3]:= u[x_, t_] = 4 ArcTan[Exp[x]]; u[x, t] // TraditionalForm

Out[3]//TraditionalForm=

\[4 \tan^{-1}(e^{x})\]

In[4]:= sinegordon\_eq[u]


From the plot below we see that \(u\) varies between 0 (for large negative \(x\)) and \(2\pi\) (for large positive \(x\)).

In[5]:= Plot[{u[x, 0], 2 \[Pi]}, {x, -4, 4}, PlotRange -> {0, 6.5}, PlotStyle ->

\{Hue[0], AbsoluteThickness[2]}, {Hue[0.7], AbsoluteThickness[2], Dashing[\{0.02, 0.02\}]}]]

Out[5]=

A more general solution with time dependence is

\[u(x, t) = 4 \tan^{-1}\left[\exp\left( x - vt \right) \sqrt{1-v^2} \right]\]
which shows that the soliton travels with velocity \( v \). Note that \( 1/\sqrt{1 - v^2} \) is a "Lorentz contraction" factor. We verify again that this satisfies the equation.

\[
\ln(6)\leftarrow u[x_\_, t\_] := 4 \text{ArcTan}\left[\exp\left(\frac{x - v t}{\sqrt{1 - v^2}}\right)\right]
\]

\[
\ln(7)\leftarrow \text{sinegordoneq}[u]
\]

\[
\text{Out}(7)\leftarrow \text{True}
\]

Now we choose \( v = 0.8 \) and produce a simple animation of the soliton. We integrate up to time \( t = 20 \), during which the soliton moves a distance \( x = 16 \).

\[
\ln(8)\leftarrow v = 0.8;
\]

\[
\ln(9)\leftarrow \text{Animate}[\text{Plot}[u[x_, t], \{x, -5, 22\}, \text{PlotRange} \to \{0, 6.5\}], \{t, 0, 20, 0.25\}]
\]

![Image of soliton animation]

- **The "Breather" Solution**

There are also multi-soliton solutions, which we don’t show, and a "breather" solution, a bound soliton-anti soliton pair, which oscillates and stays close to the origin. Here it is:

\[
\ln(10)\leftarrow ub[x_\_, t\_] = 4 \text{ArcTan}\left[\sin[t/\sqrt{2}] / \cosh[x/\sqrt{2}]\right]; \text{ub}[x, t] /\text{TraditionalForm}
\]

\[
\text{Out}(10)/\text{TraditionalForm}\leftarrow 4 \tan^{-1}\left(\text{sech}\left(\frac{x}{\sqrt{2}}\right) \sin\left(\frac{t}{\sqrt{2}}\right)\right)
\]

We verify that it satisfies the sine Gordon equation

\[
\ln(11)\leftarrow \text{sinegordoneq}[ub]
\]

\[
\text{Out}(11)\leftarrow \text{True}
\]

and show an animation of it (the period is \( 2 \sqrt{2} \pi \)). Note that the largest range of \( u \) is at \( x=0 \) where \( u \) varies between \( -\pi \) and \( \pi \).
In[12]:=
Animate[Plot[ub[x, t], {x, -6, 6}, PlotRange -> {-3.2, 3.2}],
{t, 0, NSqrt[2 Pi], NSqrt[2 Pi]/80}]

Out[12]=

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