PHYSICS 5I
Homework 4

Due in class, Wednesday November 4.

1. We know that

$$I = \int_a^b f(x)\, dx\,, \tag{1}$$

i.e. the definite integral of $f(x)$ from $x = a$ to $x = b$, is the area under the curve $y = f(x)$ for x between a and b.

Integration is the opposite of differentiation so if the indefinite integral of $f(x)$ is $F(x)$, i.e.

$$F(x) = \int f(x)\, dx\,,$$

then

$$f(x) = \frac{dF}{dx}\,.$$

Often we are able to determine definite integrals like Eq. (1) by hand (i.e. analytically). However, sometimes we are not able to do this and need to resort to *numerical* methods. In this problem we use a numerical method to evaluate a definite integral. For simplicity we will consider an example where the answer is known exactly, but realize that the important applications of numerical methods are those where the exact answer is *not* known.

The indefinite integral of $1/(1+x)$ is $\ln(1+x)$. Hence

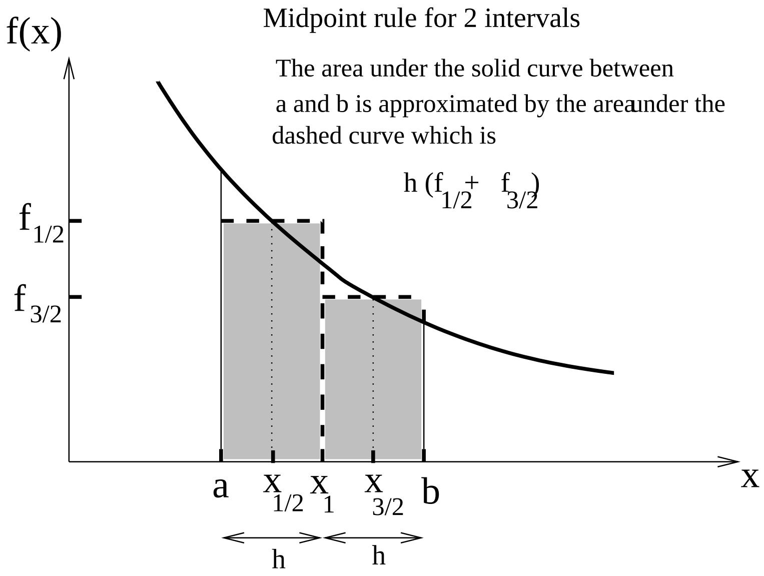
$$\int_0^1 \frac{1}{1+x}\, dx = [\ln(1+x)]_0^1 = \ln 2 - \ln 1 = \ln 2 \simeq 0.6931\,.$$

We will calculate it *numerically* by the "midpoint rule" discussed in class according to which

$$I = \int_a^b f(x)\, dx \simeq h\left(f_{1/2} + f_{3/2} + \cdots + f_{n-1/2}\right)\,,$$

in which the range from a to b is divided into n intervals, each of width $h = (b-a)/n$. The function $f(x)$ is evaluated at the midpoint of each interval, i.e. $x_{k-1/2} = a + (k - \frac{1}{2})h$ where $k = 1, 2, \cdots, n$, and we denote $f(x_{k-1/2})$ by $f_{k-1/2}$ for convenience.

The figure below is an example with $n = 2$ intervals. The integral is the area under the solid curve, $f(x)$, between a and b, and we approximate it by the midpoint rule, the (shaded) area under the dashed curve. Note that the difference between the solid curve and the dashed curve is sometimes positive and sometimes negative so the error *tends to cancel.*



Either if you are able to do elementary programming, write a computer program to compute the integral numerically for a fairly large number of intervals, e.g. $n = 32$, and show that it agrees very well with the exact answer.
Note: One can show that the error is proportional to h^2. It would be amusing for you to test this by considering different numbers of intervals, e.g. $n = 1$, 2, 4, 8, 16, 32.
I will shortly put on the class website a C program which will do midpoint integration.

Or if you are not able to do programming (in say C or Matlab) then do the midpoint rule *by hand* for $n = 1, 2, 4$ and 8 intervals, and show that even for small n it works fairly well and that the results improve when the number of intervals increases.