PHYSICS 5I
Homework 5
Due in class, Wednesday November 18 (Nov. 11 is a holiday)

1. When discussing relativity, we found that, in the presence of a *constant* force F, the velocity of a particle of mass m is given by the solution of the following equation.

$$\frac{dv}{dt} = \frac{F}{m} \left(1 - v^2/c^2\right)^{3/2} . \tag{1}$$

We would like to know how v varies with time assuming, for example, that the particle is at rest, $v = 0$, at $t = 0$.

In class we *verified* that the solution is

$$v(t) = \frac{F\,c\,t}{\left(m^2\,c^2 + F^2\,t^2\right)^{1/2}} . \tag{2}$$

Here we learn how to solve Eq. (1) *numerically*. Of course numerical methods are most useful for problems when an exact answer can not be found, but it also useful to *test* numerical methods on problems where the answer is known.

To avoid having lots of parameters, consider the case of $F = m = 1$, and also work in units where $c = 1$, so Eq. (1) becomes

$$\boxed{dv/dt = (1 - v^2)^{3/2},} \tag{3}$$

and the solution is

$$\boxed{v(t) = \frac{t}{(1 + t^2)^{1/2}}.} \tag{4}$$

If the RHS of Eq. (3) involved t, rather than v, we could obtain the solution simply by integrating with respect to t. If we were doing the integral numerically we could use, for example, the midpoint rule discussed in the previous lecture and homework assignment. However, as it stands with factors of v on the RHS, Eq. (3) is a *differential equation*. In math courses you will learn how to solve differential equations using pencil and paper. but here we will will solve Eq. (3) *numerically*, using ideas similar to those in the midpoint rule for integration.

The problem we want to solve is to start the particle off at an initial time t_0 with speed v_0 and then determine $v(t)$ over the subsequent time T, i.e. for

$$t_0 \le t \le t_0 + T\,. \tag{5}$$

We divide T into n intervals of width $h = T/n$.

To make the method more general than just solving Eq. (3), we write the equation to be solved as

$$\boxed{\frac{dv}{dt} = f(v)\,,} \tag{6}$$

so, for Eq. (3), we have $f(v) = (1 - v^2)^{3/2}$.

Consider one time interval from $t = t_0$ to $t = t_1 = t_0 + h$. During this time the velocity changes from v_0 to v_1. As for the midpoint rule, to step forward in time a good approximation is

$$v_1 = v_0 + f(v_{1/2})\, h\,, \tag{7}$$

where $f(v_{1/2})$ is dv/dt evaluated at the midstep, $t_{1/2} = t_0 + h/2$. Unfortunately we don't know the value of $v_{1/2}$. We proceed by getting a *rough* estimate of it by evaluating the derivative at v_0, which we do know, i.e.

$$v_{1/2} = v_0 + f_{v_0}\, \tfrac{h}{2}\,. \tag{8}$$

We can then substitute for $v_{1/2}$ into Eq. (7) to get v_1. Having gone from $v = v_0$ at $t = t_0$ to $v = v_1$ at $t = t_1 = t_0 + h$, we can apply the same two steps, Eq. (7) and (8), to go from t_1 to t_2 and then t_2 to t_3 etc.

To summarize, to determine v_{i+1} at time t_{i+1} given that the speed was v_i at time t_i, we do the following operations:

$$\boxed{v_{i+1/2} = v_i + \tfrac{h}{2}\, f(v_i),} \tag{9}$$

$$\boxed{v_{i+1} = v_i + h\, f(v_{i+1/2}).} \tag{10}$$

We repeat this n times to go from t_0 to $t_0 + T$. This is one of the (several) methods for integrating differential equations associated with the names of Runge and Kutta. I will denote it by RK2.

Either if you are able to do elementary programming, write a computer program to compute the solution to Eq. (3) with the starting condition $v(0) = 0$, for the interval $0 \le t \le 2$. Do this by dividing the region of time up into a fairly large number of intervals, e.g. 50 and using the RK2 method discussed above. For some of these discrete times print your estimate of $v(t)$ and compare with the exact solution given above. Your results should agree *very well* with the exact solution.

You may also want to check that using a finer mesh of times leads to a *more* accurate answer.

Or if you are not able to do programming (in say C or Matlab or Excel) then consider the time interval from 0 to 1, divide it into 8 intervals, and apply the RK2 rule *by hand.* For each of the discrete times, print your estimate of $v(t)$ and compare with the exact solution given above. Your results should agree *reasonably well* with the exact solution.

Note: More generally, a "first order" differential equation (so-called because it involves just first derivatives as well as the function) can be written as

$$\boxed{\frac{dv}{dt} = f(v, t)\,,} \tag{11}$$

rather than Eq. (6), i.e. the right hand side (RHS) can depend on the independent variable t, as well as v. The second order Runge-Kutta method (RK2) described above then becomes

$$v_{i+1/2} = v_i + \tfrac{h}{2}\, f(v_i, t_i), \tag{12}$$

$$v_{i+1} = v_i + h\, f(v_{i+1/2}, t_{i+1/2}). \tag{13}$$