

PHYSICS 5I
Homework 5

Due in class, Wednesday November 18 (Nov. 11 is a holiday)

1. When discussing relativity, we found that, in the presence of a *constant* force F , the velocity of a particle of mass m is given by the solution of the following equation.

$$\frac{dv}{dt} = \frac{F}{m} (1 - v^2/c^2)^{3/2} . \quad (1)$$

We would like to know how v varies with time assuming, for example, that the particle is at rest, $v = 0$, at $t = 0$.

In class we *verified* that the solution is

$$v(t) = \frac{F c t}{(m^2 c^2 + F^2 t^2)^{1/2}} . \quad (2)$$

Here we learn how to solve Eq. (1) *numerically*. Of course numerical methods are most useful for problems when an exact answer can not be found, but it also useful to *test* numerical methods on problems where the answer is known.

To avoid having lots of parameters, consider the case of $F = m = 1$, and also work in units where $c = 1$, so Eq. (1) becomes

$$\boxed{dv/dt = (1 - v^2)^{3/2}} , \quad (3)$$

and the solution is

$$\boxed{v(t) = \frac{t}{(1 + t^2)^{1/2}}} . \quad (4)$$

If the RHS of Eq. (3) involved t , rather than v , we could obtain the solution simply by integrating with respect to t . If we were doing the integral numerically we could use, for example, the midpoint rule discussed in the previous lecture and homework assignment. However, as it stands with factors of v on the RHS, Eq. (3) is a *differential equation*. In math courses you will learn how to solve differential equations using pencil and paper. but here we will solve Eq. (3) *numerically*, using ideas similar to those in the midpoint rule for integration.

The problem we want to solve is to start the particle off at an initial time t_0 with speed v_0 and then determine $v(t)$ over the subsequent time T , i.e. for

$$t_0 \leq t \leq t_0 + T . \quad (5)$$

We divide T into n intervals of width $h = T/n$.

To make the method more general than just solving Eq. (3), we write the equation to be solved as

$$\boxed{\frac{dv}{dt} = f(v)} , \quad (6)$$

so, for Eq. (3), we have $f(v) = (1 - v^2)^{3/2}$.

Consider one time interval from $t = t_0$ to $t = t_1 = t_0 + h$. During this time the velocity changes from v_0 to v_1 . As for the midpoint rule, to step forward in time a good approximation is

$$v_1 = v_0 + f(v_{1/2})h, \quad (7)$$

where $f(v_{1/2})$ is dv/dt evaluated at the midstep, $t_{1/2} = t_0 + h/2$. Unfortunately we don't know the value of $v_{1/2}$. We proceed by getting a *rough* estimate of it by evaluating the derivative at v_0 , which we do know, i.e.

$$v_{1/2} = v_0 + f_{v_0} \frac{h}{2}. \quad (8)$$

We can then substitute for $v_{1/2}$ into Eq. (7) to get v_1 . Having gone from $v = v_0$ at $t = t_0$ to $v = v_1$ at $t = t_1 = t_0 + h$, we can apply the same two steps, Eq. (7) and (8), to go from t_1 to t_2 and then t_2 to t_3 etc.

To summarize, to determine v_{i+1} at time t_{i+1} given that the speed was v_i at time t_i , we do the following operations:

$$\boxed{v_{i+1/2} = v_i + \frac{h}{2} f(v_i)}, \quad (9)$$

$$\boxed{v_{i+1} = v_i + h f(v_{i+1/2})}. \quad (10)$$

We repeat this n times to go from t_0 to $t_0 + T$. This is one of the (several) methods for integrating differential equations associated with the names of Runge and Kutta. I will denote it by RK2.

Either if you are able to do elementary programming, write a computer program to compute the solution to Eq. (3) with the starting condition $v(0) = 0$, for the interval $0 \leq t \leq 2$. Do this by dividing the region of time up into a fairly large number of intervals, e.g. 50 and using the RK2 method discussed above. For some of these discrete times print your estimate of $v(t)$ and compare with the exact solution given above. Your results should agree *very well* with the exact solution.

You may also want to check that using a finer mesh of times leads to a *more* accurate answer.

Or if you are not able to do programming (in say C or Matlab or Excel) then consider the time interval from 0 to 1, divide it into 8 intervals, and apply the RK2 rule *by hand*. For each of the discrete times, print your estimate of $v(t)$ and compare with the exact solution given above. Your results should agree *reasonably well* with the exact solution.

Note: More generally, a “first order” differential equation (so-called because it involves just first derivatives as well as the function) can be written as

$$\boxed{\frac{dv}{dt} = f(v, t)}, \quad (11)$$

rather than Eq. (6), i.e. the right hand side (RHS) can depend on the independent variable t , as well as v . The second order Runge-Kutta method (RK2) described above then becomes

$$v_{i+1/2} = v_i + \frac{h}{2} f(v_i, t_i), \quad (12)$$

$$v_{i+1} = v_i + h f(v_{i+1/2}, t_{i+1/2}). \quad (13)$$