Class 15 Feb

M-1

Matrix – algebra:

$N \times M$

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1M} \\ \vdots & & & \\ M_{N,1} & \cdots & \cdots & M_{N,M} \end{bmatrix} \begin{array}{l} \text{— row 1} \\ \\ \text{row } N \end{array}$$

col 1   Col M

rectangular $N \times M$ matrix

or   or

square matrix $N \times N$ is a special case

- Looks ~~like~~ similar to determinant, but different

- Scalar multiplication

$$\text{Det} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \times 5 = \begin{vmatrix} 5a_{11} & 5a_{12} & 5a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\text{Matrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times 5 = \begin{bmatrix} 5a_{11} & 5a_{12} & 5a_{13} \\ 5a_{21} & 5a_{22} & 5a_{23} \\ 5a_{31} & 5a_{32} & 5a_{33} \end{bmatrix}$$

- Add two matrices if same dimensions

- Subtract two matrices of same dimensions

- Multiply

$N \times M$ with $M \times L$ to produce $N \times L$ matrix

$$\boxed{M_1 \cdot M_2 = M_3}$$

$(N \times M) \cdot (M \times L) = (N \times L)$ ← Pnemonic

Example: $M_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} = (2 \times 3)$ matrix

$M_2 = \begin{bmatrix} 4 & 1 & 5 & 6 \\ 2 & 0 & 6 & 5 \\ 1 & 3 & 4 & 5 \end{bmatrix} = (3 \times 4)$ matrix

$$M_1 \cdot M_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 & 5 & 6 \\ 2 & 0 & 6 & 5 \\ 1 & 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1\times4+2\times2+3\times1 & 1\times1+2\times0+3\times3 & 1\times5+2\times6+3\times4 & 1\times6+2\times5+3\times15 \\ 2\times4+4\times2+1\times1 & & & \cdots \end{bmatrix}$$

$(2 \times 3) \times (3 \times 4) = [2 \times 4]$!

Notice we cannot make any sense of

$M_2 \cdot M_1$

$(3 \times 4) \cdot (2 \times 3)$ does not satisfy the matching condition!

- How about square matrices

$A = (3 \times 3)$ $\quad B = (3 \times 3)$

$$\begin{cases} A \cdot B = C \\ B \cdot A \stackrel{?}{=} C' \\ \text{Q is } \boxed{C' = C}\ ? \end{cases}$$

→ Some (but not all) matrices commute i.e. $\boxed{C' = C}$

In general $\boxed{C' \neq C}$ – i.e. non commuting matrices.

M2.1

$A = \begin{pmatrix} 2 & 4 \\ 6 & 1 \end{pmatrix}$

- Powers of matrices $\qquad A^2 = A.A = \begin{pmatrix} 2 & 4 \\ 6 & 1 \end{pmatrix} . \begin{pmatrix} 2 & 4 \\ 6 & 1 \end{pmatrix} = \begin{pmatrix} 28 & 12 \\ 18 & 25 \end{pmatrix} \neq \begin{bmatrix} 2^2 & 4^2 \\ 6^2 & 1^2 \end{bmatrix}$

$A^2 = A.A, \qquad A^3 = A.A.A$, etc

- Functions of matrices.

$$e^A = \mathbb{1} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

- Special matrices

$\rightarrow \quad \mathbb{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 \\ 0 & & \ddots & 1 \end{bmatrix} \qquad \left.\begin{matrix} A.\mathbb{1} = A \\ B.\mathbb{1} = B \end{matrix}\right\}$

$\rightarrow \quad \mathbb{O} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \end{bmatrix} \qquad A - A = \mathbb{O}$

Example $Q = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \qquad Q.Q = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\boxed{Q \neq \mathbb{O}} \qquad \boxed{Q^2 = \mathbb{O}}$

- $(A+B)^2 = (A+B).(A+B)$

$= A^2 + B^2 + (A.B + B.A)$

$\hookrightarrow$ not $\underline{2A.B}$ Because?

- $(A + 3B)^2 = A^2 + 9B^2 + 3(A.B + BA)$

etc

$$\underline{(A + iB)^2 = A^2 - B^2 + i\,(AB + BA)}$$

Matrices acting on vectors!

Linear algebra name derives from this type of operation

$$\left.\begin{matrix} M = n\times n \\ V = n\times 1 \end{matrix}\right\} \qquad [M\cdot V] = (n\times 1) \quad \therefore \text{ matches}$$

$$\therefore \boxed{M\cdot V = V'}$$

Example

$(3\times 3)\cdot(3\times 1)$

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} m_{11}v_1 + m_{12}v_2 + m_{13}v_3 \\ m_{21}v_1 + m_{22}v_2 + m_{23}v_3 \\ m_{31}v_1 + m_{32}v_2 + m_{33}v_3 \end{bmatrix} \equiv \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix}$$

- Linearity

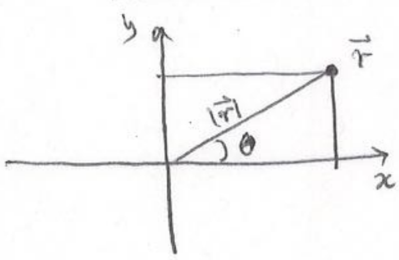
v_1, v_2 we can linearly $c_1 v_1 + c_2 v_2$

$\longrightarrow \quad M\cdot(c_1 v_1 + c_2 v_2) = c_1 M v_1 + c_2 M v_2 = c_1 v_1' + c_2 v_2'$!

$\longrightarrow \quad (\alpha M + \alpha' M')\cdot V = \alpha (M\cdot V) + \alpha' (M'\cdot V)$

c_1 and c_2 are arbitrary numbers.

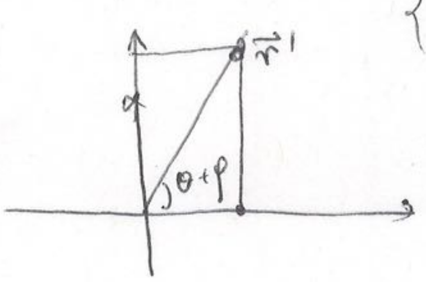
- Physical meaning: Example: 2-d vectors



$\longleftrightarrow \quad \vec{r} = (x, y) \qquad \left.\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}\right\} \quad \boxed{r = |\vec{r}|}$

Now rotate about $\hat{z}$ axis by an angle φ

i.e. $\theta \to \theta + \varphi \qquad \vec{r} \to \vec{r}\,' =$ rotated version



$\vec{r}\,' = (x', y') \qquad |\vec{r}\,'| = r$ (rotation)

$$\begin{cases} x' = r\cos(\theta+\varphi) = x\cos\varphi - y\sin\varphi \\ y' = r\sin(\theta+\varphi) = y\cos\varphi + x\sin\varphi \end{cases}$$

We saw from this that

$$\boxed{\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}$$

$\therefore$ $\boxed{V' = M.V}$ Matrix transforms V to V'

But.

Is it a rotation?

Preparation: $|\vec{r}'| = |\vec{r}|$

- $|r| = \sqrt{x^2+y^2} =$

We can use a "new idea"! $N=2$

Column vector

$$r \to v = \begin{bmatrix} x \\ y \end{bmatrix}$$

$N \times 1$

Let us define

$r^T = v^T = \begin{bmatrix} x & y \end{bmatrix}$ ← row vector

Transpose $\qquad 1 \times N$

We can define matrix product

$$V^T . V = (1 \times N) . (N \times 1) = (1 \times 1) = \text{a number.}$$

scalar, i.e.

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2$$

$\therefore$ $\boxed{|\vec{r}| = \sqrt{V^T . V}}$ Thus we can define a transposed vector and construct the length.

How about matrix

$$M^T = \text{transpose}(M)$$

or

$$\boxed{(M^T)_{ij} = (M)_{ji}}$$

ex $N=2$

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$M^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

How about taking transpose

$M \cdot V = V'$?

$$V'^T = (M \cdot V)^T = V^T \cdot M^T \quad !! \quad (\text{not } (M^T \cdot V^T))$$

Proof / $\cancel{V'_i = \sum_j M_{ij} V_j}$

$\boxed{\cancel{V'}^T}$

$$v = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$M \cdot V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \underbrace{\begin{bmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{bmatrix}}_{V'}$$

Check out $V'^T \stackrel{?}{=} (V^T \cdot M^T) = \begin{bmatrix} \alpha & \beta \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a\alpha + b\beta & c\alpha + d\beta \end{bmatrix}$

✓ ok !!

$$(V^T \cdot M) \neq V'^T$$

So: How do we preserve length of a vector

$$\boxed{V^T \cdot V = V'^T \cdot V'}$$ Guarantees length conservation

RHS
$$\left.\begin{aligned} V' &= M \cdot V \\ V'^T &= V^T \cdot M^T \end{aligned}\right\}$$

$\therefore$
$$V'^T \cdot V' = V^T M^T M V = V^T (M^T M) V = V^T V$$

$\Rightarrow$ $M^T \cdot M = \mathbb{1}$ is identity

$$\mathbb{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$d=3$ $\quad$ $d=2$

Orthogonal Matrices preserve length

$$\boxed{M^T \cdot M = \mathbb{1}}$$

Check:

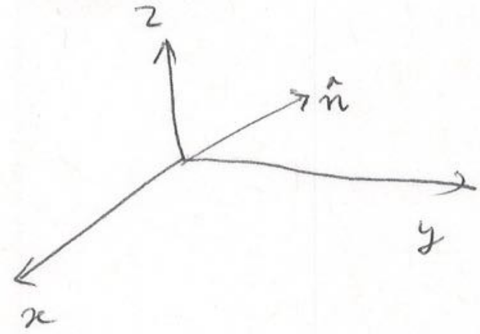
$$M = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

$$M^T = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$$

$$M^T \cdot M = \left.\begin{bmatrix} \cos^2\varphi + \sin^2\varphi & \cos\varphi\sin\varphi - \sin\varphi\cos\varphi \\ -\sin\varphi\cos\varphi + \cos\varphi\sin\varphi & \sin^2\varphi + \cos^2\varphi \end{bmatrix}\right\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

! QED

3-d rotation



We can rotate about any direction $\hat{n}$!

X-axis

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

Z-axis

$$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

etc

- $M_1 \cdot M_2 \neq M_2 \cdot M_1$ except in special cases
- $\det(M_1 M_2) = (\det M_1)(\det M_2)$ square matrix. Always true!

↑ Class on 15 Feb

↓ Class on 20 Feb

- $MV = V'$

$\Rightarrow \boxed{V = M^{-1} \cdot V'}$ Inverse matrix $\boxed{\text{if } M^{-1} \text{ exists}}$ $\quad M \cdot M^{-1} = \mathbb{1} = M^{-1} M$

→ In linear equations whenever Eqns are consistent a solution exists

Related fact: M^{-1} exists if eqns are consistent.