Class Feb 20, 2018

MM.1

* With rotations, we have seen that in 3-dimensions

Eqn:

$$M \cdot V = V'$$

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \qquad V' = \begin{bmatrix} V_1' \\ V_2' \\ V_3' \end{bmatrix}$$

corresponds to a mapping (Linear) of a point in 3-d

$$\left.\begin{aligned} \vec{r} &= V_1 \hat{x} + V_2 \hat{y} + V_3 \hat{z} \\ \text{to} \quad \vec{r}' &= V_1' \hat{x} + V_2' \hat{y} + V_3' \hat{z} \end{aligned}\right\}$$

$$\therefore \left.\begin{aligned} \vec{r} &\leftrightarrow V_i \\ \vec{r}' &\leftrightarrow V_i' \end{aligned}\right\}$$

× Linearity

$$\Rightarrow M \cdot (V_a + V_b) = M \cdot V_a + M \cdot V_b$$

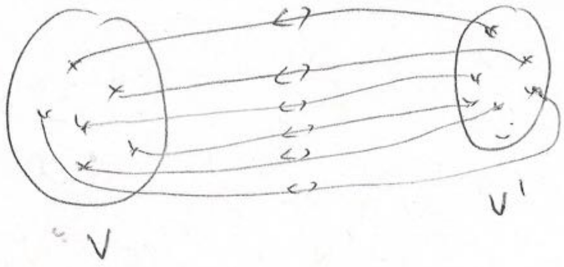
Questions

(a) Under what conditions does the inverse map exist?

(b) What property of M guarantees a rotation of space?

(c) " " " Inversion / Reflection of space?

(a) If Map is $1 \leftrightarrow 1$ we can invert!



If $M \cdot V = V'$
then: $V = M^{-1} \cdot V'$

$M \cdot M^{-1} = M^{-1} \cdot M = \mathbb{1}$

$M^{-1} \cdot M \cdot V = M^{-1} \cdot V'$

Since $\mathbb{1} \cdot V = V$, we are ok!

If map is not $1 \leftrightarrow 1$, cannot invert!

- Condition for inversion to make sense =

$\boxed{\det M \neq 0}$

- How do we find the inverse of a matrix?

Recall some properties of determinants

Example of 3×3 matrix A

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\|A\| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}.$$

{Calculation proceeds by forming Cofactors!

What are Cofactors?

Let us expand $\|A\|$ along first row — (say)

$$\|A\| = A_{11}(A_{22}A_{33} - A_{32}A_{23}) - A_{12}(A_{21}A_{33} - A_{31}A_{23}) + A_{13}(A_{21}A_{32} - A_{31}A_{22})$$

notice the sign!

$$\equiv A_{11} \overbrace{\begin{Vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{Vmatrix}}^{M_{11}} - A_{12} \overbrace{\begin{Vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{Vmatrix}}^{M_{12}} + A_{13} \overbrace{\begin{Vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{Vmatrix}}^{M_{13}}$$

We give these sub-determinants a name $\rightarrow$ Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

or Minor M_{ij}

1st row

φ_{11} $\boxed{\|A\| \equiv A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13}}$ or $\boxed{\|A\| = (A_{11}M_{11} - A_{12}M_{12} + A_{13}M_{13})}$

$\therefore (-1)^{i+j}$ takes out the sign, giving a simple reln.

We could also have expanded along a different row, or column

Example. 2nd column

$$\|A\| = A_{12}C_{12} + A_{22}C_{22} + A_{32}C_{32}$$

3rd row

$$\|A\| = A_{31}C_{31} + A_{32}C_{32} + A_{33}C_{33}$$

φ_{33}

Rules to get C_{ij}

(a) Strike out i^{th} row and j^{th} column in A, which gives a lower dimensional matrix.

(b) ~~Cofactor~~ Determinant of this lower dimnl matrix is M_{ij} (the minor)

(c) $(-1)^{i+j} M_{ij} = C_{ij}$

• Now next take matrix of cofactors

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Keeping in mind $\quad \|A\| = A_{11} C_{11} + A_{12} C_{12} + A_{13} C_{13}$

we take transpose of C $\quad (C^T)_{ij} = C_{ji}$

$$C^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Next compute

$$A \cdot C^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \cdot \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{bmatrix} = \varphi$$

$$\phi_{ij} = \sum_{k=1}^{3} A_{ik} C^T_{kj} = \sum_{k=1}^{3} A_{ik} C_{jk}$$

Now $\quad \varphi_{11} = A_{11} C_{11} + A_{12} C_{12} + A_{13} C_{13} = \|A\|$

$\quad \varphi_{33} = A_{31} C_{31} + A_{32} C_{32} + A_{33} C_{33} = \|A\|$

Thus $\varphi_{ii} = \|A\| \qquad i=1,2,3$

What about

φ_{ij} for $i \neq j$!

Recall

$$A = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

Try $\phi_{12} = A_{11} C_{21} + A_{12} C_{22} + A_{13} C_{23}$

$= - A_{11} M_{21} + A_{12} M_{22} - A_{13} M_{23}$

$= \cancel{A_{11}}$

$$= - A_{11} \left(\underset{\mathrm{I}}{A_{12} A_{33}} - \underset{\mathrm{II}}{A_{32} A_{13}} \right) + A_{12} \left(\underset{\mathrm{I'}}{A_{11} A_{33}} - \underset{\mathrm{III}}{A_{31} A_{13}} \right) - A_{13} \left(\underset{\mathrm{II'}}{A_{11} A_{32}} - \underset{\mathrm{III'}}{A_{12} A_{31}} \right)$$

$= 0!$

In fact $\boxed{\varphi_{ij} = \delta_{ij} \|A\|}$ i.e. all off diagonals are zero

$$A . C^T = \varphi = \begin{bmatrix} \|A\| & 0 & 0 \\ 0 & \|A\| & 0 \\ 0 & 0 & \|A\| \end{bmatrix} = \|A\| \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\mathbb{1}}$$

Therefore $\boxed{A^{-1} \equiv \frac{1}{\|A\|} C^T}$ Important result!

$$\boxed{A . A^{-1} = \mathbb{1} = A^{-1} . A}$$

Question (5)

Rotations?

$M . V = V'$ we want $\|V'\| = \|V\|$

i.e. $V'^T . V' = V^T . V$

Now $\boxed{V'^T = V^T . M^T}$ $\boxed{(A.B)^T = B^T . A^T}$

$$\therefore V'^T . V' = V^T . (M^T . M) V = V^T V$$

$$\therefore \boxed{M^T . M = \mathbb{1} = M . M^T} \quad \boxed{M^{-1} . M = M . M^{-1} = \mathbb{1}} \quad \boxed{M^T = M^{-1}}$$

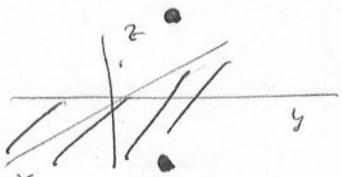
orthogonal matrix

$\cdot$ Det $|M^T . M| = 1 = |\det M|^2$ (using $\det M^T = \det M$ Prove!)

$\therefore \boxed{\det M = \pm 1}$ $\boxed{\det(A.B) = (\det A) \times (\det B)}$

$+1 \rightarrow$ rotation

$-1 \rightarrow$ reflections / and inversion in odd dimensions only

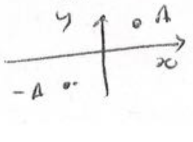


• 3-d $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix}$ reflection in x_3 plane

$\therefore M \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\therefore \det |M| = -1$ $\Bigg\}$ $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\det = -1$

• 2-d $\det \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = 1$

$\therefore$ In 2-d inversions are not different from rotation



$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{bmatrix} -x \\ -y \end{bmatrix}$ Also a rotation of π (about z axis).

Linear Eqns and matrices: Revisited

Two classes of problems

(1) General Linear

$$a_{11}c_1 + a_{12}c_2 + a_{13}c_3 + \cdots + a_{1N}c_N = b_1$$

$$a_{21}c_1 + a_{22}c_2 + \qquad + a_{2N}c_N = b_2$$

$$a_{N1}c_1 + a_{N2}c_2 + \cdots + a_{NN}c_N = b_N$$

$$[A].[c_i] = [b_i]$$

$$\{(N\times N).(N\times 1) = (N\times 1)\}$$

(2) Homogeneous Eqns: $b_1 = 0 = b_2 = \cdots = b_N$

$\therefore [b_i] = 0$

Solutions require the inversion of A, $\boxed{A^{-1} = \left(\frac{1}{\det A}\right) C^T}$

(1) If $|\det A| \neq 0$, we can find A^{-1}. Hence

$[c] = A^{-1}[b]$ can be performed for any b.

$\therefore$ given any b and A, st $\det|A| \neq 0$

we can find non-trivial [c].

Here non-trivial $\Rightarrow c_i \neq 0$.

$\therefore$ Called a consistent set of LE's.

(2) Homogeneous eqn:

If $[b] = 0$ we can always solve the eqns by $c_i = 0$.

This is ~~not too~~ interesting. and requires $|\det A| \neq 0$

$\boxed{\text{Interesting case}}$ $\boxed{\text{If } |\det A| \neq 0}$ $\longrightarrow$ Linear Independence of Eqns.

If $\det A| = 0$, we can see an example below

MM.7

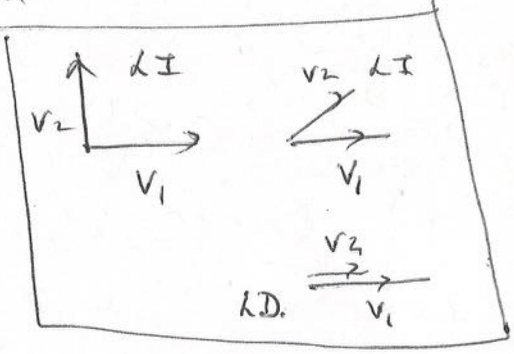
Linear independence

given v_1, v_2, are they L.I ?

If c_1, c_2 $\exists$, s.t. $\boxed{c_1 v_1 + c_2 v_2 = 0}$ then ~~not~~ L.I

$v_2 = \left(-\frac{c_1}{c_2}\right) v_1$

2 d



3-d example

$V_1 \ V_2 \ V_3$ given

$c_1 V_1 + c_2 V_2 + c_3 V_3 = 0$

If the only solution is $c_1 = 0 = c_2 = c_3$

Then $V_1 V_2 V_3$ are linearly Ind.

If non-trivial solution $\exists$, (i.e c_j's non zero)

Then Linearly dependent!

$$V_1 = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \quad V_2 = \begin{bmatrix} 2 \\ 6 \\ 7 \end{bmatrix} \quad V_3 = \begin{bmatrix} 11 \\ 5 \\ 4 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 6 \\ 7 \end{bmatrix} + c_3 \begin{bmatrix} 11 \\ 5 \\ 4 \end{bmatrix} = 0$$

$\therefore$

$$\underbrace{\begin{bmatrix} 1 & 4 & 9 \\ 2 & 6 & 7 \\ 11 & 5 & 4 \end{bmatrix}^{T}}_{m} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Trm T

$\boxed{\det m = 239} \neq 0$

$$m^{-1} = \frac{1}{239} \begin{bmatrix} 11 & -29 & 26 \\ -69 & 95 & -11 \\ 56 & -39 & 2 \end{bmatrix}$$

Row redn. $\boxed{c = m^{-1} \cdot W}$

$\therefore$ If we put $W = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\boxed{c_1 = 0 = c_2 = c_3}$

$\therefore$ vectors are L.I.

Let us change

$$V_3 = \begin{bmatrix} 3 \\ 10 \\ 16 \end{bmatrix}$$

Then $m = \begin{bmatrix} 1 & 4 & 9 \\ 2 & 6 & 7 \\ 3 & 10 & 16 \end{bmatrix}$ m^{-1} does not exist

$\boxed{\det M = 0}$

By row reduction

$$\sim \begin{bmatrix} 1 & 4 & 9 \\ 2 & 6 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $= 2 < 3$

$\therefore$ Linearly dependent

$V_3 = V_1 + V_2$

$\boxed{\text{Need } \cancel{\text{to put } c_3 = 0}}$! ~~But~~ $\therefore$ Linearly dependent LD.

$c_1 V_1 + c_2 V_2 + c_3 (V_1 + V_2) = 0$

$\Rightarrow (c_1 + c_3) V_1 + (c_2 + c_3) V_2 = 0$

$\therefore \boxed{c_3 = -c_1 = -c_2}$

* Linear dependence of functions! "Wronskian"

$\boxed{-\frac{d^2 f}{dx^2} = k^2 f}$

$$\begin{cases} f_1(x) = \sin x \\ f_2(x) = \cos x \end{cases}$$

$\boxed{\alpha \doteq c_1 \sin x + c_2 \cos x = 0}$ $\boxed{\forall x}$

Can I find c_1, c_2 st. this eqn is true with non trivial c_1, c_2. ($c_1 = 0 = c_2$ is trivial)

If $\alpha = 0$ $\frac{d}{dx}\alpha = 0$ as well

$$\left.\begin{aligned} \alpha &= c_1 \sin x + c_2 \cos x = 0 \\ \frac{d\alpha}{dx} &= +c_1 \cos x - c_2 \sin x = 0 \end{aligned}\right\} \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\boxed{\det = -1}$

$\therefore c_1 = 0 = c_2$ is only solution.