Complexity of the quantum adiabatic algorithm

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This talk can be downloaded at http://physics.ucsc.edu/˜peter/talks/ucsc.pdf

Introduction

- What is a **Quantum Computer**?
- What is the **"Quantum Adiabiatic Algorithm"** proposed for quantum computers.
- Motivation for studying the **complexity** of the Quantum Adiabatic Algorithm for **much larger sizes** than has been studied before.
- The **Monte Carlo** method that will be used to do this.
- **Results** for a particular problem (Exact Cover).
- **Conclusions**.

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Many proposed implementations:

- Superconductor-based (Josephson junctions)
- Trapped ions
- Quantum dots
- NMR-based (e.g. phosphorous-doped silicon)
- $\cdots$

Quantum Computer II

And also many experimental difficulties:

- Need to be able to couple to the qubits in order to **manipulate** them.
- But otherwise need to prevent coupling of bits to outside world because this causes **decoherence**.
- **Scalability** to large number of bits.

So far, quantum computing operations have only been successfully carried out on very small numbers of bits. However, it still interesting to consider ...

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The best known is **Shor's factoring algorithm** which factors an integer of $\mathbf{n}$ bits in a time which is of order $\mathbf{n^3}$, i.e. **polynomial** in $\mathbf{n}$, as opposed to the best classical algorithm which takes a time of order $\mathbf{\exp(c\, n^{1/3}\, \log^{2/3} n)}$.

Relevant for **encryption**: $\Longrightarrow$ Important in commerce and for the military.

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Here we are interested in a **general** class of problems: **"optimization problems"** in which we need to minimize a function of $\mathbf{N}$ binary variables, $z_i = \mathbf{0}, \mathbf{1}$, with constraints.

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Could a quantum computer solve **typical** instances of NP-Hard problems with just **polynomial complexity**, i.e.

$$\text{complexity} \propto N^{\sigma}\,,$$

for some value of σ?

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Add a “driver Hamiltonian”, which is simple and does not commute with $\mathcal{H}_P$. The simplest is a “transverse field” $\mathcal{H}_D = -h \sum_i \sigma_i^x$.

The total Hamiltonian is

$$\boxed{\mathcal{H} = [1 - \lambda(t)]\,\mathcal{H}_D + \lambda(t)\,\mathcal{H}_P\,,}$$

where the **“control parameter”** $\lambda(t)$ varies from **0** at $t = 0$ to **1** at $t = \mathcal{T}$, the **running time**, or **complexity**.

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At $t = \mathcal{T}$, just have $\mathcal{H}_P$. If the evolution is adiabatic, the system is in the ground state of $\mathcal{H}_P$ and **the problem is solved**.

Quantum Adiabatic Algorithm

The **Quantum Adiabatic Algorithm** is less demanding on the hardware than algorithms like Shor’s.

The QAA **gradually** evolves the Hamiltonian, which is hardwired into the connections in the computer, e.g. by changing a magnetic field, whereas Shor’s algorithm proceeds by a series of **discrete** unitary transformations.

It is easier to avoid interference between the bits and to maintain quantum coherence if changes are made gradually, rather than in a series of discrete jumps.

Here there is **real interest in the quantum computing community** in building a quantum computer which uses the QAA.

However, even if one can build one **will it be more efficient than a classical computer** for NP-hard problems?

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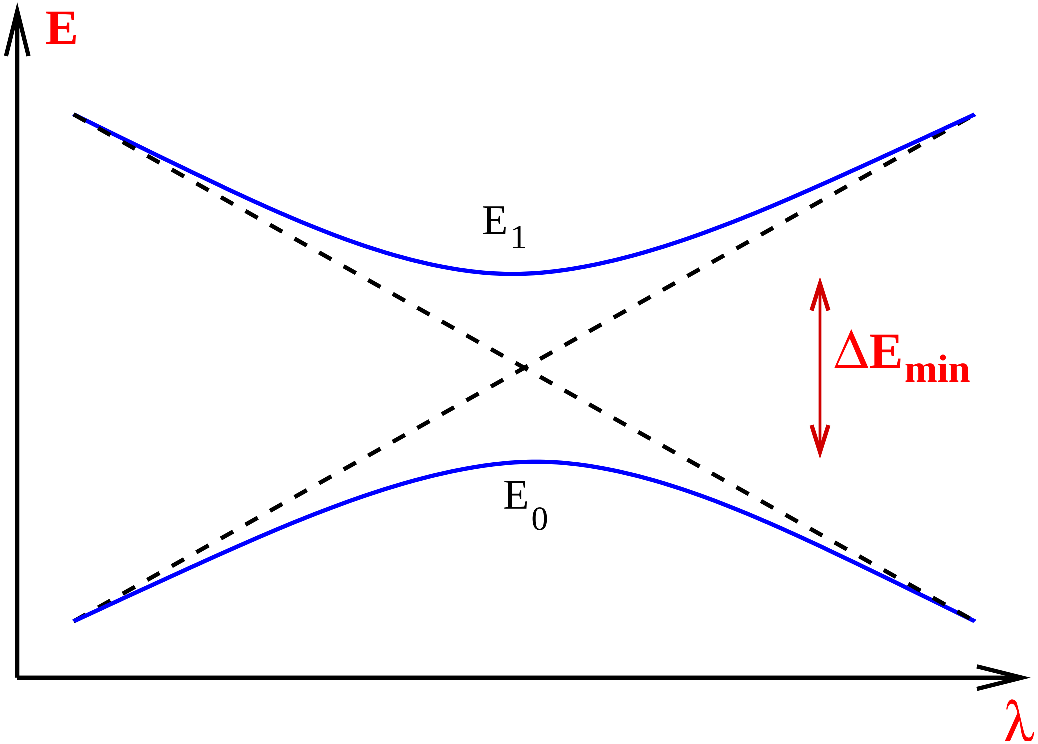

The dashed lines show a crossing that the ground state and first excited would have in the absence of any coupling between them. However, there is actually “level repulsion” so the two levels, shown by the solid lines, do not cross but have a minimum gap $\boldsymbol{\Delta E_{\mathrm{min}}}$.

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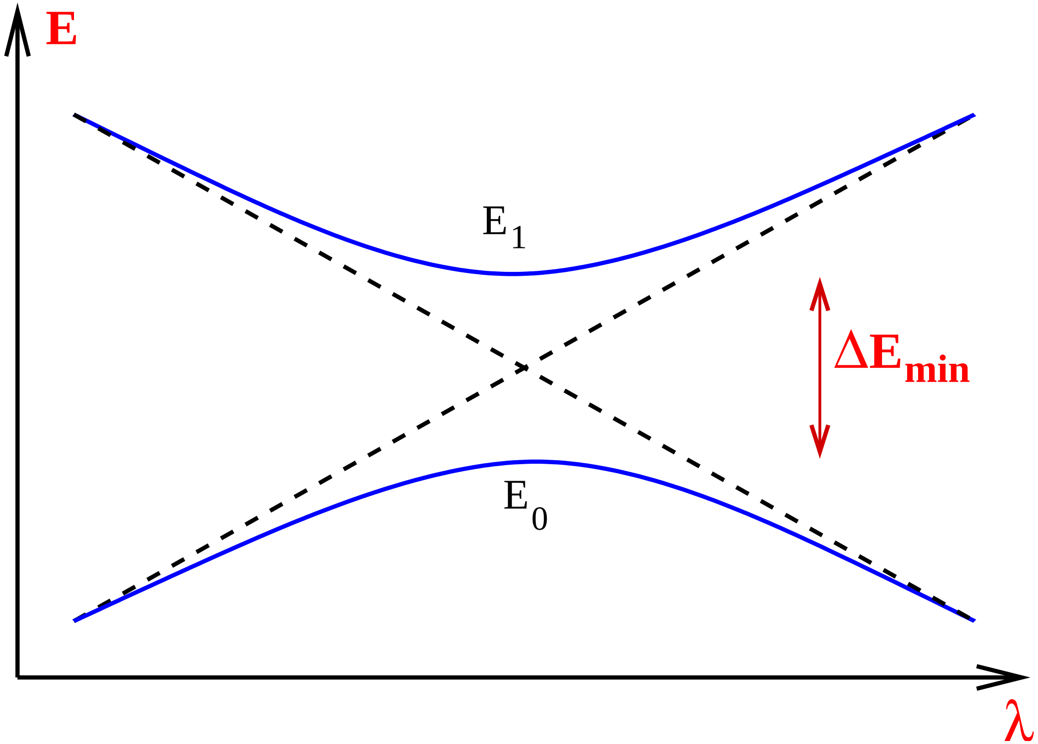

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Landau-Zener theory. To stay in ground state, time $\propto (\mathbf{\Delta E_{min}})^{-2}$.

Quantum Phase Transition

As $\lambda(t)$ is varied the system is likely to go through a **Quantum Phase Transition** where the gap will be particularly small.

Hence we are, effectively interested in:

The Size Dependence of the Energy Gap at a Quantum Phase Transition

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$\Longrightarrow$ **“Monte Carlo" methods**

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QMC depends on the correspondence between the time evolution operator in quantum mechanics $\boxed{e^{-i\mathcal{H}t}}$ and the Boltzmann operator in statistical mechanics $\boxed{e^{-\mathcal{H}\beta}}$. We see that $\beta \equiv T^{-1}$ is like **imaginary time**.

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Working through the details, one ends up with a **Classical Action** comprising copies of the system at different values of imaginary time τ where $0 \le \tau < \beta$. One discretizes imaginary time (Trotter decomposition) into L_τ "time slices" separated by the time-slice width $\Delta\tau$. We have

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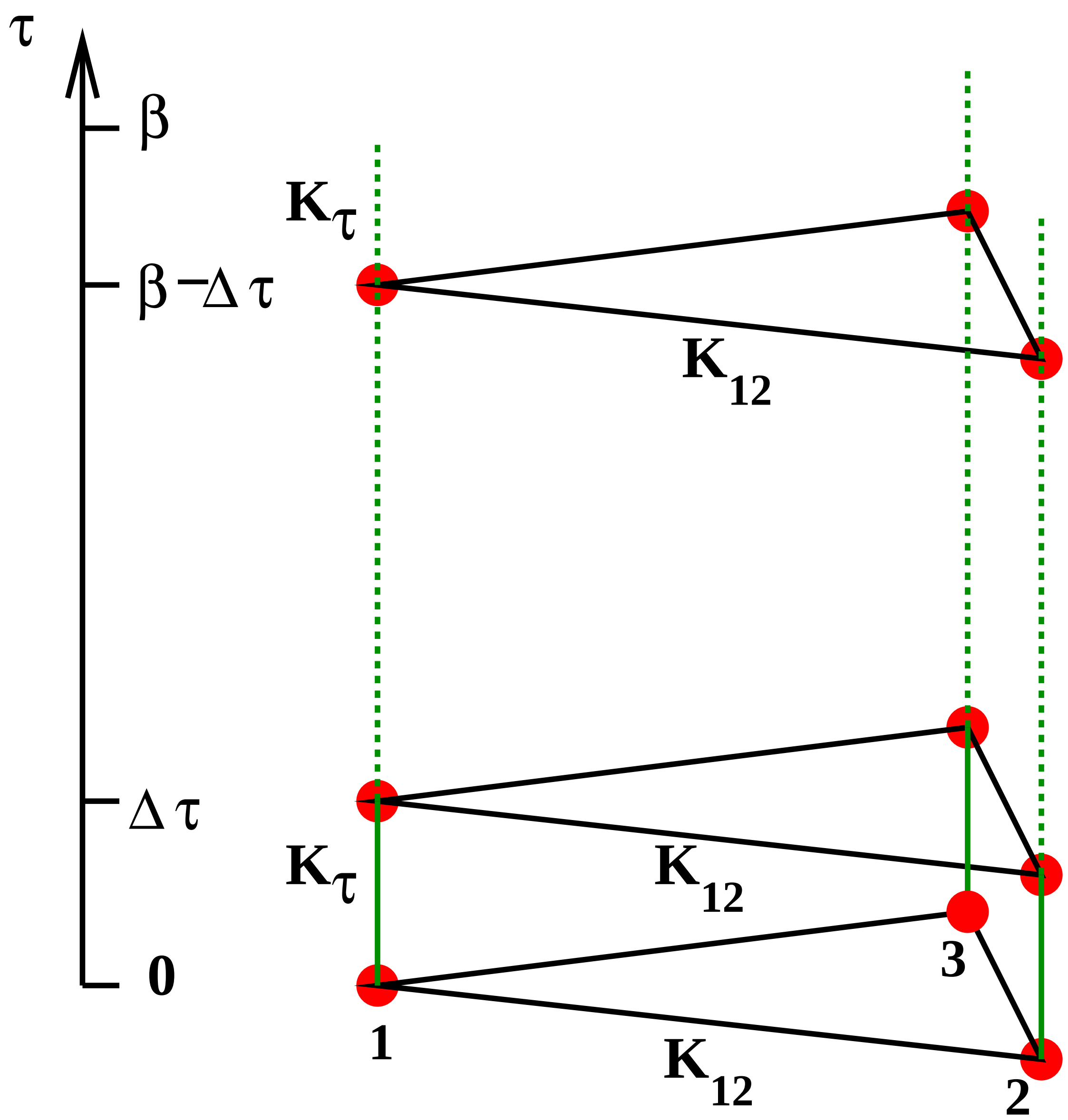
$$T^{-1} \equiv \beta = L_\tau / \Delta\tau \, .$$

The exact quantum mechanical Hamiltonian is reproduced in the limit $\Delta\tau \to 0$. However, this limit is not necessary for our purposes.

Quantum Monte Carlo: II



Trotter decomposition in QMC.

At each time slice 3 sites are shown. An independent spin $\sigma_i^z(\tau)$ lives at each site and each of the L_τ time slices. If spins i and j have an interaction in $\mathcal{H}_P$, then, each time slice, these spins interact with a coupling K_{ij}, the same for each slice. Spins on the same site but at neighboring time slices are coupled by an interaction K_τ, again the same for all slices. (Details on next slide.)

The slice at time $\tau = \beta$ is identified with the slice at $\tau = 0$ (i.e. we have periodic boundary conditions in the imaginary time direction).

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2. ferromagnetic couplings between different spins at the same site but neighboring time slices arising from the driver Hamiltonian

$$\mathcal{H}_D = -\sum_i \sigma_i^x \Longrightarrow -\sum_{m=0}^{L_\tau - 1} K_\tau \sigma_i^z(\tau_m)\sigma_i^z(\tau_{m+1})$$

   where $e^{-2K_\tau} = \tanh(\Delta\tau\, h)$.

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In quantum mechanics, correlations between a spin at an initial (real) time t_0 and a later time $t_0 + t$ have the form

$$C(t) \equiv \frac{1}{N} \sum_{i=1}^{N} \langle \sigma_i^z(t_0) \sigma_i^z(t_0 + t) \rangle = \frac{1}{N} \sum_{i=1}^{N} \left[\sum_n \left| \langle 0 | \sigma_i^z | n \rangle \right|^2 \right] e^{i(E_n - E_0)t} .$$

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In imaginary time, the complex exponentials are replaced by real, decaying exponentials:

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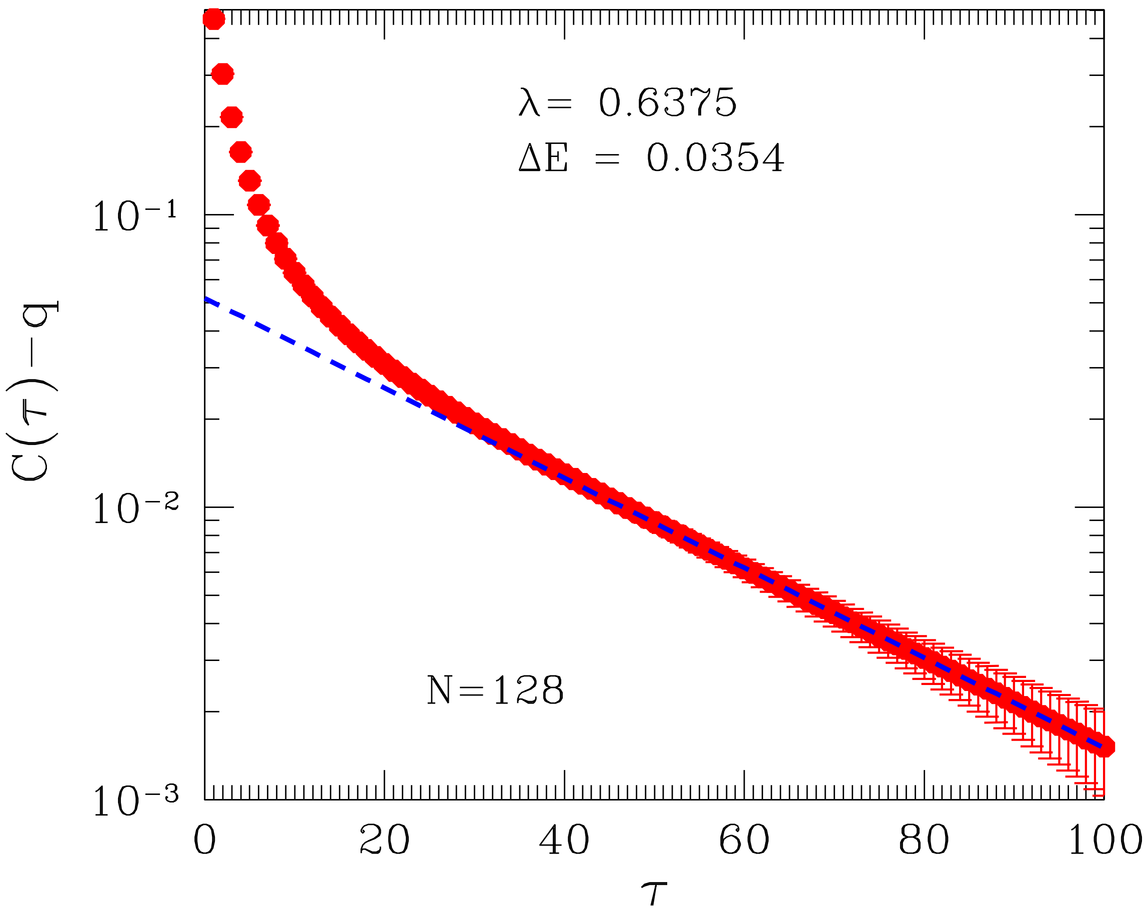
Hence, at large τ, we have

$$\boxed{C(\tau) = q + \frac{1}{N}\sum_{i=1}^{N}\left|\langle 0|\sigma_i^z|1\rangle\right|^2 e^{-(E_1-E_0)\tau} ,}$$

where $q = N^{-1}\sum_i \langle\sigma_i^z\rangle^2$. (See next slide for some results.)

Sample results for $C(\tau)$



Results for the time dependent correlation function against τ for one instance of the Exact Cover problem with $N = 128$ near the location of the minimum gap. Note that the vertical axis is logarithmic. Fitting to the straight line region gives a slope (equal to the gap ΔE) equal to 0.0354.

We took $L_\tau = 300, \Delta\tau = 1$, so $T^{-1} \equiv \beta = 300$. Hence the condition $T \ll \Delta E$ is well satisfied.

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$$\begin{aligned} \mathcal{H}_\mathrm{P} &= \frac{1}{8} \sum_{\alpha=1}^{M} \Big(5 - \sigma_{\alpha_1}^z - \sigma_{\alpha_2}^z - \sigma_{\alpha_3}^z + \sigma_{\alpha_1}^z \sigma_{\alpha_2}^z \\ &+ \sigma_{\alpha_2}^z \sigma_{\alpha_3}^z + \sigma_{\alpha_3}^z \sigma_{\alpha_1}^z + 3\, \sigma_{\alpha_1}^z \sigma_{\alpha_2}^z \sigma_{\alpha_3}^z \Big), \end{aligned} \qquad (1)$$

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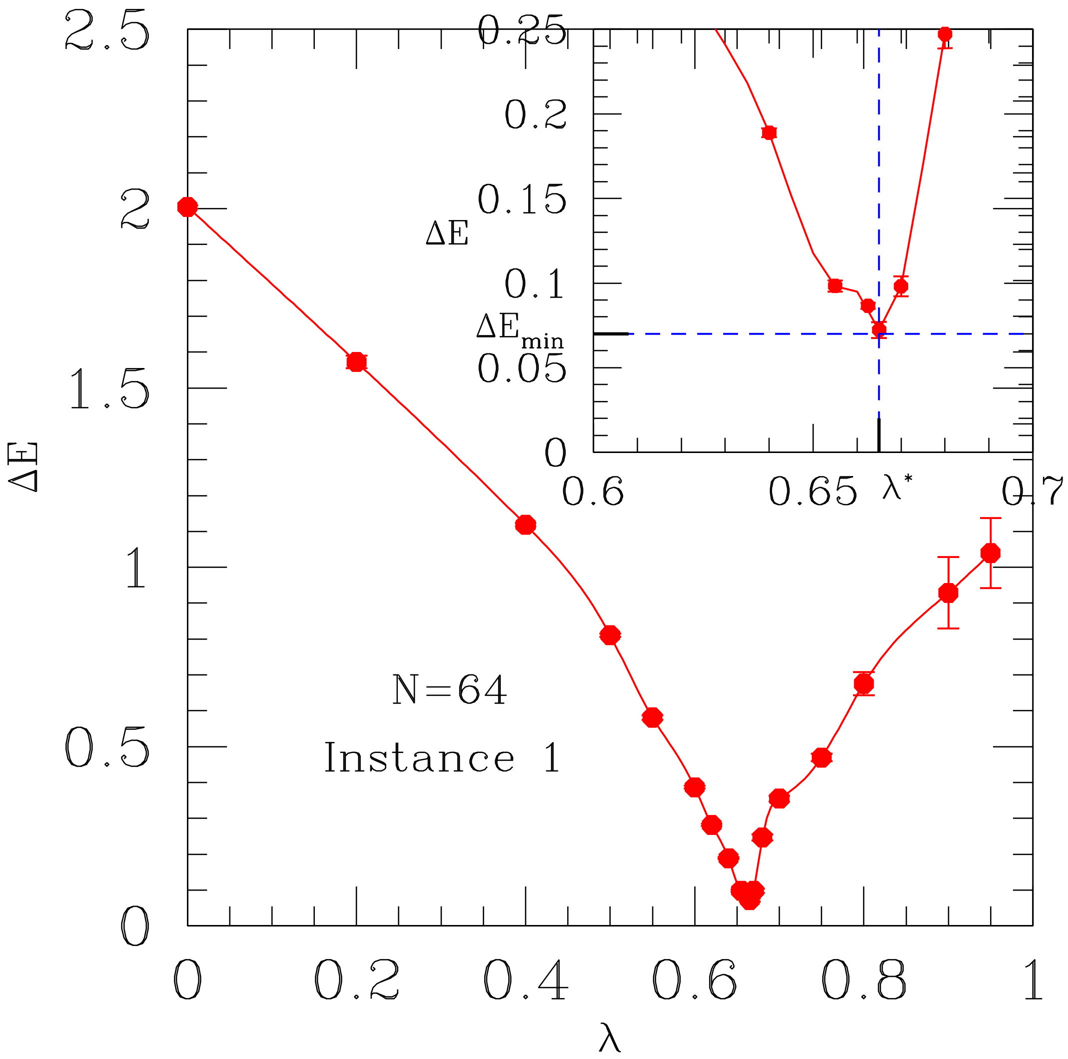
Following Farhi et al. we take instances with a **“Unique Satisfying Assignment”** (USA). To find these with reasonable probability, we adjust the ratio M/N for each size N.

Dependence of gap on λ



Results for the gap to the first excited state $\boldsymbol{\Delta E}$ as a function of the control parameter $\boldsymbol{\lambda}$ for one instance with $\boldsymbol{N = 64}$. The gap has is finite for $\boldsymbol{\lambda = 0}$ (this is due to the driver Hamiltonian, $\sum_i \sigma_i^x$). It is also finite for $\boldsymbol{\lambda = 1}$ because we chose instances with this property (Unique Satisfying Assignment). There is a minimum of the gap at an intermediate value of $\boldsymbol{\lambda}$, presumably close to a

quantum phase transition.

We compute $\boldsymbol{\Delta E_{\min}}$ for many (50) instances for several different sizes, $\boldsymbol{N = 16, 32, 64, 128}$.

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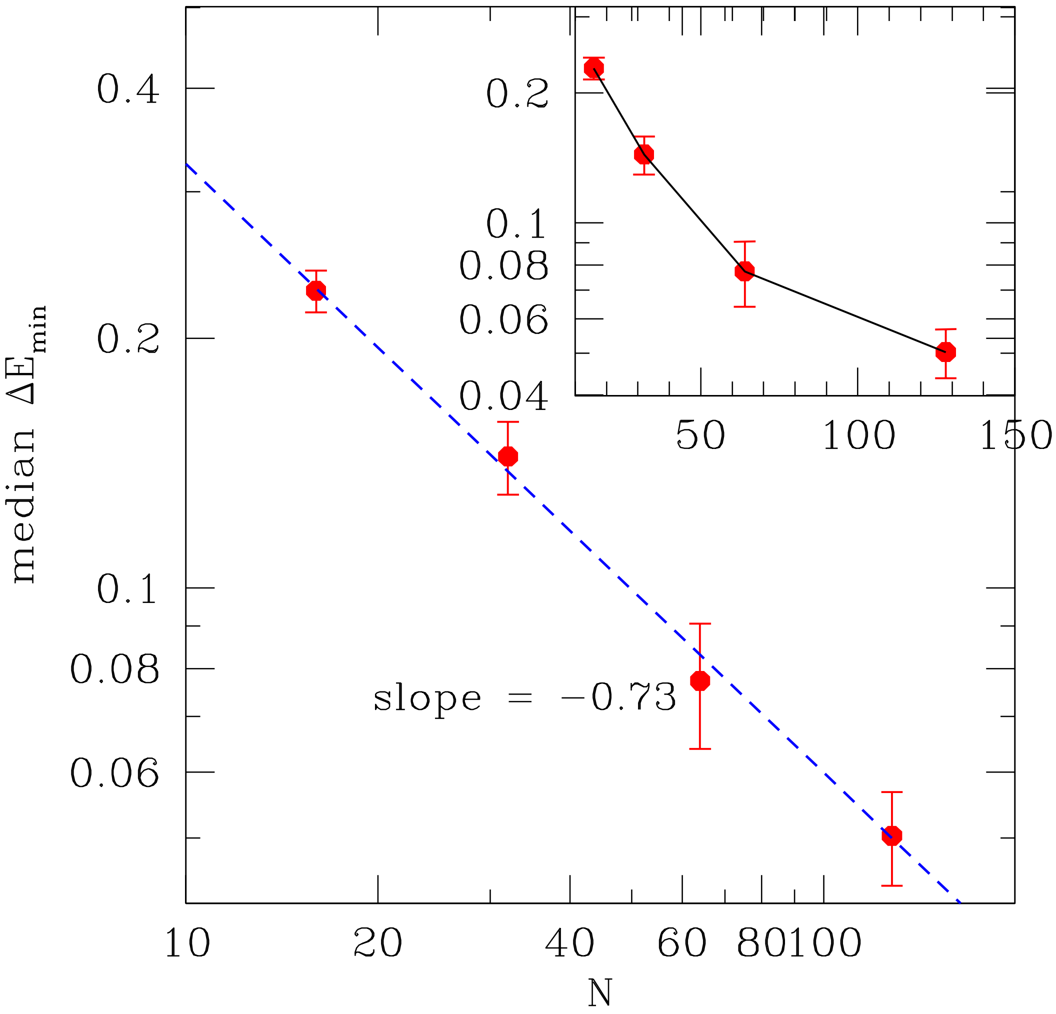

50 instances for each size.

A log-log plot of the **median** of the minimum gap as a function of the number of bits N up to $N = 128$. From the satisfactory straight line fit, it is seen that the median $\Delta E_{\min}$ *decreases as a power law*,

$$\text{median } \Delta E_{\min} \propto N^{-\mu},$$

for these sizes, with

$$\mu = 0.73 \pm 0.06.$$

The inset shows a log-linear plot. The pronounced curvature shows that the behavior is *not exponential* for this range of sizes.

Expect $\text{complexity} \propto N^{2\mu}$ (if matrix element effects are small).

Imaginary time discretization

Note: The discretization of imaginary time does not affect the way the complexity varies with N, though it does affect the precise value of the energy gap for given N and λ. Once the relaxation time $(\Delta E)^{-1}$ is much larger than the "lattice spacing" $\Delta\tau$ the lattice discretization is unimportant. Hence, whether the minimum gap varies exponentially with N or as a power law **will not depend on the value of $\Delta\tau$**.

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Example: Exact solution of the Ising model in two dimensions. The magnetization tends to zero as $T \to T_c^-$, like $(T_c - T)^\beta$. With a lot of work, this can be calculated on different lattices, e.g. square and triangular. The value of T_c depends on the lattice (it is **"non-universal**) but $\beta = 1/8$, the same for all lattice structures, i.e. it is **"universal"**.

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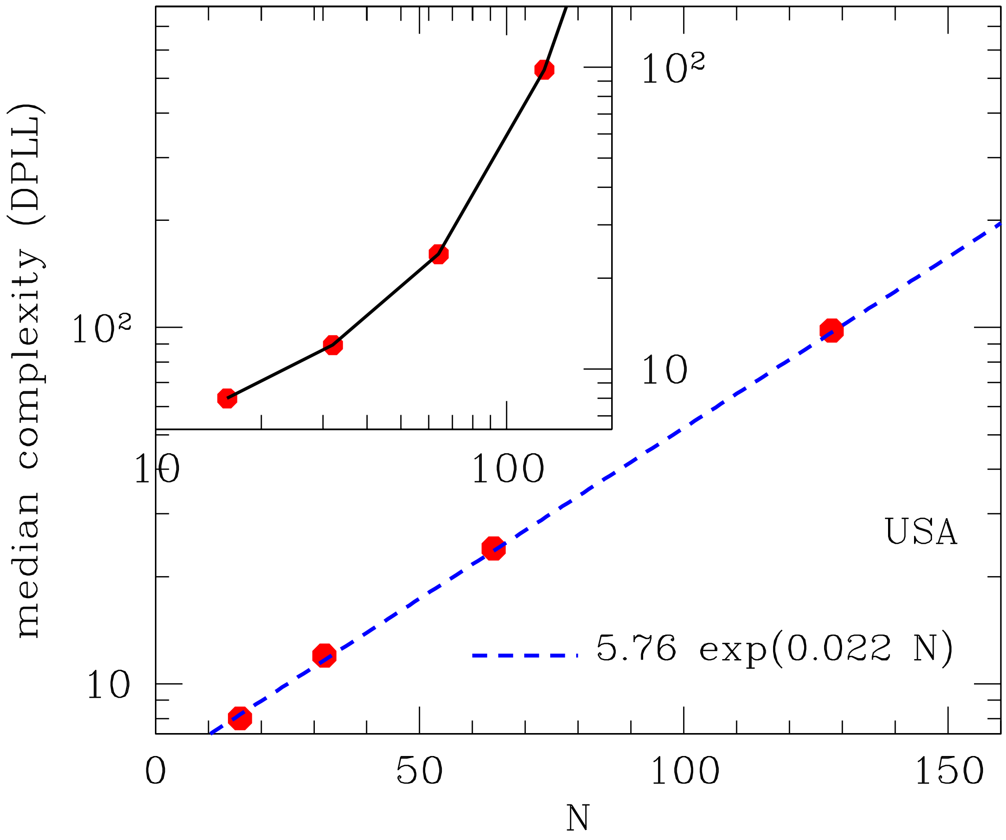
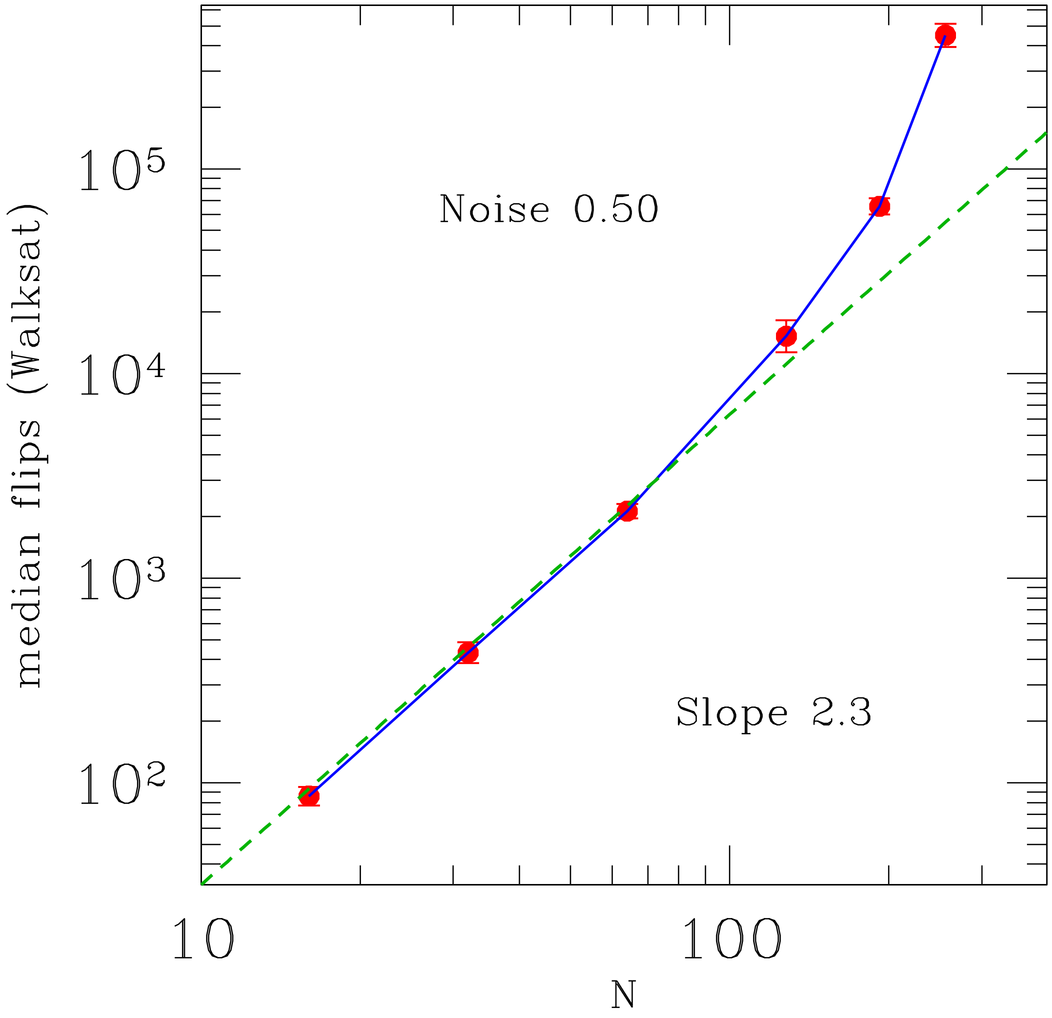
Note: One can simulate the $\Delta\tau \to 0$ limit, but this is more complicated.

Classical Algorithms: I



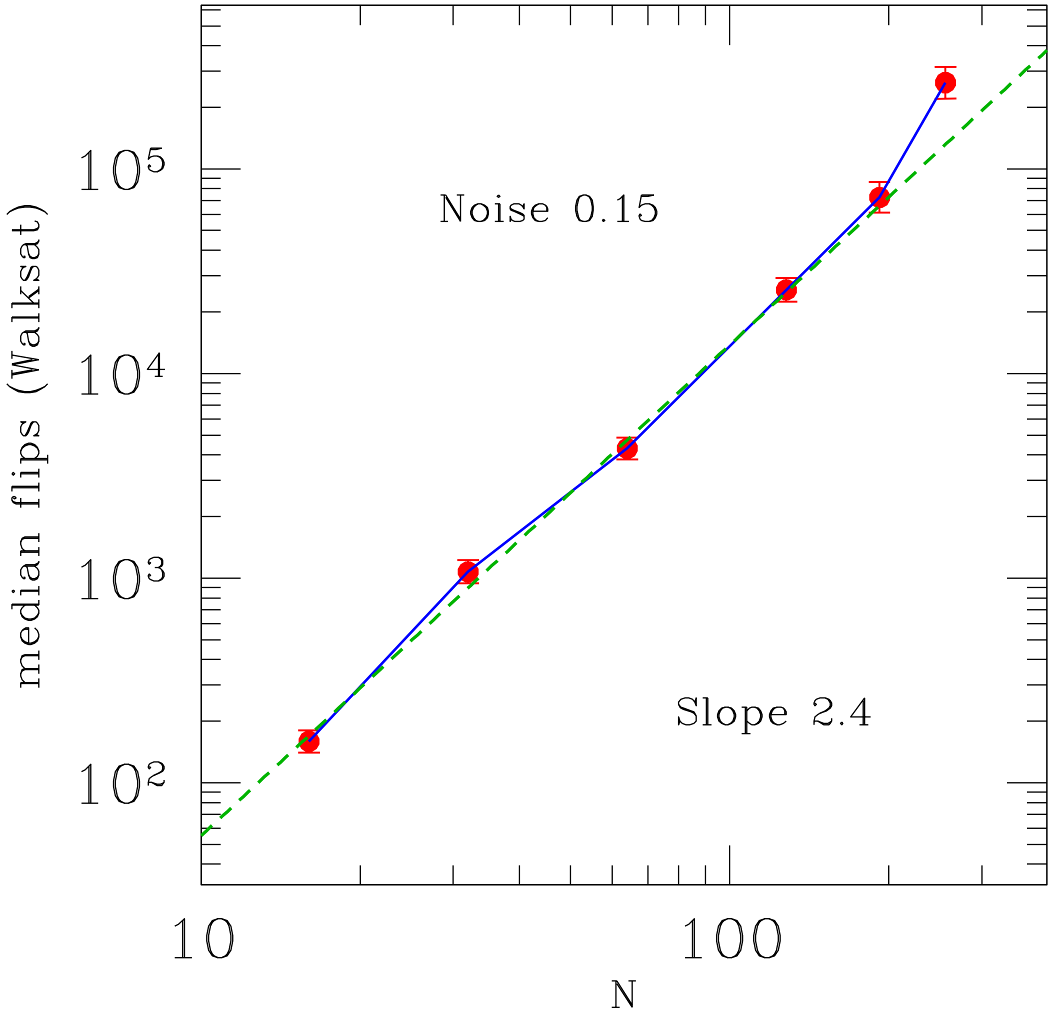
A commonly used classical algorithm for satisfiability problems is the Davis Putnum algorithm. This is guaranteed to correctly say whether or not there is a satisfying assignment. The figure shows the complexity for the instances used in the QMC simulations. It is clearly exponential for the range of sizes studied.

Classical Algorithms: II



A classical algorithm which is more analgous to QAA is WALKSAT, a local **heuristic** search algorithm. Like simulated annealing, it includes “up-hill" moves in a stochastic way. Using the default value of the “noise parameter" the complexity for the QAA instances with USA crosses over from power-law to (presumably) exponential for $N \gtrsim 100$. (But note the QMC is so far only for $N \le 128$).

Classical Algorithms: IIb



Adjusting the noise parameter, the crossover to exponential behavior is pushed to larger sizes $N \gtrsim 200$. (Remember: the QMC is so far only for $N \leq 128$).

Conclusions

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Thank you