

# Physics 250

## Singular Fourier transforms and the Integral Representation of the Dirac Delta Function

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### I. INTRODUCTION AND FOURIER TRANSFORM OF A DERIVATIVE

One can show that, for the Fourier transform

$$g(k) = \int_{-\infty}^{\infty} f(x)e^{ikx} dx \quad (1)$$

to converge as the limits of integration tend to  $\pm\infty$ , we must have  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Roughly, if  $f(x)$  varies slowly at large  $|x|$ , the integral over one period of the  $e^{ikx}$  is proportional to  $f'(x)$ . If  $|f(x)| \rightarrow 0$  smoothly as  $|x| \rightarrow \infty$  then  $|f'(x)| \rightarrow 0$  faster than  $1/|x|$  and the sum over the periods is a convergent series. (Note that Jordan's lemma, which we use when we do this type of integral by contour integration, also requires only that  $|f(x)| \rightarrow 0$ .)

Another result depending on the vanishing of  $f(x)$  for  $|x| \rightarrow \infty$  is the expression for the Fourier Transform of a derivative. From Eq. (1), if  $g(k)$  is the Fourier Transform of  $f(x)$  and  $g_1(k)$  is the transform of  $f'(x)$  we have

$$g_1(k) = \int_{-\infty}^{\infty} f'(x)e^{ikx} dx = \left[ f(x)e^{ikx} \right]_{-\infty}^{\infty} - ik \int_{-\infty}^{\infty} f(x)e^{ikx} dx \quad \boxed{= -ikg(k)}, \quad (2)$$

where we integrated by parts and, in the last expression, assumed that  $f(x)$  vanishes at  $\pm\infty$ .

Hence *standard* Fourier transforms only apply to functions which vanish at infinity.

Nonetheless, Fourier transforms are so useful that it is desirable to apply them to some functions which do not satisfy this condition. These transforms are known as "*singular* Fourier transforms" and will need some form of "*regularization*" to make the integrals converge.

### II. A SINGULAR FOURIER TRANSFORM INVOLVING A DELTA FUNCTION

As an example consider  $f(x) = 1$ . In order that the Fourier transform  $g(k)$  exists, we regularize the integral by putting in the "convergence factor"  $e^{-\epsilon|x|}$  where  $\epsilon$  is small and positive. Hence we determine the Fourier transform of

$$f_\epsilon(x) = e^{-\epsilon|x|}, \quad (3)$$

which is

$$g_\epsilon(k) = \int_{-\infty}^{\infty} e^{ikx} e^{-\epsilon|x|} dx.$$

We separate the integral into the negative- $x$  region and the positive- $x$  region to find

$$g_\epsilon(k) = \int_{-\infty}^0 e^{ikx} e^{\epsilon x} dx + \int_0^{\infty} e^{ikx} e^{-\epsilon x} dx = \frac{1}{ik + \epsilon} + \frac{1}{-ik + \epsilon} = \frac{2\epsilon}{\epsilon^2 + k^2}. \quad (4)$$

For  $\epsilon \rightarrow 0$   $g_\epsilon(k)$  becomes a narrow high peak, the area under which is

$$\int_{-\infty}^{\infty} \frac{2\epsilon}{\epsilon^2 + k^2} dk = 2 [\tan^{-1}(k/\epsilon)]_{-\infty}^{\infty} = 2\pi.$$

It is therefore convenient to define a quantity  $\delta_\epsilon(k)$  by

$$\boxed{\delta_\epsilon(k) = \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + k^2}}, \quad (5)$$

which has unit area under it. Hence, for  $\epsilon \rightarrow 0^+$ ,  $\delta_\epsilon(k)$  is a representation of what is known as the Dirac delta function  $\delta(k)$ . This is an “infinitely high, infinitely narrow” peak with unit area under it. It is defined by the two relations

$$\delta(x) = 0, \quad (x \neq 0), \quad (6)$$

$$\int \delta(x) dx = 1, \quad (\text{if region of integration includes } x = 0). \quad (7)$$

From these, it is straightforward to prove the following results:

$$\int \delta(x - a) f(x) dx = f(a), \quad (8)$$

$$\delta(cx) = \frac{\delta(x)}{|c|}, \quad (9)$$

where the region of integration in Eq. (8) includes  $x = a$ . If you are unfamiliar with Eqs. (8) and (9), you should take the trouble to derive them.

From Eqs. (4) and (5) we have

$$\int_{-\infty}^{\infty} e^{ikx} e^{-\epsilon|x|} dx = 2\pi\delta_\epsilon(k). \quad (10)$$

This is a Fourier transform, for which I use the following notation:

$$\boxed{e^{-\epsilon|x|} \xrightarrow{FT} 2\pi\delta_\epsilon(k)}. \quad (11)$$

The integral in Eq. (10) is well defined because the  $e^{-\epsilon|x|}$  factor ensures convergence.

Equations like Eq. (10) are generally used in situations when they are multiplied on both sides by a smooth function of  $k$ ,  $u(k)$  say, and integrated, *i.e.*

$$\int_{-\infty}^{\infty} u(k) \left[ \int_{-\infty}^{\infty} e^{ikx} e^{-\epsilon|x|} dx \right] dk = 2\pi \int u(k) \delta_{\epsilon}(k) dk. \quad (12)$$

It turns out that this equation is well behaved if  $\epsilon$  is set to zero on the LHS (and the limit  $\epsilon \rightarrow 0$  is taken on the RHS). This gives

$$\begin{aligned} \int_{-\infty}^{\infty} u(k) \left[ \int_{-\infty}^{\infty} e^{ikx} dx \right] dk &= 2\pi \int_{-\infty}^{\infty} u(k) \delta(k) dk \\ &= 2\pi u(0), \end{aligned} \quad (13)$$

(which is known as Fourier's integral). We used Eq. (8) to get the last expression. One is often tempted to set  $\epsilon$  to zero also in Eq. (10) (*i.e. without* multiplying it by a smooth function and integrating), in which case we write

$$\boxed{\int_{-\infty}^{\infty} e^{ikx} dx = 2\pi\delta(k)}. \quad (14)$$

However, as it stands, Eq. (14) *does not make sense* because the integral does not exist. We therefore have to understand Eq. (14) in one of the following two senses:

- As a stand-alone equation, in which case it has to be regularized by the convergence factor  $e^{-\epsilon|x|}$ , so Eq. (14) really means Eq. (10) for  $\epsilon$  *tending* to zero (but not strictly zero).
- Multiplied by a smooth function  $u(k)$  and integrated over  $k$  as in Eq. (12), in which case the convergence factor is unnecessary. Equation (14) is then really a shorthand for Eq. (13). This is normally the sense in which we understand Eq. (14).

Since Eq. (14) is a Fourier transform, we can write it as

$$\boxed{1 \xrightarrow{FT} 2\pi\delta(k)}, \quad (15)$$

which should be compared with Eq. (11). The inverse transform then gives

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(k) e^{-ikx} dk = 1,$$

as required.

### III. APPLICATIONS OF THE INTEGRAL REPRESENTATION OF THE DELTA FUNCTION

In this section we give some applications of the integral representation of the delta function, Eq. (14).

### A. Convolution Theorem

In class, we have already met the convolution theorem, that is, if

$$F(x) = \int_{-\infty}^{\infty} f(y)f(x-y) dy, \quad (16)$$

which is the convolution of  $f$  with itself, then  $G(k)$ , the Fourier transform of  $F(x)$ , is simply related to  $g(k)$ , the Fourier transform of  $f(x)$ , by

$$G(k) = g(k)^2. \quad (17)$$

We will now give a simple alternative derivation of this result using the integral representation of the delta function, and then use this method to obtain a *generalized* convolution theorem. We can write Eq. (16) as

$$F(x) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 f(x_1)f(x_2)\delta(x_1 + x_2 - x). \quad (18)$$

Using Eq. (14) we have

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dk f(x_1)f(x_2)e^{ik(x_1+x_2-x)}. \quad (19)$$

The integrals over  $x_1$  and  $x_2$  are now independent of each other and can be carried out, with the result that

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(t)e^{ikt} dt \right]^2 e^{-ikx} dk, = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k)^2 e^{-ikx} dk, \quad (20)$$

which shows that  $F(x)$  is the inverse transform of  $g(k)^2$ , *i.e.* that  $g(k)^2$  is the Fourier transform of  $F(x)$ . Hence we have obtained Eq. (17).

We can now generalize this result to the case where the convolution, rather than involving two variables as in Eq. (18), involves  $n$  variables,  $x_1, x_2, \dots, x_n$ , but with the same constraint that the sum must equal some prescribed value  $x$ , *i.e.*

$$\boxed{F(x) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n f(x_1) \cdots f(x_n) \delta(x_1 + \cdots + x_n - x)}. \quad (21)$$

Using the integral representation of the delta function, as before, the integrals over the  $x_i$  decouple, and we find

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(t)e^{ikt} dt \right]^n e^{-ikx} dk, = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k)^n e^{-ikx} dk, \quad (22)$$

which shows that the Fourier transform of  $F(x)$  is

$$\boxed{G(k) = g(k)^n}, \quad (23)$$

a remarkably simple result. Equation (23) is the desired generalization of the convolution theorem, Eq. (17), to  $n$  variables. We shall use this result in class to solve a problem in statistics.

*Note:* With the alternative definition of Fourier transforms, which puts in factors of  $\sqrt{2\pi}$  to make the transform and the inverse transform symmetric with respect to each other, the convolution theorem for  $n$  variables is

$$\sqrt{2\pi}G(k) = \left[ \sqrt{2\pi}g(k)^n \right]. \quad (24)$$

### B. Parseval's Theorem

Related to the convolution theorem is another useful theorem associated with the name of Parseval. (You may recall that there is a Parseval's theorem for Fourier series, which is actually closely related.)

Using the Fourier transform

$$g(k) = \int_{-\infty}^{\infty} f(x)e^{ikx} dx \quad (25)$$

we have

$$\int_{-\infty}^{\infty} g(k)g^*(k) dk = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dk f(x_1)f^*(x_2)e^{ik(x_1-x_2)}. \quad (26)$$

Doing the integral over  $k$  using Eq. (14) gives  $2\pi\delta(x_1 - x_2)$ , so

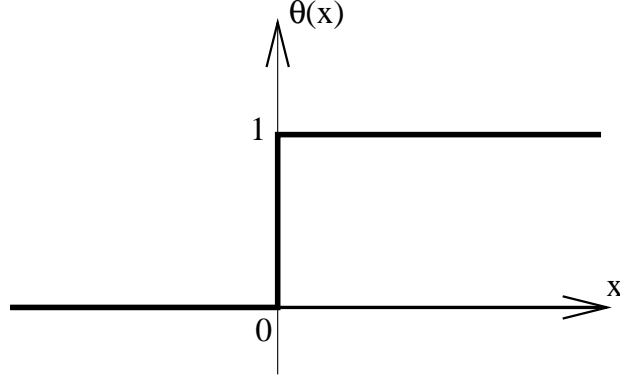
$$\begin{aligned} \int_{-\infty}^{\infty} |g(k)|^2 dk &= 2\pi \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 f(x_1)f^*(x_2)\delta(x_1 - x_2) \\ &= \boxed{2\pi \int_{-\infty}^{\infty} |f(x)|^2 dx}, \end{aligned} \quad (27)$$

which is Parseval's theorem.

*Note:* With the alternative definition of Fourier transforms, the factor of  $2\pi$  in Eq. (27) is missing, so there is complete symmetry between the two sides.

## IV. EXAMPLES OF SINGULAR FOURIER TRANSFORMS INVOLVING A STEP FUNCTION

It is also interesting to consider singular Fourier transforms of functions involving the (Heaviside) step function



$$\theta(x) = \begin{cases} 0, & (x < 0) \\ 1, & (x > 0) \end{cases},$$

which is denoted  $H(x)$  in the book. Putting in the convergence factors, the Fourier transform is just given by the  $x > 0$  part of the transform of unity in Eq. (4), *i.e.* it is given by  $(-ik + \epsilon)^{-1}$  with  $\epsilon \rightarrow 0^+$ , which we write as

$$\boxed{\theta(x) \xrightarrow{FT} \frac{i}{k + i\epsilon}} \quad (28)$$

This equation is to be understood in the same sense as Eq. (14), which is described in the two “bullets” after that equation. The small imaginary part in the denominator gives the prescription for doing the inverse transform,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{ie^{-ikx}}{k + i\epsilon} dk, \quad (29)$$

which should, of course, be  $\theta(x)$ . The integrand has a pole just below the axis at  $-i\epsilon$ . We complete the contour by a great semicircle. Noting that  $e^{-ikx} = e^{-ik_r x} e^{k_y x}$ , if  $x > 0$  we need to complete the contour in the lower half plane ( $k_y < 0$ ). The residue is 1, and there is a minus sign because the contour is in a clockwise sense, Hence we get

$$\theta(x) = \frac{i}{2\pi} 2\pi i (-1) = 1 \quad (x > 0),$$

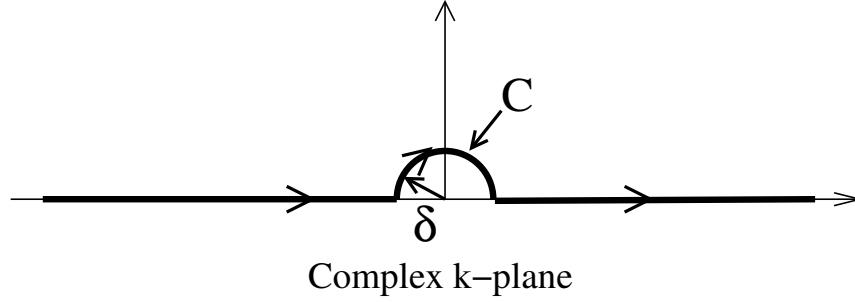
which is correct. If  $x < 0$  we complete in the upper half plane so the contour does not include the singularity, and we get zero, *i.e.*

$$\theta(x) = 0 \quad (x < 0),$$

which is also correct.

It is useful to consider further the expression for the Fourier Transform of  $\theta(x)$  in Eq. (28). Since the integrand has a pole at  $-i\epsilon$  and  $\epsilon > 0$ , the contour passes *above* the pole. In the

limit of  $\epsilon \rightarrow 0^+$ , the pole is arbitrarily close to the origin and it is convenient to deform the path of integration so it forms a small semicircle of radius  $\delta$  above the origin as shown.



We will take  $\delta \rightarrow 0$  (though  $\delta \gg \epsilon$ ). Hence, for  $\epsilon \rightarrow 0$

$$\lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{u(k) dk}{k + i\epsilon} = \lim_{\delta \rightarrow 0} \left[ \int_{-\infty}^{-\delta} \frac{u(k) dk}{k} + \int_{\delta}^{\infty} \frac{u(k) dk}{k} \right] + \int_C \frac{u(z) dz}{z}, \quad (30)$$

where  $C$  is the semicircular contour around the origin shown in the above figure, and  $u(k)$  is some smooth function. The integral in the square brackets, where we integrate *up to* a small distance below a singularity and *from the same* distance above the singularity, is known as the principal value integral. It is denoted by the symbol  $\mathcal{P}$ , *i.e.*

$$\mathcal{P} \int_{-\infty}^{\infty} f(k) dk \equiv \lim_{\delta \rightarrow 0} \left[ \int_{-\infty}^{-\delta} f(k) dk + \int_{\delta}^{\infty} f(k) dk \right].$$

Along the semicircle of radius  $\delta$ , we have  $z = \delta e^{i\theta}$  and so, for  $\delta \rightarrow 0$ ,

$$\int_C \frac{u(z) dz}{z} = \int_{\pi}^0 u(\delta e^{i\theta}) \frac{i\delta e^{i\theta} d\theta}{\delta e^{i\theta}} = iu(0) \int_{\pi}^0 d\theta = -i\pi u(0).$$

Consequently we can write Eq. (30) as

$$\lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{u(k) dk}{k + i\epsilon} = \mathcal{P} \int_{-\infty}^{\infty} \frac{u(k) dk}{k} - i\pi u(0). \quad (31)$$

It is frequently useful to forget about the smooth function  $u(k)$  and the integration, and write (with  $\epsilon \rightarrow 0^+$  assumed)

$$\boxed{\frac{1}{k + i\epsilon} = \mathcal{P} \left( \frac{1}{k} \right) - i\pi\delta(k)}. \quad (32)$$

Similarly, we find

$$\frac{1}{k - i\epsilon} = \mathcal{P} \left( \frac{1}{k} \right) + i\pi\delta(k). \quad (33)$$

It follows from Eqs. (32) and (28) that the Fourier transform of  $\theta(x)$  is given by

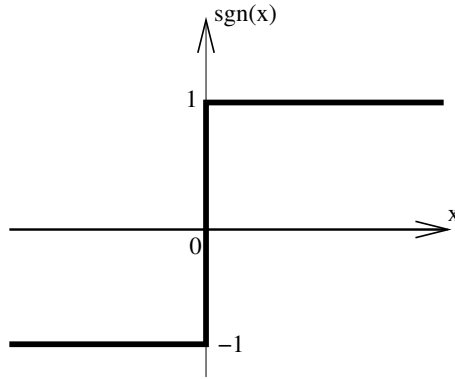
$$\boxed{\theta(x) \xrightarrow{FT} iP \left( \frac{1}{k} \right) + \pi\delta(k)}. \quad (34)$$

Similarly the Fourier transform of  $1 - \theta(x)$ , which takes value 1 for  $x < 0$  and 0 for  $x > 0$  is given by the negative  $x$  region of the integral in Eq. (4), *i.e.*

$$1 - \theta(x) \xrightarrow{FT} -i\mathcal{P}\left(\frac{1}{k}\right) + \pi\delta(k). \quad (35)$$

Note that adding Eqs. (34) and (35) the Fourier transform of unity is found to be  $2\pi\delta(k)$  as obtained earlier.

Finally the Fourier transform of the sign function



$$\text{sgn}(x) = \begin{cases} -1, & (x < 0) \\ 1, & (x > 0) \end{cases} = 2\theta(x) - 1$$

is the *difference* between the results in Eqs. (34) and (35), *i.e.*

$$\boxed{\text{sgn}(x) \xrightarrow{FT} 2i\mathcal{P}\left(\frac{1}{k}\right)}. \quad (36)$$

It is interesting to verify that the inverse FT of Eq. (36) gives  $\text{sgn}(x)$ . We have

$$f(x) = \frac{1}{2\pi} 2i\mathcal{P} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{k} dk. \quad (37)$$

By symmetry only the imaginary part of the complex exponential contributes so

$$f(x) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\sin kx}{k} dk, \quad (38)$$

where the “principal part” symbol can now be taken away because the singularity at  $k = 0$  is removed once the cosine part of the integrand in Eq. (37) is eliminated. Changing variables  $k' = kx$ , and noting that if  $x < 0$  the order of limits is inverted, we get

$$f(x) = \frac{1}{\pi} \begin{cases} \int_{-\infty}^{\infty} \frac{\sin k'}{k'} dk', & (x > 0) \\ \int_{\infty}^{-\infty} \frac{\sin k'}{k'} dk', & (x < 0) \end{cases} = \begin{cases} 1 & (x > 0) \\ -1 & (x < 0) \end{cases},$$

*i.e.*  $f(x) = \text{sgn}(x)$  as required, where we used the result derived in class that

$$\int_{-\infty}^{\infty} \frac{\sin k}{k} dk = \pi.$$

Finally, if we set  $x = 0$  in Eq. (38) we get  $f(0) = 0$ , in agreement with the general result that, at a discontinuity, the value obtained by a Fourier transform is the average of the limiting values on either side.

## V. OTHER SINGULAR FOURIER TRANSFORMS

One can also regularize the FT of functions which grow with a *power* of  $x$  at large  $x$ , since  $x^n e^{-\epsilon|x|} \rightarrow 0$  for any finite  $n$  and any non-zero (positive) value of  $\epsilon$ . Typically the result involves a derivative of the delta function. For example, suppose we want the Fourier Transform of  $f(x) = x$ . Assuming convergence factors where necessary, we have

$$\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx$$

and so, differentiating with respect to  $k$ , gives

$$\delta'(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ix e^{ikx} dx$$

from which we obtain that

$$\boxed{x \xrightarrow{FT} -2\pi i \delta'(k)}. \quad (39)$$

However, it is not possible to regularize functions which diverge *exponentially* at large  $x$ , because this divergence is too strong to be canceled by a regularization factor  $e^{-\epsilon|x|}$  in the limit  $\epsilon \rightarrow 0$ . For these problems the related technique of *Laplace transforms*, which *can* treat such functions, may be useful.

## VI. SUMMARY

We have discussed several improper Fourier transforms, such as Eqs. (15), (34), (36) and (39). Taken literally, the integrals do not exist and so, as discussed in the text, these equations have to be understood in one of the following senses:

- As a stand-alone equation, in which case the integral in the FT has to be regularized by a convergence factor like  $e^{-\epsilon|x|}$ .

- Both sides of the equation are multiplied by a smooth function of  $k$  and integrated, in which case the convergence factor is unnecessary.

Equation (15) corresponds to the integral representation of the Dirac delta function, Eq. (14), which is very useful as shown in Sec. III. As a byproduct we also obtained in Eqs. (32) and (33) the useful result,

$$\frac{1}{x \pm i\epsilon} = \mathcal{P} \left( \frac{1}{x} \right) \mp i\pi\delta(x), \quad (40)$$

for  $\epsilon \rightarrow 0^+$ , which also needs to be multiplied by a smooth function and integrated in order to make sense. We have also obtained the singular Fourier transforms for the step function, Eq. (28) and for  $f(x) = x$ , Eq. (39).

### Acknowledgments

I'm grateful to Onuttom Narayan for emphasizing to me the importance of singular Fourier transforms.